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Recurrent Geodesics in Flat Lorentz 3-Manifolds

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Abstract. Let M be a complete flat Lorentz 3-manifold M with purely hyperbolic holonomy Γ. Recurrent geodesic rays are completely classified when Γ is cyclic. This implies that for any pair of periodic geodesics γ_1, γ_2, a unique geodesic forward spirals towards γ_1 and backward spirals towards γ_2.

1 Introduction

This note concerns the dynamical properties of geodesics in a flat Lorentz 3-manifold M. We assume M is geodesically complete, that is, M is the quotient $\mathbb{A}^{2,1}/\Gamma$ of 3-dimensional Minkowski spacetime $\mathbb{A}^{2,1}$ by a discrete group Γ of affine isometries acting properly on $\mathbb{A}^{2,1}$.

A *recurrent* geodesic ray is a nonproper affine map from $\mathbb{R}^+$ into M. We mainly focus on the case when Γ is cyclic. This basic example already displays rich and interesting behavior. Theorem 3.3 implies that recurrent geodesics lie in one of two codimension-one submanifolds of M, which intersect in the unique periodic (and birecurrent) geodesic.

Section 2 develops preliminaries on Minkowski space and its isometries. We describe a measure of signed Lorentzian distance, called the *Margulis invariant*, which we use to classify recurrent rays. We define what it means for a geodesic to be recurrent and spiralling.

Section 3 is devoted to the particular case of *cylinders*: quotients of spacetime by cyclic hyperbolic groups $\langle\gamma\rangle$. By means of a $\langle\gamma\rangle$-invariant function, we show in Lemma 3.1 that recurrent geodesic rays must lie in one of two codimension-one submanifolds. Not every geodesic ray in those submanifolds is recurrent: Theorem 3.3 provides a characterization of recurrent geodesic rays in cylinders. The notion of a geodesic ray spiralling towards a periodic geodesic is introduced.

The most interesting examples are *Margulis spacetimes*, when Γ is a free purely hyperbolic discrete subgroup of the isometry group of $\mathbb{A}^{2,1}$, that is, a *Schottky group*. Drumm [2, 3] showed that every noncocompact discrete subgroup of $\mathrm{SO}(2,1)$ admits proper affine deformations. Non-periodic birecurrent geodesics can only be found when the rank of the fundamental group is greater than one. We discuss how Theorem 3.3 extends to Margulis spacetimes in Section 4.

Some of the results in this paper first appeared in [1].

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2 Geometry of Minkowski Spacetime

2.1 Minkowski 2 + 1-Spacetime

Let $\mathbb{R}^{2,1}$ denote a three-dimensional real vector space equipped with the standard symmetric bilinear form of signature $(2,1)$:

$$\mathbb{B}(\mathrm{x},\mathrm{y}) = x_1 y_1 + x_2 y_2 - x_3 y_3,$$

where $\mathrm{x} = (x_1, x_2, x_3)$ and $\mathrm{y} = (y_1, y_2, y_3)$. A vector $\mathrm{v} \in \mathbb{R}^{2,1}$ is *spacelike* (resp. *timelike*, *lightlike*) if $\mathbb{B}(\mathrm{v},\mathrm{v}) > 0$ (resp. $\mathbb{B}(\mathrm{v},\mathrm{v}) < 0$, $\mathbb{B}(\mathrm{v},\mathrm{v}) = 0$). (Lightlike vectors are also called *null*.)

Denote by $\mathbb{A}^{2,1}$ the affine space modeled on $\mathbb{R}^{2,1}$: for every $p \in \mathbb{A}^{2,1}$, the tangent space

$$\mathbb{A}^{2,1}_p = \{q - p : q \in \mathbb{A}^{2,1}\}$$

is endowed with the bilinear form $\mathbb{B}(\,\cdot\,,\,\cdot\,)$. Clearly, $\mathbb{A}^{2,1}_p \cong \mathbb{R}^{2,1}$.

We adopt the following convention to distinguish vectors from points in affine space: vectors in $\mathbb{R}^{2,1}$ will be written in bold face $\mathrm{x}, \mathrm{y}, \mathrm{v}$ *etc.*, whereas points in $\mathbb{A}^{2,1}$ will be denoted p, q, *etc.*

Any line in $\mathbb{A}^{2,1}$ can be described as $p + \mathbb{R}\mathrm{v}$, where $p \in \mathbb{A}^{2,1}$ and $\mathrm{v} \in \mathbb{R}^{2,1}$. Two lines $p + \mathbb{R}\mathrm{v}$, $q + \mathbb{R}\mathrm{w}$ are *parallel* if $\mathrm{v} = k\mathrm{w}$ for some $k \neq 0$; we also say that the line is *parallel to* v. The line $p + \mathbb{R}\mathrm{v}$ is called *spacelike*, *timelike* or *lightlike* according to the causal character of v.

The set of non-spacelike vectors, with the origin removed, has two connected components. A choice of component is a *time orientation* on $\mathbb{R}^{2,1}$. We will adopt the standard time orientation: a non-spacelike vector $\mathrm{v} = (v_1, v_2, v_3)$ is *future-pointing* if $v_3 > 0$ and *past-pointing* otherwise.

The *Lorentz-orthogonal plane* of $\mathrm{v} \in \mathbb{R}^{2,1}$ at p is the set of all vectors based at p which are *Lorentz-orthogonal* to v:

$$\begin{aligned} p + \mathrm{v}^{\perp} &= \{q \in \mathbb{A}^{2,1} : \mathbb{B}(q - p, \mathrm{v}) = 0\} \\ &= \{p + \mathrm{x} : \mathbb{B}(\mathrm{x},\mathrm{v}) = 0, \mathrm{x} \in \mathbb{R}^{2,1}\}. \end{aligned}$$

2.2 Isometries

An *affine isometry* of $\mathbb{A}^{2,1}$ is an affine transformation γ whose linear part preserves the bilinear form $\mathbb{B}(\,\cdot\,,\,\cdot\,)$. Thus the linear part of an affine isometry of $\mathbb{A}^{2,1}$ lies in $O(2,1)$. The isometry group of $\mathbb{A}^{2,1}$ is denoted $\mathrm{Isom}(\mathbb{A}^{2,1})$. The connected component of the identity of $O(2,1)$, denoted $\mathrm{SO}(2,1)^0$, consists of those linear isometries which preserve orientation and time orientation.

An affine isometry is said to be *hyperbolic* if its linear part is hyperbolic: that is, it is an element of $\mathrm{SO}(2,1)^0$ which has three real distinct eigenvalues.

If $g \in \mathrm{SO}(2,1)^0$ is hyperbolic, then its eigenvalues are $\lambda < 1 < \lambda^{-1}$, for some positive $\lambda \in \mathbb{R}$. The λ- and λ^{-1}-eigendirections are lightlike and the 1-eigendirection is spacelike.

Let $g \in \mathrm{SO}(2,1)^0$ be a hyperbolic isometry with smallest eigenvalue $\lambda < 1$. Set $\mathsf{x}^+(g)$, $\mathsf{x}^-(g)$ to be future-pointing eigenvectors of λ^{-1}, λ, respectively, normalized so that $\|\mathsf{x}^\pm(g)\| = 1$, where $\|\cdot\|$ denotes Euclidean length.

Choose $\mathsf{x}^0(g)$ to be the unique spacelike 1-eigenvector satisfying $\mathbb{B}(\mathsf{x}^0(g), \mathsf{x}^0(g)) = 1$, such that the *null frame*

$$\{\mathsf{x}^0(g), \mathsf{x}^-(g), \mathsf{x}^+(g)\}$$

is a positively oriented basis. For hyperbolic $\gamma \in \mathrm{Isom}(\mathbb{A}^{2,1})$ with linear part g, set:

$$\{\mathsf{x}^0(\gamma), \mathsf{x}^-(\gamma), \mathsf{x}^+(\gamma)\} = \{\mathsf{x}^0(g), \mathsf{x}^-(g), \mathsf{x}^+(g)\}.$$

The following facts are well known:

Lemma 2.1 *Let $\gamma \in \mathrm{Isom}(\mathbb{A}^{2,1})$ be hyperbolic. Then there exists a line $C_\gamma \subset \mathbb{A}^{2,1}$, parallel to $\mathsf{x}^0(\gamma)$, which is invariant under the action of γ. Moreover, γ acts by translation on C_γ. Furthermore, C_γ is the unique γ-invariant line if and only if γ acts freely on $\mathbb{A}^{2,1}$.*

Proof Since γ is affine, it acts on N, the space of lines parallel to $\mathsf{x}^0(\gamma)$. Observe that N is isomorphic to the Lorentz-orthogonal plane $\mathsf{x}^0(\gamma)^\perp$, which, in turn, is isomorphic to two-dimensional Minkowski space. The eigenvalues of the induced action of γ each differ from 1, so this action has a fixed point.

Let $C_\gamma \subset \mathbb{A}^{2,1}$ be the line parallel to $\mathsf{x}^0(\gamma)$ corresponding to this fixed point in N; clearly, C_γ is invariant under the action of γ: if $p \in C_\gamma$,

$$\gamma(p) = p + \alpha \mathsf{x}^0(\gamma), \tag{1}$$

where $\alpha \in \mathbb{R}$. Any other point q on C_γ can be written as $p + t\mathsf{x}^0(\gamma)$, for some $t \in \mathbb{R}$. Since $\mathsf{x}^0(\gamma)$ is fixed by γ,

$$\gamma(q) = \gamma\big(p + t\mathsf{x}^0(\gamma)\big) = \gamma(p) + t\mathsf{x}^0(\gamma) = q + \alpha\mathsf{x}^0(\gamma).$$

Thus γ acts by translation on C_γ.

Finally, γ fixes a point p if and only if the line $p + \mathbb{R}\mathsf{x}^0(\gamma)$ is pointwise fixed, so C_γ is the unique γ-invariant line if and only if γ acts freely. ■

2.3 Margulis's Invariant

In his construction of properly discontinuous affine groups, Margulis [4, 5] introduced the following invariant.

Definition 2.2 For a hyperbolic isometry $\gamma \in \mathrm{Isom}(\mathbb{A}^{2,1})$, set $\alpha(\gamma)$ to be the parameter α in (1).

Since $\alpha(\gamma)$ represents the displacement of γ along C_γ, the proof of Lemma 2.1 implies:

Lemma 2.3 *Let $\gamma \in \mathrm{Isom}(\mathbb{A}^{2,1})$ be a hyperbolic isometry. Then γ acts freely on $\mathbb{A}^{2,1}$ if and only if $\alpha(\gamma) \neq 0$.* ■

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2.4 Stable and Unstable Planes

Definition 2.4 Let $\gamma \in \mathrm{Isom}(\mathbb{A}^{2,1})$ be hyperbolic and let $p \in C_\gamma$. The planes

$$E^+_\gamma = p + \langle \mathrm{x}^0(\gamma), \mathrm{x}^+(\gamma)\rangle = p + \mathrm{x}^+(\gamma)^\perp$$
$$E^-_\gamma = p + \langle \mathrm{x}^0(\gamma), \mathrm{x}^-(\gamma)\rangle = p + \mathrm{x}^-(\gamma)^\perp,$$

are the *weak-unstable plane* and the *weak-stable plane* of γ respectively.

Note that $E^+_\gamma = E^-_{\gamma^{-1}}$, $E^+_\gamma \cap E^-_\gamma = C_\gamma$, and $E^\pm_\gamma$ is γ-invariant.

Consider the orbit of a point q in $\mathbb{A}^{2,1}$, under the action of γ. We can write

$$q = p + k^+\mathrm{x}^+(\gamma) + k^-\mathrm{x}^-(\gamma),$$

where $p \in C_\gamma$ and $k^\pm \in \mathbb{R}$. Thus for every n,

$$\gamma^n(q) = p + n\alpha(\gamma)\mathrm{x}^0(\gamma) + k^+\lambda^{-n}\mathrm{x}^+(\gamma) + k^-\lambda^n\mathrm{x}^-(\gamma).$$

If $k^+ = 0$, the orbit converges towards C_γ as n increases. When $k^+ \neq 0$, the sequence approaches E^+_γ, but eventually leaves every compact set intersecting the weak-unstable plane. When $n \to -\infty$, $\gamma(q)$ approaches C_γ if $k^- = 0$ and approaches E^-_γ otherwise.

2.5 Geodesics

Let $M = \mathbb{A}^{2,1}/\Gamma$, where $\Gamma < \mathrm{Isom}(\mathbb{A}^{2,1})$ acts properly on $\mathbb{A}^{2,1}$. Then M is a complete Lorentz manifold and its fundamental group is isomorphic to Γ. Let $\pi\colon \mathbb{A}^{2,1} \to M$ denote the quotient projection.

A *geodesic* in M is a nonconstant affine map $l\colon \mathbb{R} \to M$, that is, the composition $\pi \circ \tilde{l}$ where $\tilde{l}\colon \mathbb{R} \to \mathbb{A}^{2,1}$ is a nonconstant affine map. The *reverse* of a geodesic l is the geodesic $-l$ defined by

$$-l(t) := l(-t).$$

A *geodesic ray* in M is a nonconstant affine map $l\colon \mathbb{R}^+ \to M$, that is, the composition $\pi \circ \tilde{l}$ where $\tilde{l}\colon \mathbb{R}^+ \to \mathbb{A}^{2,1}$ is a nonconstant affine map. The *forward ray* of a geodesic l is the restriction of l to $\mathbb{R}^+$ and the *backward ray* of l is the forward ray of $-l$.

A geodesic ray l is *parallel* to a line (respectively vector or ray), if it can be lifted to a line in $\mathbb{A}^{2,1}$ that is parallel to the line (respectively, vector or ray).

2.6 Periodic Geodesics

A geodesic l is *periodic* if for some $T > 0$, and all $t > 0$,

$$l(t + T) = l(t). \tag{2}$$

The smallest positive T satisfying (2) is the *period* of l. Let $l\colon \mathbb{R} \to M$ be a periodic geodesic with period T. The restriction of l to $[0, T]$ determines an element

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$\gamma_l \in \pi_1\big(M; l(0)\big)$. The *cylinder associated to* l is the quotient $M_l := \mathbb{A}^{2,1}/\langle\gamma_l\rangle$, which is the total space of a covering space $M_l \to M$.

In particular, if $\pi_1\big(M, l(0)\big)$ is cyclic and the linear holonomy is purely hyperbolic, then M is a cylinder.

We will often identify a periodic geodesic with its image, which is an immersed S^1 in M.

2.7 Recurrence

The following conditions are equivalent:

- $l\colon \mathbb{R}^+ \to M$ is a proper map: for every compact $K \subset M$, the inverse image $l^{-1}(K) \subset \mathbb{R}^+$ is compact;
- For every increasing sequence $t_k \to +\infty$, the sequence $l(t_k)$ has no accumulation point;
- l is not periodic and the image $l(\mathbb{R}^+ \cup \{0\})$ is closed.

A geodesic ray l is *recurrent* if the mapping $l\colon \mathbb{R}^+ \to M$ is not proper. Equivalently, its image $l(\mathbb{R}^+)$ has compact closure in M.

A geodesic is also said to be recurrent if its forward ray or its backward ray is recurrent. For example, a periodic geodesic is recurrent.

A geodesic ray r *spirals towards a periodic geodesic* l if for every neighborhood N of l, there exists $T = T(N) > 0$ such that $r(t) \in N$ for $t > T$. In particular, such a geodesic is recurrent.

A geodesic l is *birecurrent* if both its forward ray and its backward ray are recurrent. A geodesic l *bispirals* if both its forward ray and backward ray spiral towards periodic geodesics.

3 Cylinders

We now classify recurrent geodesic rays in cylinders. Let $M = \mathbb{A}^{2,1}/\langle\gamma\rangle$, where γ is a hyperbolic affine isometry of $\mathbb{A}^{2,1}$.

The *stable surface* $M^- = \pi(E^-_\gamma)$ and the *unstable surface* $M^+ = \pi(E^+_\gamma)$ are each diffeomorphic to an annulus to which M deformation retracts. Similarly M^+, M^- each deformation retract to the *core geodesic* $M^0 = M^+ \cap M^-$, which is the unique periodic geodesic in M.

Lemma 3.1 *A recurrent geodesic ray* $\mathbb{R}^+ \to M$ *lies in either* M^+ *or* M^-.

Proof Let p be an arbitrary point on the invariant line C_γ. For every point $q \in \mathbb{A}^{2,1}$, write

$$q = p + k^0 \mathrm{x}^0(\gamma) + k^- \mathrm{x}^-(\gamma) + k^+ \mathrm{x}^+(\gamma)$$

where $k^-, k^+, k^0 \in \mathbb{R}$ and $\{\mathrm{x}^-(\gamma), \mathrm{x}^+(\gamma), \mathrm{x}^0(\gamma)\}$ is the null frame associated to γ. Then

$$\gamma(q) = p + \big(k^0 + \alpha(\gamma)\big)\mathrm{x}^0(\gamma) + \lambda k^- \mathrm{x}^-(\gamma) + \lambda^{-1} k^+ \mathrm{x}^+(\gamma)$$

and

$$\mathbb{B}\big(\gamma(q) - p, \mathsf{x}^{\pm}(\gamma)\big) = \mathbb{B}\big(\gamma(q - p), \mathsf{x}^{\pm}(\gamma)\big) = \lambda^{\pm 1}\mathbb{B}\big(q - p, \mathsf{x}^{\pm}(\gamma)\big).$$

Thus the quadratic function $\tilde{f}\colon \mathbb{A}^{2,1} \to \mathbb{R}$ given by

$$\tilde{f}(q) = \mathbb{B}\big(q - p, \mathsf{x}^{-}(\gamma)\big)\,\mathbb{B}\big(q - p, \mathsf{x}^{+}(\gamma)\big) \tag{3}$$

is independent of the choice of $p \in C_\gamma$, and is $\langle\gamma\rangle$-invariant. Define $f\colon M \to \mathbb{R}$ as $f := \pi \circ \tilde{f}$.

Suppose that $l\colon \mathbb{R}^+ \to M$ is a recurrent geodesic ray. Write $l = \pi \circ \tilde{l}$ where

$$\begin{aligned} \tilde{l}\colon \mathbb{R}^+ &\to \mathbb{A}^{2,1} \\ t &\mapsto q + t\mathsf{v} \end{aligned} \tag{4}$$

is a lift of l in $\mathbb{A}^{2,1}$. Then

$$(f \circ l)(t) = (\tilde{f} \circ \tilde{l})(t) = a + bt + ct^2 \tag{5}$$

where p is an arbitrary point on C_γ and

$$\begin{gathered} a = f(l(0)), \\ b = \mathbb{B}\big(q - p, \mathsf{x}^{+}(\gamma)\big)\,\mathbb{B}\big(\mathsf{v}, \mathsf{x}^{-}(\gamma)\big) + \mathbb{B}\big(q - p, \mathsf{x}^{-}(\gamma)\big)\,\mathbb{B}\big(\mathsf{v}, \mathsf{x}^{+}(\gamma)\big), \\ c = \mathbb{B}\big(\mathsf{v}, \mathsf{x}^{+}(\gamma)\big)\,\mathbb{B}\big(\mathsf{v}, \mathsf{x}^{-}(\gamma)\big). \end{gathered}$$

Unless $c = 0$, the function $f \circ l\colon \mathbb{R}^+ \to \mathbb{R}$ in (5) is quadratic and is unbounded as $t \to +\infty$. If $c = 0$ but $b \neq 0$, then $f \circ l$ is a nonconstant affine function, also tending to $\pm\infty$ as $t \to +\infty$. In either case $f \circ l$ defines a proper map $\mathbb{R}^+ \to \mathbb{R}$, contradicting recurrence of l. Thus $f \circ l$ is constant, that is $b = c = 0$.

Since $c = 0$, at least one of $\mathbb{B}\big(\mathsf{v}, \mathsf{x}^{+}(\gamma)\big), \mathbb{B}\big(\mathsf{v}, \mathsf{x}^{-}(\gamma)\big)$ must vanish. Thus $\tilde{l}$ is parallel to either E^-_γ or E^+_γ respectively. If it is parallel to both, then v is parallel to $\mathsf{x}^0(\gamma)$. We postpone the discussion of this case to the end of the proof.

Suppose that $\tilde{l}$ is parallel to E^+_γ, but not E^-_γ. Thus $\mathbb{B}\big(\mathsf{v}, \mathsf{x}^{+}(\gamma)\big) = 0$ but $\mathbb{B}(\mathsf{v}, \mathsf{x}^{-}(\gamma)) \neq 0$. Then $b = 0$ implies that $\mathbb{B}\big(q - p, \mathsf{x}^{+}(\gamma)\big) = 0$. Therefore $\tilde{l}\colon \mathbb{R}^+ \to E^+_\gamma$.

In the same fashion, if $\tilde{l}$ is parallel to E^-_γ, but not E^+_γ, then $\tilde{l}(\mathbb{R}^+) \subset E^-_\gamma$.

Finally, suppose that $\tilde{l}$ is parallel to $\mathsf{x}^0(\gamma)$, but lies on neither E^-_γ nor E^+_γ. Let

$$U = \mathbb{A}^{2,1} - (E^-_\gamma \cup E^+_\gamma).$$

Consider the quotient space N of $\mathbb{A}^{2,1}$ by the one-dimensional foliation parallel to C_γ. The restriction of the quotient map $\Pi\colon \mathbb{A}^{2,1} \to N$ to U is a $\langle\gamma\rangle$-equivariant mapping with respect to a *proper* action of $\langle\gamma\rangle \cong \mathbb{Z}$ on $U' = \Pi(U)$. Specifically, U'

is the complement of two intersecting lines in the plane N and $\langle\gamma\rangle$ acts by hyperbolic linear maps with discrete orbits.

Thus the composition

$$\langle\gamma\rangle \times \{q\} \to \mathbb{A}^{2,1} \xrightarrow{\Pi} N$$
$$(\gamma^n, q) \mapsto \gamma^n q$$

is proper. Consequently the mapping

$$\langle\gamma\rangle \times \mathbb{R}^+ \to \mathbb{A}^{2,1}$$
$$(\gamma^n, t) \mapsto \gamma^n(q + t\mathsf{v})$$

is proper. Therefore $l\colon \mathbb{R}^+ \to M$ is proper, a contradiction. ∎

Lemma 3.2 *Suppose $l = \pi \circ \tilde{l}$ is a geodesic ray in M, with $\tilde{l}(t) = p + t\mathsf{v}$. Suppose one of the following conditions holds:*

(i) $l \subset M^+$ *and* $\mathbb{B}\big(\alpha(\gamma)\mathsf{x}^0(\gamma), \mathsf{v}\big) > 0$,

(ii) $l \subset M^-$ *and* $\mathbb{B}\big(\alpha(\gamma)\mathsf{x}^0(\gamma), \mathsf{v}\big) < 0$.

Then l spirals towards M^0.

Proof First suppose that $l \subset M^+$, so that $\tilde{l}$ lies in the weak-unstable plane E^+_γ. Let N be a neighborhood of M^0.

We can choose $\tilde{V}$ in a lift of N such that $\tilde{V} \cap E^+_\gamma$ is a quadrilateral with vertices $p \pm k^0\mathsf{x}^0(\gamma) \pm k^+\mathsf{x}^+(\gamma)$, with $p \in C_\gamma$. Then:

$$\gamma^n\tilde{V} \cap E^+_\gamma = p + \big(\alpha(\gamma) \pm k^0\big)\,\mathsf{x}^0(\gamma) \pm k^+\lambda^{-n}\mathsf{x}^+(\gamma).$$

As n increases, the γ^n-translates of $\tilde{V}$ are dilated in the $\mathsf{x}^+(\gamma)$-direction at the rate of λ^{-n}, whereas the $\mathsf{x}^+(\gamma)$-coefficient of $\tilde{l}$ grows linearly. Furthermore, since $\mathbb{B}\big(\alpha(\gamma)\mathsf{x}^0(\gamma), \mathsf{v}\big) > 0$, the ray $\tilde{l}\big([T, \infty)\big)$ is entirely contained in $\bigcup_{n>n_0} \gamma^n(\tilde{V})$, for some $n_0 > 0$ and $T \in \mathbb{R}^+$. Therefore $l\big([T, \infty)\big) \subset N$.

Since the $E^+_\gamma = E^-_{\gamma^{-1}}$, the proof for condition (ii) reduces to the proof considered previously: $\alpha(\gamma^{-1}) = \alpha(\gamma^{-1})$ and $\mathsf{x}^0(\gamma^{-1}) = -\mathsf{x}^0(\gamma)$ imply $\mathbb{B}(\alpha(\gamma^{-1})\mathsf{x}^0(\gamma^{-1}), \mathsf{v}) > 0$, as desired. ∎

In particular, every geodesic in $M^+ \cup M^-$ parallel to a spacelike vector contains a recurrent ray: either its forward ray or its backward ray spirals towards M^0.

We show these are the only recurrent geodesic rays in a cylinder. To this end, apply a coordinate change on $E^\pm_\gamma$, so that γ acts by translation. (Compare [1], where the approach is in the same spirit as Lemma 3.2). Although Theorem 3.3 does not require Lemma 3.2, Lemma 3.2 suggests how recurrence arises.

The restriction of γ to E^+_γ is represented by the affine map

$$\gamma^+\colon \begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 0 \\ 0 & \lambda^{-1} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} \alpha \\ 0 \end{bmatrix},$$

where $0 < \lambda < 1$ and $\alpha \neq 0$.

Apply the coordinate change

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} x \\ \eta \end{bmatrix},$$

where

$$\eta(x, y) = \lambda^{x/\alpha} y,$$
$$y(x, \eta) = \lambda^{-x/\alpha} \eta.$$

The action of $\langle\gamma\rangle$ in (x, η)-coordinates is given by horizontal translation by $n\alpha$:

$$(\gamma^+)^n \colon \begin{bmatrix} x \\ \eta \end{bmatrix} \mapsto \begin{bmatrix} x + n\alpha \\ \eta \end{bmatrix}.$$

This defines a $\langle\gamma\rangle$-invariant diffeomorphism of the stable surface M^+ with the Cartesian product $(\mathbb{R}/\alpha\mathbb{Z}) \times \mathbb{R}$

$$\xi \colon M^+ \to (\mathbb{R}/\alpha\mathbb{Z}) \times \mathbb{R}$$
$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} x \bmod \alpha\mathbb{Z} \\ \eta(x, y) \end{bmatrix}.$$

A geodesic in the unstable plane falls into one of two categories, depending on whether it is parallel to the eigenvector $\mathrm{x}^+(\gamma)$ or not. If the geodesic is parallel to $\mathrm{x}^+(\gamma)$, then we call it a *vertical geodesic*.

Theorem 3.3 *Suppose $l = \pi \circ \tilde{l}$ is a geodesic ray in M, with $\tilde{l}(t) = p + t\mathrm{v}$. Then l is recurrent if and only if one of the following holds:*

(i) $l \subset M^+$ *and* $\mathbb{B}\big(\alpha(\gamma)\mathrm{x}^0(\gamma), \mathrm{v}\big) > 0$

(ii) $l \subset M^-$ *and* $\mathbb{B}\big(\alpha(\gamma)\mathrm{x}^0(\gamma), \mathrm{v}\big) < 0$.

Proof By Lemma 3.1, any recurrent geodesic must lie in $M^+ \cup M^-$. Lemma 3.2 shows that conditions (i) and (ii) are sufficient. We will now show that condition (i) is necessary when $l \subset M^+$. The fact that condition (ii) must hold when $l \subset M^-$ is proved as in Lemma 3.2, by substituting γ^{-1} for γ.

Since in (x, η)-coordinates, γ acts by translation along the x-axis, a geodesic ray is proper if and only if its η-coordinate is unbounded.

Let $\tilde{\beta} \colon \mathbb{R} \to M^+$ be the geodesic containing $\tilde{l}$ as either its forward or backward ray.

If $\tilde{\beta}$ is vertical, it admits the following representation:

$$\tilde{\beta}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 0 \\ c \end{bmatrix}.$$

In the (x, η)-coordinate system:

$$\xi \circ \tilde{\beta}(t) = \begin{bmatrix} x_0 \bmod \alpha\mathbb{Z} \\ \eta_0 + c't \end{bmatrix},$$

where

$$\eta_0 = \lambda^{x_0/\alpha} y_0,$$
$$c' = \lambda^{x_0/\alpha} c.$$

As $t \to +\infty$, the η-coordinate is unbounded. Hence the geodesic ray $\xi \circ \tilde{\beta}(t)$ does not recur.

If $\tilde{\beta}$ is not vertical, then

$$\tilde{\beta}(t) = \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + t \begin{bmatrix} 1 \\ m \end{bmatrix},$$

where $m \in \mathbb{R}$ is the slope of $\tilde{\beta}$. Then

$$\eta \circ \tilde{\beta}(t) = e^{\mu t}(y_0' + tm'),$$

where

$$y_0' = \lambda^{x_0/\alpha} y_0,$$
$$m' = \lambda^{x_0/\alpha} m,$$
$$\mu = \log(\lambda)/\alpha.$$

Suppose now that $\tilde{l}$ is the forward ray of $\tilde{\beta}$. The η-coordinate is bounded exactly when $\mu < 0$. Since $0 < \lambda < 1$, $\alpha > 0$. If $\tilde{l}$ is the backward ray of $\tilde{\beta}$, then the η-coordinate is bounded if and only if $\mu > 0$, that is, when $\alpha < 0$. In either case, l is recurrent if and only if $\mathbb{B}\big(\alpha(\gamma)\mathsf{x}^0(\gamma), \mathsf{v}\big) > 0$. ■

Figure 1 illustrates the proof of the theorem, by showing the orbit of a recurrent geodesic in (x, η)-coordinates. Figure 2 shows the result in the quotient: the geodesic $l \subset M^+$ may cross the periodic geodesic M^0. In that case l crosses M^0 transversely, and then spirals back towards M^0, intersecting itself infinitely many times.

If $m = 0$, then l never intersects itself, but spirals towards M^0 on one side.

Corollary 3.4 *The only birecurrent geodesic in M is the periodic geodesic M^0.*

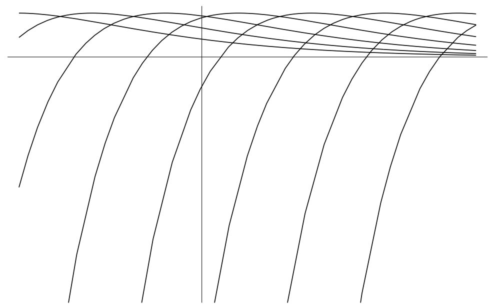

Figure 1: The orbit of a recurrent geodesic on the unstable surface in (x, η)-coordinates: the horizontal line projects to M^0, and all recurrent geodesics in M^+ spiral towards it.

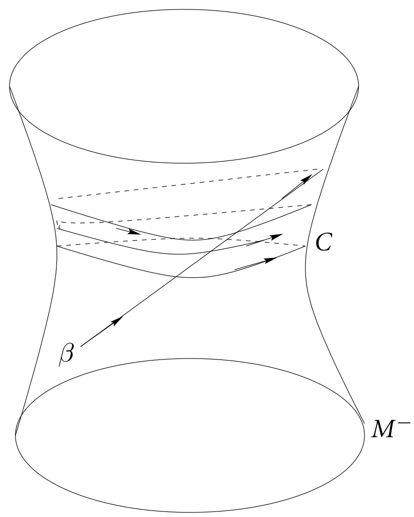

Figure 2: How a ray spirals towards a periodic geodesic.

4 Recurrent Geodesics in Flat Lorentz 3-Manifolds

Theorem 4.1 *Let M be a complete flat Lorentz 3-manifold. For any oriented periodic geodesics $l_1, l_2 \colon \mathbb{R} \to M$, there exists a birecurrent geodesic l whose forward ray spirals towards l_1 and whose backward ray spirals towards l_2.*

Such geodesics correspond to equivalence classes of arcs a whose endpoints lie on l_1 and l_2, under the equivalence relation of homotopy relative to l_1 and l_2.

Proof Choose a basepoint $x \in M$ and the corresponding universal covering space $\Pi \colon \mathbb{A}^{2,1} \to M$. Join l_1, l_2 to x by arcs a_1, a_2 respectively such that the composition $a_1 a_2^{-1}$ is homotopic to a by a homotopy relative to l_1 and l_2. Let γ_i be the holonomy of the based loops corresponding to l_i, $i = 1, 2$. Thus l_i lifts to C_{γ_i} in the universal cover.

For $\gamma \in \Gamma$, denote its associated stable and unstable surfaces in M by $M_\gamma^{\pm}$. Every geodesic in M is the projection of a geodesic in the cylinder M_{C_γ}, since it is a covering space.

Now suppose l is a geodesic in $M_{\gamma_i}^+ \subset M$. By Theorem 3.3, its forward ray spirals towards l_i. Similarly if $l \subset M_{\gamma_i}^-$, then its backward ray spirals towards l_i.

Bispiralling geodesics are obtained as follows. The intersection $M_{\gamma_1}^+ \cap M_{\gamma_2}^-$ is the image of the intersection of the two planes $E_{\gamma_1}^+$ and $E_{\gamma_2}^-$, which is a line. Choose a linear parametrization

$$l\colon \mathbb{R} \to E_{\gamma_1}^+ \cap E_{\gamma_2}^-.$$

The forward ray of l spirals towards l_1 and the backward ray of l spirals towards l_2 as claimed. ■

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