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ESTIMATES FOR KERNELS OF INTERTWINING OPERATORS ON SL(n, $\mathbb{R}$)

MICHAEL COWLING and STEFANO MEDA

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Abstract

In this paper we study the kernels and the L^p–L^q boundedness properties of some intertwining operators associated to representations of SL(n, $\mathbb{R}$).

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1. Introduction

In this paper, we estimate the size of the kernels and study the L^p–L^q mapping properties of certain 'potential' operators on the group V of all lower triangular unipotent $n \times n$ matrices. These operators arise naturally in studying the analytic continuation of the unitary principal series of the group SL(n, $\mathbb{R}$), or its extension G, defined to be the group of all real $n \times n$ matrices of determinant ± 1. They may be described as follows.

The noncompact semisimple Lie group G has finite centre and real rank $n-1$. We write Θ for the standard Cartan involution of G, that is,

$$\Theta(x) = (x^{-1})^t \quad \text{for all } x \in G,$$

where ^{t} denotes transpose. The fixed point set of Θ is the orthogonal group O(n), which we denote by K; it is a maximal compact subgroup of G. We denote by A

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the abelian subgroup of diagonal matrices in G with positive entries and by N the nilpotent group of all upper triangular unipotent matrices. Then KAN is an Iwasawa decomposition of G. The group ΘN, which we denote by V, is then the group of lower triangular unipotent matrices.

The centraliser and normaliser of A in K are denoted by M and M'. Then M consists of all diagonal matrices in K and is normal in M'. We denote by P the minimal parabolic subgroup MAN of G.

The group M'/M, denoted by W, is finite; we denote a typical element by w_i. Choose a representative $\bar{w}_i$ of w_i in M', that is, an element of the appropriate coset of M in M'. The Bruhat decomposition asserts that G is the disjoint union of the sets $MAN\bar{w}_i MAN$, all but one of which, $MAN\bar{w}^* MAN$ say, are of dimension less than $n^2 - 1$, the dimension of G. We may take the representative $\bar{w}^*$ in M' of the 'longest element' w^* in W to be

$$E_{1,n} + E_{2,n-1} + \cdots + E_{n,1}, \tag{1.1}$$

where $E_{i,j}$ is the matrix whose (i, j)th entry is 1 and whose other entries are 0. Further, $MAN\bar{w}^* MAN = \bar{w}^* VMAN$, so that $\bar{w}^* VMAN$ is a dense open subset of G whose complement is a finite union of submanifolds of lower dimension. The mapping $(v, m, a, n) \mapsto vman$ is a diffeomorphism from $V \times M \times A \times N$ onto the Zariski open subset $VMAN$ of G. For almost all x in G, we may write

$$x = V(x)M(x)A(x)N(x),$$

where $V(x) \in V$, $M(x) \in M$, $A(x) \in A$, and $N(x) \in N$.

Denote by $\mathfrak{a}$ the Lie algebra of A, that is, the set of diagonal matrices of trace 0; we write $\operatorname{diag}(x_1, \ldots, x_n)$ for the diagonal matrix with diagonal entries $x_1, \ldots, x_n$. Define ρ to be the linear functional

$$\operatorname{diag}(x_1, \ldots, x_n) \mapsto \sum_{1 \le i < j \le n} \frac{x_i - x_j}{2}$$

on $\mathfrak{a}$ (then ρ is the usual half-sum of the positive roots with multiplicities).

Suppose that λ is in $\mathfrak{a}'_{\mathbb{C}}$. Define the character χ_λ of P by

$$\chi_\lambda(man) = a^{\lambda+\rho}$$

for all $m \in M$, $a \in A$, and $n \in N$, where a^λ is short for $\exp(\lambda(\log a))$. We induce the corresponding character of P to G. Explicitly, let B_λ^∞ denote the space of all C^∞ functions ξ on G with the property that

$$\xi(xp) = \xi(x)\, \chi_\lambda(p^{-1})$$

for all $x \in G$ and $p \in P$. We denote by π_λ the left translation representation of G on B_λ^∞. A common notation for π_λ is

$$\operatorname{ind}_{MAN}^{G}(1 \otimes \chi_\lambda \otimes 1).$$

Since $VMAN$ is Zariski open in G, every function in B_λ^∞ is determined by its restriction to the nilpotent group V. The representation π_λ may be realised in the so-called noncompact picture thus:

$$[\pi_\lambda(x)\xi](v) = \xi\big(V(x^{-1}v)\big)\, A(x^{-1}v)^{-\lambda-\rho} \quad \text{for all } x \in G,\ v \in V,$$

for all ξ in B_λ^∞. For p in $[1, \infty)$, define $\delta(p)$ to be $2/p - 1$. Then $-1 < \delta(p) \leq 1$. For λ in $\delta(p)\rho + i\mathfrak{a}'$, we endow B_λ^∞ with the norm

$$\|\xi\|_p = \left[\int_V |\xi(v)|^p \,\mathrm{d}v\right]^{1/p}.$$

Then π_λ extends to an isometric representation (unitary if $p = 2$) on the completion of B_λ^∞ in this norm. For ξ in B_λ^∞, we define $I_\lambda\xi$ by

$$I_\lambda\xi(x) = \int_V \xi(x\bar{w}^* v)\,\mathrm{d}v \quad \text{for all } x \in G.$$

At least formally, if this integral makes sense, then

$$I_\lambda\xi(xman) = I_\lambda\xi(x)\, a^{-\lambda-\rho}$$

for all $x \in G$, $m \in M$, $a \in A$, and $n \in N$, so that $I_\lambda\xi$ ought to lie in $B_{-\lambda}^\infty$, and further, I_λ commutes with left translations, so that $I_\lambda\pi_\lambda = \pi_{-\lambda}I_\lambda$. In particular, $I_\lambda\xi$ is fixed by K if ξ is.

Knapp and Stein [7] (developing earlier work of Kunze and Stein [9, 10], of Schiffmann [11], and of Gindikin and Karpelevič [2]) showed that, if $\mathrm{Re}(\lambda) > 0$ (in an appropriate sense), then I_λ does indeed make sense, and that I_λ continues meromorphically into $\mathfrak{a}'_{\mathbb{C}}$. Furthermore, they showed that if z is a purely imaginary complex number, then $I_{z\rho}$ extends to a bounded operator on $L^2(V)$. It is easy to show that if $\mathrm{Re}(z) = 1$, then $I_{z\rho}$ extends to a bounded operator from $L^1(V)$ to $L^\infty(V)$. Also, the operator norms of $I_{z\rho}$ grow admissibly when $\mathrm{Im}(z)$ tends to infinity. Hence Stein's complex interpolation theorem applies to the analytic family of operators $\{I_{z\rho} : \mathrm{Re}(z) \in [0, 1]\}$, and it follows that $I_{\delta(p)\rho}$ is bounded from $L^p(V)$ to $L^{p'}(V)$, where p' denotes the conjugate index $p/(p-1)$ of p.

In the case where $n = 2$, this result has been known for a long time. Indeed, in this case the operator $I_{\delta(p)\rho}$ may be realised as the convolution operator on the real line with kernel $v \mapsto |v|^{-2/p'}$ (see [8] or the proof of our main result). This is the kernel of the classical Riesz potential operator of homogeneous degree $-2/p'$, which is bounded from $L^r(\mathbb{R})$ to $L^q(\mathbb{R})$ whenever $1/r - 1/q = 2/p'$; in particular, it is bounded from $L^p(\mathbb{R})$ to $L^{p'}(\mathbb{R})$. To prove this result, one first shows that the kernel of $I_{\delta(p)\rho}$ is in the Lorentz space $L^{p'/2,\infty}(\mathbb{R})$, and then uses Hunt's convolution theorem [5]. A similar result holds for all real rank one simple groups. This fact was crucial to the improved version of the Kunze–Stein phenomenon proved by the authors of this paper and Setti (described in [1]).

For groups of higher rank, however, the situation is more complicated. To illustrate the problems which may arise, consider the case of $\mathrm{SL}(2,\mathbb{R}) \times \mathrm{SL}(2,\mathbb{R})$. The intertwining operator $I_{\delta(p)\rho}$ may be realised as convolution on $\mathbb{R}^2$ with the kernel

$$k_s(x,y) = \frac{1}{|xy|^{1/s}},$$

where $s = p'/2$. This kernel is singular on the union of two one-dimensional submanifolds of $\mathbb{R}^2$. It is straightforward to check that

$$\left|\{(x,y) \in \mathbb{R}^2 : |k_s(x,y)| > t\}\right| = \infty \quad \text{for all } t \in \mathbb{R}^+.$$

Hence k_s is not in the Lorentz space $L^{s,\infty}(\mathbb{R}^2)$, and Hunt's convolution theorem does not apply. However,

$$|\{y \in \mathbb{R} : k_s(x,y) > t\}| = \left|\left\{y \in \mathbb{R} : \frac{1}{|x||y|} > t^s\right\}\right| = \frac{2}{|x|\,t^s},$$

so that $k_s(x,\cdot)$ is in $L^{s,\infty}(\mathbb{R})$ when $x \neq 0$. Furthermore,

$$\|k_s(x,\cdot)\|_{L^{s,\infty}(\mathbb{R})} = \sup_{t>0} t\left|\{y \in \mathbb{R} : |k_s(x,y)| > t\}\right|^{1/s} = \left(\frac{2}{|x|}\right)^{1/s};$$

thus, the function $x \mapsto \|k_s(x,\cdot)\|_{L^{s,\infty}(\mathbb{R})}$ is in $L^{s,\infty}(\mathbb{R})$. We say that k is in the iterated Lorentz space $L^{s,\infty}(\mathbb{R}_{(1)}; L^{s,\infty}(\mathbb{R}_{(2)}))$. By applying Hunt's convolution theorem on $\mathbb{R}$ twice, we conclude that convolution with k_s is bounded from $L^p(\mathbb{R}^2)$ to $L^q(\mathbb{R}^2)$.

The point of this paper is that the convolution kernels of the intertwining operators for $\mathrm{SL}(n,\mathbb{R})$ in the noncompact picture have a 'product structure' similar to that of k_s, and belong to 'iterated Lorentz spaces'. Consequently, certain intertwining operators are L^p–L^q bounded. This can also be proved by extending the results of Knapp and Stein [7] to show that the intertwining operators between the unitary principal series representations are L^r bounded whenever $1 < r < \infty$ (this involves writing the operators as a composition of rank-one operators and showing that each of these is L^r bounded), and then using interpolation. However our approach also yields information about the sizes of the kernels of the intertwining operators.

The same idea applies, *mutatis mutandis*, to the group $\mathrm{SL}(n,\mathbb{C})$ and can be extended to $\mathrm{SL}(n,\mathbb{H})$. We are also able to deal with other semisimple Lie groups on a case-by-case basis. This will appear elsewhere.

2. Convolution operators on nilpotent groups

We say that a Lie group H is the unimodular semidirect product of subgroups H_1 and H_2, and we write $H = H_1 \ltimes H_2$, if H, H_1, and H_2 are unimodular, H_2 is normal in H, and the map $(h_1, h_2) \mapsto h_1 h_2$ is a homeomorphism from $H_1 \times H_2$ onto H. Thus H_1 and H_2 are closed, $H_1 \cap H_2 = \{I\}$ and $H_1 H_2 = H$, and conjugation on H_2 by elements of H preserves measure.

Suppose that H is a unimodular Lie group and that p and q are in $[1, \infty)$. Denote by $Cv_p^q(H)$ the Banach space of all distributions k on H such that the operator $f \mapsto f * k$, initially defined from $C_c^\infty(H)$ to $C^\infty(H)$, extends continuously to a bounded operator from $L^p(H)$ to $L^q(H)$; we endow $Cv_p^q(H)$ with the operator norm. If $H = H_1 \ltimes H_2$, we say that f is in $Cv_p^q\big(H_1; Cv_p^q(H_2)\big)$ if for all fixed h_1 in H_1, the function $h_2 \mapsto f(h_1 h_2)$ is in $Cv_p^q(H_2)$ and $h_1 \mapsto \|f(h_1 \cdot)\|_{Cv_p^q(H_2)}$ is in $Cv_p^q(H_1)$.

PROPOSITION 2.1. *Suppose that H is a unimodular semidirect product $H_1 \ltimes H_2$ and that $1 \le p, q < \infty$. If k is in $Cv_p^q\big(H_1; Cv_p^q(H_2)\big)$, then k is in $Cv_p^q(H)$, and*

$$\|k\|_{Cv_p^q(H)} \le \|k\|_{Cv_p^q(H_2; Cv_p^q(H_1))}.$$

PROOF. Suppose that f is a continuous function with compact support. By a standard result about integration on groups (see, for example, [6, Proposition 5.26]),

$$\int_H f(u)\, du = \int_{H_1} \int_{H_2} f(u_1 u_2)\, du_2\, du_1 = \int_{H_1} \int_{H_2} f(u_2 u_1)\, du_2\, du_1.$$

Write u^v for $v^{-1} u v$, and define the function g_{u_1, v_1} on H_2 by

$$g_{u_1, v_1}(s_2) = f(u_1 v_1 s_2^{v_1}) \quad \text{for all } s_2 \in H_2.$$

Then

$$\begin{aligned}
&\left(\int_H |(f * k)(u)|^q\, du \right)^{1/q} \\
&\quad = \left(\int_{H_1} \int_{H_2} \left| \int_{H_1} \int_{H_2} f(u_1 u_2 v_2 v_1)\, k((v_2 v_1)^{-1})\, dv_2\, dv_1 \right|^q du_2\, du_1 \right)^{1/q} \\
&\quad = \left(\int_{H_1} \int_{H_2} \left| \int_{H_1} \int_{H_2} f\big(u_1 v_1 (u_2 v_2)^{v_1}\big)\, k(v_1^{-1} v_2^{-1})\, dv_2\, dv_1 \right|^q du_2\, du_1 \right)^{1/q} \\
&\quad = \left(\int_{H_1} \left[\int_{H_2} \left| \int_{H_1} \big[g_{u_1, v_1} *_{H_2} k(v_1^{-1} \cdot)\big](u_2)\, dv_1 \right|^q du_2 \right]^{q/q} du_1 \right)^{1/q}.
\end{aligned}$$

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By Minkowski's inequality,

$$\begin{aligned}\|f * k\|_q &\le \left(\int_{H_1}\left[\int_{H_1}\|g_{u_1,v_1} *_{H_2} k(v_1^{-1}\cdot)\|_{L^q(H_2)}\,\mathrm{d}v_1\right]^q \mathrm{d}u_1\right)^{1/q}\\ &\le \left(\int_{H_1}\left[\int_{H_1}\|g_{u_1,v_1}\|_{L^p(H_2)}\,\|k(v_1^{-1}\cdot)\|_{Cv_p^q(H_2)}\,\mathrm{d}v_1\right]^q \mathrm{d}u_1\right)^{1/q}.\end{aligned}$$

Since conjugations by elements of H_1 are measure-preserving on H_2,

$$\begin{aligned}\|g_{u_1,v_1}\|_{L^p(H_2)} &= \left(\int_{H_2}|f(u_1v_1s_2^{v_1})|^p\,\mathrm{d}s_2\right)^{1/p} = \left(\int_{H_2}|f(u_1v_1s_2')|^p\,\mathrm{d}s_2'\right)^{1/p}\\ &= \|f(u_1v_1\cdot)\|_{L^p(H_2)}.\end{aligned}$$

Therefore,

$$\|f * k\|_q \le \left(\int_{H_1}\left[\int_{H_1}\|f(u_1v_1\cdot)\|_{L^p(H_2)}\,\|k(v_1^{-1}\cdot)\|_{Cv_p^q(H_2)}\,\mathrm{d}v_1\right]^q \mathrm{d}u_1\right)^{1/q}.$$

Now the inner integral is the convolution on H_1 of the functions

$$v_1 \mapsto \|f(v_1\cdot)\|_{L^p(H_2)} \quad \text{and} \quad v_1 \mapsto \|k(v_1\cdot)\|_{Cv_p^q(H_2)},$$

evaluated at u_1. Since $v_1 \mapsto \|k(v_1\cdot)\|_{Cv_p^q(H_2)}$ is in $Cv_p^q(H_1)$ by assumption,

$$\begin{aligned}\|f * k\|_q &\le \left(\int_{H_1}\|f(u_1v_1\cdot)\|_{L^p(H_2)}{}^p\,\mathrm{d}u_1\right)^{1/p}\|k\|_{Cv_p^q(H_1;Cv_p^q(H_2))}\\ &= \|f\|_p\,\|k\|_{Cv_p^q(H_1;Cv_p^q(H_2))}.\end{aligned}$$

The required conclusion follows. □

Suppose that H is a unimodular Lie group and that $H_1, H_2, \ldots, H_d$ are closed unimodular subgroups of H, whose dimensions sum to that of H. Suppose (backward recursively) that H_{i-1} normalises $H_iH_{i+1}\cdots H_d$, the conjugation action of H_{i-1} on $H_iH_{i+1}\cdots H_d$ is unimodular, and $H_{i-1}\cap H_iH_{i+1}\cdots H_d = \{I\}$ when $i = d, \ldots, 2$. Then $H_{i-1}H_i\cdots H_d = H_{i-1} \ltimes H_i\cdots H_d$ when $i = d, \ldots, 2$. We say (again) that H is a unimodular semidirect product, and, abusing notation a little (forgetting significant parentheses), we write

$$H = H_1 \ltimes H_2 \ltimes \cdots \ltimes H_d. \tag{2.1}$$

We now define recursively an iterated version of the Lorentz space $L^{s,\infty}$, which will be important for our main result (Theorem 3.4). Suppose that (2.1) holds.

When $d = 1$, we define $L^{s,\infty}(H_1)$ to be the usual Lorentz space, that is, the set of all measurable functions f on H_1 such that, for some (f-dependent) number C,

$$|\{h \in H_1 : |f(h)| > \lambda\}|^{1/s} \leq C/\lambda \quad \text{for all } \lambda \in \mathbb{R}^+.$$

We define the 'norm' of f to be the minimum possible value of C. When $d > 1$, we define $L^{s,\infty}(H_1; L^{s,\infty}(H_2; \dots L^{s,\infty}(H_d)\dots))$ to be the space of all measurable functions f on H such that, for almost all h_1 in H_1, $(h_2, \dots, h_d) \mapsto f(h_1 h_2 \cdots h_d)$ is in $L^{s,\infty}(H_2; \dots L^{s,\infty}(H_d)\dots)$, and the function $h_1 \mapsto \|f(h_1 \cdot)\|_{L^{s,\infty}(H_2;\dots L^{s,\infty}(H_d)\dots)}$ is in $L^{s_1,\infty}(H_1)$. The 'norm' of f is defined to be the $L^{s,\infty}(H_1)$ 'norm' of this last function. We have written 'norm' because, in general, the triangle inequality does not hold, and we do not have a true norm.

We write $\mathbb{R}_{(i)}$ for the range of the one-parameter subgroup $t \mapsto t\mathbf{e}_i$ of the group $\mathbb{R}^2$, where $\{\mathbf{e}_1, \mathbf{e}_2\}$ is the standard basis of $\mathbb{R}^2$. Then

$$\mathbb{R}^2 = \mathbb{R}_{(1)} \ltimes \mathbb{R}_{(2)} = \mathbb{R}_{(2)} \ltimes \mathbb{R}_{(1)}.$$

We now show that, for this example, the order of the variables in the iterated space matters and that the iterated spaces are different to the Lorentz space $L^{s,\infty}(\mathbb{R}^2)$. We observe that f is in $L^{s,\infty}(\mathbb{R}^2)$ if and only if $|f|^s$ is in $L^{1,\infty}(\mathbb{R}^2)$, and similarly, f is in $L^{s,\infty}(\mathbb{R}_{(i)}; L^{s,\infty}(\mathbb{R}_{(j)}))$ if and only if $|f|^s$ is in $L^{1,\infty}(\mathbb{R}_{(i)}; L^{1,\infty}(\mathbb{R}_{(j)}))$. Thus it suffices to consider the case where $s = 1$.

Define the functions g, h and k on $\mathbb{R}^2$ by

$$g(x, y) = e^{-y}\chi_E(x, y), \quad h(x, y) = g(y, x), \quad \text{and} \quad k(x, y) = \frac{1}{|xy|},$$

where $E = \{(x, y) \in \mathbb{R}^+ \times \mathbb{R}^+ : x < e^y\}$.

If $\lambda > 0$ and $x > 0$, then

$$\begin{aligned}\{y \in \mathbb{R} : g(x, y) > \lambda\} &= \{y \in \mathbb{R} : e^{-y} > \lambda\} \cap \{y \in \mathbb{R} : \chi_E(x, y) > 0\} \\ &= \{y \in \mathbb{R} : y > 0,\ y > \log x,\ y < -\log\lambda\},\end{aligned}$$

which is empty if $\lambda \geq 1$ or if $\lambda \leq 1/x$, and nonempty otherwise. Thus

$$\|g(x, \cdot)\|_{L^{1,\infty}(\mathbb{R})} = \sup_{\lambda>0} \lambda|\{y \in \mathbb{R} : g(x, y) > \lambda\}| = \sup_{0<\lambda<1} \lambda \log(\lambda^{-1}) = e^{-1},$$

if $0 < x \leq 1$, while if $x > 1$, then

$$\|g(x, \cdot)\|_{L^{1,\infty}(\mathbb{R})} = \sup_{0<\lambda<1/x} \lambda\left[\log(\lambda^{-1}) - \log x\right] = \sup_{0<x\lambda<1} \lambda \log\left((x\lambda)^{-1}\right) = (ex)^{-1}.$$

Now $x \mapsto \|g(x, \cdot)\|_{L^{1,\infty}(\mathbb{R})}$ is in $L^{1,\infty}(\mathbb{R})$, and g is in $L^{1,\infty}(\mathbb{R}_{(1)}; L^{1,\infty}(\mathbb{R}_{(2)}))$. It is easy to check that $\|g(\cdot, y)\|_{L^{1,\infty}(\mathbb{R})} = \chi_{\mathbb{R}^+}(y)$, which is not in $L^{1,\infty}(\mathbb{R})$. Thus g does not belong to $L^{1,\infty}(\mathbb{R}_{(2)}; L^{1,\infty}(\mathbb{R}_{(1)}))$.

Similarly, h is in $L^{1,\infty}(\mathbb{R}_{(2)}; L^{1,\infty}(\mathbb{R}_{(1)}))$, but not $L^{1,\infty}(\mathbb{R}_{(1)}; L^{1,\infty}(\mathbb{R}_{(2)}))$.

If $\lambda \geq 1$, then the set $\{(x, y) \in \mathbb{R}^2 : g(x, y) > \lambda\}$ is empty, while if $0 < \lambda < 1$, then the set is $\{(x, y) \in E : y < -\log \lambda\}$. Therefore,

$$|\{(x, y) \in \mathbb{R}^2 : g(x, y) > \lambda\}| = \left(\frac{1}{\lambda} - 1\right) \chi_{(0,1)}(\lambda),$$

whence $\|g\|_{L^{1,\infty}(\mathbb{R}^2)} = \sup_{\lambda>0} \lambda \,|\{(x, y) \in \mathbb{R}^2 : g(x, y) > \lambda\}| = 1$, and g is in the standard Lorentz space $L^{1,\infty}(\mathbb{R}^2)$. Similarly, h also belongs to this space. As already remarked, k belongs to the iterated space (with either ordering of the variables), but not to the Lorentz space $L^{1,\infty}(\mathbb{R}^2)$, so the iterated space (with either ordering of the variables) is not contained in the standard Lorentz space. Consideration of the functions g and h shows that the standard Lorentz space is not included in the iterated space (with either ordering of the variables).

PROPOSITION 2.2. *Suppose that H is a unimodular semidirect product*

$$H = H_1 \ltimes H_2 \ltimes \cdots \ltimes H_d,$$

in the sense of (2.1), *where $1 < p, q, s < \infty$, and that $1/p + 1/s = 1/q + 1$. Then the iterated space $L^{s,\infty}(H_1; \ldots L^{s,\infty}(H_d) \ldots)$ is contained in $Cv_p^q(H)$ and there exists a constant C_d such that $\|k\|_{Cv_p^q(H)} \leq C_d \|k\|_{L^{s,\infty}(H_1;\ldots L^{s,\infty}(H_d)\ldots)}$.*

PROOF. We argue by induction on the number of factors. If there is only one factor, the result is Hunt's well known convolution theorem [5].

Suppose that the result holds when the number of factors is less than d. The subgroup $H_2 H_3 \cdots H_d$ is normal in H. By definition of the iterated Lorentz space, for almost all h_1 in H_1, $(h_2, \ldots, h_d) \mapsto f(h_1 h_2 \cdots h_d)$ is in $L^{s,\infty}(H_2; \ldots L^{s,\infty}(H_d) \ldots)$, and the function $h_1 \mapsto \|f(h_1 \cdot)\|_{L^{s,\infty}(H_2;\ldots L^{s,\infty}(H_d)\ldots)}$ is in $L^{s_1,\infty}(H_1)$. By the inductive hypothesis, $L^{s,\infty}(H_2; \ldots L^{s,\infty}(H_d) \ldots)$ is contained in $Cv_p^q(H_2 \cdots H_d)$ and

$$\|k'\|_{Cv_p^q(H_2\ldots H_d)} \leq C_{d-1} \|k'\|_{L^{s,\infty}(H_2;\ldots L^{s,\infty}(H_d)\ldots)}$$

for all k' in $L^{s,\infty}(H_2; \ldots L^{s,\infty}(H_d) \ldots)$. In particular,

$$\|f(h_1 \cdot)\|_{Cv_p^q(H_2\ldots H_d)} \leq C_{d-1} \|f(h_1 \cdot)\|_{L^{s,\infty}(H_2;\ldots L^{s,\infty}(H_d)\ldots)},$$

so $h_1 \mapsto \|f(h_1 \cdot)\|_{Cv_p^q(H_2\ldots H_d)}$ is in $L^{s,\infty}(H_1)$, and hence in $Cv_p^q(H_1)$, by Hunt's convolution theorem applied to H_1. The required result now follows from Proposition 2.1, with $H_2 \cdots H_d$ in place of H_2. $\square$

3. Structural properties of intertwining operators

Let V^* denote the set of all $v \in V$ such that $\bar{w}^* v$ is in $VMAN$. Then V^* is Zariski open in V, and in particular is of full measure in V. The next lemma describes some properties of the map $v \mapsto V(\bar{w}^* v)$ from V^* to V. These properties are essentially known (see, for instance, [7] or [9, 10]).

LEMMA 3.1. *The following hold*:

(i) $V(\bar{w}^* V(\bar{w}^* v)) = v$ *and* $A(\bar{w}^* V(\bar{w}^* v)) = A(\bar{w}^* v)^{-1}$ *for every* v *in* V^*;

(ii) *the Jacobian of the mapping* $v \mapsto V(\bar{w}^* v)$ *is* $A(\bar{w}^* v)^{-2\rho}$ *for every* v *in* V^*;

(iii) $A(\bar{w}^* v^{-1}) = A(\bar{w}^* v)^{-\bar{w}^*}$ *for every* v *in* V^*;

(iv) $A(\bar{w}^* v^a) = a^{-\bar{w}^*} A(\bar{w}^* v) a$ *for every* v *in* V^* *and* a *in* A;

(v) *the intertwining operator* I_λ *may be realised as a convolution operator in the noncompact picture*:

$$I_\lambda \xi = \xi * r_\lambda,$$

where $r_\lambda : V^* \to \mathbb{R}^+$ *is defined by* $r_\lambda(v) = A(\bar{w}^* v^{-1})^{\lambda - \rho}$.

PROOF. To prove (i), we write $\bar{w}^* v = V(\bar{w}^* v)\, M(\bar{w}^* v)\, A(\bar{w}^* v)\, N(\bar{w}^* v)$. Multiplying both sides by $\bar{w}^*$, and observing that $\bar{w}^{*2} = e$, we see that

$$v = \bar{w}^*\, V(\bar{w}^* v)\, M(\bar{w}^* v)\, A(\bar{w}^* v)\, N(\bar{w}^* v),$$

so that

$$\bar{w}^*\, V(\bar{w}^* v) = v\, M(\bar{w}^* v)^{-1}\, A(\bar{w}^* v)^{-1}\, n'.$$

This implies that $A(\bar{w}^* V(\bar{w}^* v)) = A(\bar{w}^* v)^{-1}$ and $V(\bar{w}^* V(\bar{w}^* v)) = v$, as required.

To prove (ii), recall that the Haar measures on G, V, M, A and N may be normalised so that

$$\int_G u(g)\, dg = \int_V \int_M \int_A \int_N u(vman)\, dn\, a^{2\rho}\, da\, dm\, dv \quad \text{for all } u \in C_c(G),$$

by [6, Proposition 5.26]. We will express the invariance of the Haar measure on G in terms of this 'Bruhat decomposition for the Haar measure'. To do so, we first observe that if $\bar{w}^* v = V(\bar{w}^* v) M(\bar{w}^* v) A(\bar{w}^* v) N(\bar{w}^* v)$, then

$$\bar{w}^* vman = V(\bar{w}^* v)[M(\bar{w}^* v)m][A(\bar{w}^* v)a][(ma)^{-1} N(\bar{w}^* v)(ma)n].$$

Next, we take u on G such that $u(vman) = u_1(v) u_2(a) u_3(n)$, where

$$\int_A u_2(a)\, a^{2\rho}\, da = 1 \quad \text{and} \quad \int_N u_3(n)\, dn = 1.$$

Then

$$\begin{aligned}\int_V u_1(v)\,\mathrm{d}v &= \int_V\int_M\int_A\int_N u_1(v)\,u_2(a)\,u_3(n)\,\mathrm{d}n\,a^{2\rho}\,\mathrm{d}a\,\mathrm{d}m\,\mathrm{d}v\\ &= \int_G u(g)\,\mathrm{d}g = \int_G u(\bar{w}^* g)\,\mathrm{d}g\\ &= \int_V\int_M\int_A\int_N u_1(V(\bar{w}^*v))u_2(A(\bar{w}^*v)a)u_3(N(\bar{w}^*v)^{ma}n)\,\mathrm{d}n a^{2\rho}\,\mathrm{d}a\,\mathrm{d}m\,\mathrm{d}v\\ &= \int_V\int_A u_1(V(\bar{w}^*v))\,u_2(a)\,A(\bar{w}^*v)^{-2\rho}\,a^{2\rho}\,\mathrm{d}a\,\mathrm{d}v\\ &= \int_V u_1(V(\bar{w}^*v))\,A(\bar{w}^*v)^{-2\rho}\,\mathrm{d}v,\end{aligned}$$

so that the Jacobian of the transformation $v \mapsto V(\bar{w}^*v)$ is $A(\bar{w}^*v)^{-2\rho}$, as required.

To prove (iii), note that $\bar{w}^*v = V(\bar{w}^*v)M(\bar{w}^*v)A(\bar{w}^*v)N(\bar{w}^*v)$, whence

$$v^{-1}\bar{w}^{*-1} = N(\bar{w}^*v)^{-1}A(\bar{w}^*v)^{-1}M(\bar{w}^*v)^{-1}V(\bar{w}^*v)^{-1}.$$

Now $\bar{w}^* = \bar{w}^{*-1}$, so $\bar{w}^*v^{-1}$ is equal to

$$\begin{aligned}&\bar{w}^*N(\bar{w}^*v)^{-1}\bar{w}^{*-1}\bar{w}^*M(\bar{w}^*v)^{-1}\bar{w}^{*-1}\bar{w}^*A(\bar{w}^*v)^{-1}\bar{w}^{*-1}\bar{w}^*V(\bar{w}^*v)^{-1}\bar{w}^{*-1}\\ &\quad= v_1 m_1 \bar{w}^*A(\bar{w}^*v)^{-1}\bar{w}^{*-1}n_1,\end{aligned}$$

say, where $v_1 \in V$, $m_1 \in M$ and $n_1 \in N$, since $N^{\bar{w}^*} = V$, $V^{\bar{w}^*} = N$, and M is a normal subgroup of M'. Then $A(\bar{w}^*v^{-1}) = \bar{w}^*A(\bar{w}^*v)^{-1}\bar{w}^{*-1}$, as required.

To prove (iv), suppose that $\bar{w}^*v = v'm'a'n'$. Then for $a \in A$,

$$\bar{w}^*v^a = \bar{w}^*a^{-1}va = a^{-\bar{w}^*}\bar{w}^*va = a^{-\bar{w}^*}v'm'a'n'a = (v')^{a^{\bar{w}^*}}m'a^{-\bar{w}^*}a'a(n')^a,$$

so $A(\bar{w}^*v^a) = a^{-\bar{w}^*}A(\bar{w}^*v)a$, as required.

Finally we give a nonrigorous proof of (v), and refer the reader to [7] for more details. Recall that

$$I_\lambda\xi(v') = \int_V \xi(v'\bar{w}^*v)\,\mathrm{d}v = \int_V \xi(v'V(\bar{w}^*v))\,A(\bar{w}^*v)^{-(\rho+\lambda)}\,\mathrm{d}v.$$

By virtue of (i) and (ii), the change of variables $v'' = V(\bar{w}^*v)$ transforms the last integral into

$$\begin{aligned}&\int_V \xi(v'v'')\,A(\bar{w}^*V(\bar{w}^*v''))^{-(\rho+\lambda)}\,A(\bar{w}^*v'')^{-2\rho}\,\mathrm{d}v''\\ &\quad= \int_V \xi(v'v'')\,A(\bar{w}^*v'')^{\rho+\lambda}\,A(\bar{w}^*v'')^{-2\rho}\,\mathrm{d}v''\\ &\quad= \xi * r_\lambda(v'),\end{aligned}$$

as required. □

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In view of Lemma 3.1 (v), when $\operatorname{Re}\lambda = \delta(p)\rho$, the intertwining operator I_λ reduces to the (right) convolution operator on V whose kernel is $A(\bar{w}^* \cdot)^{(\delta(p)-1)\rho}$. Thus, it may be useful to compute $A(\bar{w}^* \cdot)^\rho$.

For x in $\mathrm{SL}(n, \mathbb{R})$, denote by $D_k(x)$ the determinant of the submatrix $(x_{ij})_{1 \le i, j \le k}$.

LEMMA 3.2. *For any x in $VMAN$, $A(x)^\rho = \prod_{k=1}^{n-1} |D_k(x)|$.*

PROOF. It is well known [3, page 434] that

$$VMAN = \{x \in \mathrm{SL}(n, \mathbb{R}) : D_1(x) \neq 0, \ldots, D_{n-1}(x) \neq 0\}.$$

Moreover,

$$M(x)A(x) = \operatorname{diag}\left(D_1(x), \frac{D_2(x)}{D_1(x)}, \ldots, \frac{D_n(x)}{D_{n-1}(x)}\right),$$

and

$$A(x) = \operatorname{diag}\left(|D_1(x)|, \frac{|D_2(x)|}{|D_1(x)|}, \ldots, \frac{|D_n(x)|}{|D_{n-1}(x)|}\right).$$

Recall that

$$\rho\big(\operatorname{diag}(x_1, \ldots, x_n)\big) = \sum_{1 \le i < j \le n} \frac{x_i - x_j}{2} = \sum_{j=1}^{n} \frac{n+1-2j}{2} x_j.$$

Then

$$\begin{aligned} A(x)^{2\rho} &= |D_1(x)|^{n-1} \left|\frac{D_2(x)}{D_1(x)}\right|^{n-3} \cdots \left|\frac{D_{n-1}(x)}{D_{n-2}(x)}\right|^{3-n} \left|\frac{1}{D_{n-1}(x)}\right|^{1-n} \\ &= |D_1(x)|^2 |D_2(x)|^2 \cdots |D_{n-2}(x)|^2 |D_{n-1}(x)|^2, \end{aligned}$$

and the lemma follows. □

Before we can use this information, we need some structural information about V. For $k \in \{1, 2, \ldots, n(n-1)/2\}$, let $m(k)$ be the integer part of $((8k-7)^{1/2}+1)/2$. Let ϕ_k be the one-parameter subgroup $t \mapsto I + tE_{i,j}$ of V, where

$$i = m(k) + 1 \quad \text{and} \quad j = k - m(k)(m(k)-1)/2,$$

and let H_k denote the range of ϕ_k. We refer to the (i, j)th place in the matrix as the place indexed by k. For the case where $n = 5$, the number k is written in the place indexed by k in the following matrix:

$$\begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \cdot & \cdot & \cdot & \cdot \\ 2 & 3 & \cdot & \cdot & \cdot \\ 4 & 5 & 6 & \cdot & \cdot \\ 7 & 8 & 9 & 10 & \cdot \end{pmatrix}.$$

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Then (backward recursively) H_{k-1} normalises $H_k \cdots H_{n(n-1)/2}$, and

$$V = H_1 \ltimes H_2 \ltimes \cdots \ltimes H_{n(n-1)/2},$$

in the sense of (2.1). Indeed, $H_k \cdots H_{n(n-1)/2}$ is the subgroup of all v in V whose only nonzero, nondiagonal entries are in the places in the matrix indexed by $k, \ldots, n(n-1)/2$, and H_{k-1} is the subgroup of all v in V whose only nonzero, nondiagonal entries are in the place in the matrix indexed by $k-1$, which lies to the right of, or above, the places indexed by $i, \ldots, n(n-1)/2$. It is easy to see that H_{k-1} normalises the subgroup of all v in V whose only nonzero, nondiagonal entries are in the rows below the place in the matrix indexed by $k-1$, and centralises the subgroup of all v in V whose only nonzero, nondiagonal entries are in the same row as the place in the matrix indexed by $k-1$.

Note that $H_i \ltimes H_{i+1} \ltimes \cdots \ltimes H_{n(n-1)/2}$ may be identified with the subgroup of lower unipotent matrices in $\mathrm{SL}(n, \mathbb{R})$ with zero entries in the places indexed by $1, \ldots, i-1$, and that $H_1 \ltimes H_2 \ltimes \cdots \ltimes H_{(n-1)(n-2)/2}$ (which has zero entries in the places indexed by $(n-1)(n-2)/2+1, \ldots, n(n-1)/2$) is isomorphic to the group of lower unipotent matrices in $\mathrm{SL}(n-1, \mathbb{R})$.

The crucial result is the following proposition, which provides a sharp estimate for the size of the intertwining kernel r_λ when $\operatorname{Re}\lambda$ is a multiple of ρ.

PROPOSITION 3.3. *Let V be the group of lower triangular real unipotent $n \times n$ matrices. For k in $\{1, \ldots, n-1\}$, define $\tilde{D}_k(v)$ by*

$$\tilde{D}_k(v) = \det(v_{ij})_{\substack{n-k \le i \le n \\ 1 \le j \le k}} \quad \textit{for all} \quad v \in V.$$

(i) *If v is in V, then v is in V^* if and only if $D_k(v) \neq 0$ when $k = 1, \ldots, k-1$.*

(ii) *For every v in V^*,*

$$A(\bar{w}^* v)^{-\rho} = \prod_{k=1}^{n-1} \frac{1}{|\tilde{D}_k(v)|}.$$

(iii) *The function $v \mapsto A(\bar{w}^* v)^{-\rho}$ is in $L^{1,\infty}(H_1; \ldots L^{1,\infty}(H_{n(n-1)/2}) \ldots)$, and*

$$\| A(\bar{w}^* \cdot)^{-\rho} \|_{L^{1,\infty}(H_1; \ldots L^{1,\infty}(H_{n(n-1)/2}) \ldots)} = 2^{n(n-1)/2}.$$

PROOF. First, an element v of V is in V^* if and only if $\bar{w}^* v$ is in $VMAN$, and this holds if and only if $D_k(\bar{w}^* v) \neq 0$ when $k = 1, \ldots, n-1$, by [3, page 434]. Multiplying the lower triangular unipotent matrix v on the left by $\bar{w}^*$, as defined in (1.1), just reverses the order of the rows of v. Therefore $D_k(\bar{w}^* v) = \pm \tilde{D}_k(v)$, and (i) follows.

Now we prove (ii). For convenience, we write $\Phi(v)$ for $A(\bar{w}^* v)^{-\rho}$. By Lemma 3.2 and part (i) of this proposition,

$$\Phi(v) = \prod_{k=1}^{n-1} \frac{1}{|D_k(\bar{w}^* v)|} = \prod_{k=1}^{n-1} \frac{1}{|\tilde{D}_k(v)|}.$$

Now we prove (iii), by induction on n. We write V_n for the group of lower unipotent real $n \times n$ matrices, and Φ_n rather than just Φ.

Suppose first that $n = 2$. Write v_x for the lower unipotent 2×2 matrix whose lower left entry is x. Then $\Phi_2(v_x) = |x|^{-1}$, and the result is evident.

For the inductive step, we let $\pi_n : \mathrm{SL}(n, \mathbb{R}) \to \mathrm{SL}(n+1, \mathbb{R})$ be the injection

$$\pi(g)_{ij} = \begin{cases} g_{ij} & \text{if } i \le n \text{ and } j \le n; \\ 0 & \text{if } i = n+1 \text{ and } j \le n; \\ 0 & \text{if } i \le n \text{ and } j = n+1; \\ 1 & \text{if } i = n+1 \text{ and } j = n+1. \end{cases}$$

To shorten the notation a little, we denote the subgroup $H_{n(n-1)/2+m}$ by H'_m where $m = 1, 2, \ldots, n$, and the corresponding one parameter subgroup by ϕ_m. Thus the only nonzero nondiagonal entries of elements of H'_m are in the $(n+1, m)$th place in the matrix. We will show that for all v in V_n^*, the function

$$h'_1 \ldots h'_n \mapsto \Phi(\pi_n(v) h'_1 \cdots h'_n)$$

is in $L^{1,\infty}(H'_1 : \ldots L^{1,\infty}(H'_n) \ldots)$ and

$$\|\Phi_{n+1}(\pi_n(v)\cdot)\|_{L^{1,\infty}(H'_1 : \ldots L^{1,\infty}(H'_n) \ldots)} = 2^n \Phi_n(v). \tag{3.1}$$

The result then follows by induction.

To prove (3.1), we take v in V_n^*. The range of ϕ_m is H'_m, and

$$\pi_n(v)\phi_1(x_1) \cdots \phi_n(x_n) = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ v_{2,1} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ v_{n,1} & v_{n,2} & \cdots & 1 & 0 \\ x_1 & x_2 & \cdots & x_n & 1 \end{pmatrix}.$$

It is easy to check that, if $k < n$, then $\pi_n(v)\phi_1(x_1) \ldots \phi_k(x_k)$ is the matrix obtained by setting $x_{k+1}, \ldots, x_n$ equal to 0 in the matrix above. Since $\tilde{D}_k$ is the determinant of the bottom left $k \times k$ submatrix,

$$\tilde{D}_k(\pi_n(v)\phi_1(x_1) \cdots \phi_n(x_n)) = \tilde{D}_k(\pi_n(v)\phi_1(x_1) \cdots \phi_k(x_k)),$$

and

$$\Phi_{n+1}(\pi_n(v)\phi_1(x_1)\cdots\phi_n(x_n)) = \prod_{k=1}^{n} \frac{1}{\left|\tilde{D}_k(\pi_n(v)\phi_1(x_1)\cdots\phi_k(x_k))\right|}.$$

It is clear that $\Phi_{n+1}(\pi_n(v)\phi_1(x_1)\cdots\phi_n(x_n))$ depends on x_n in only one determinant, namely $\tilde{D}_n(\pi_n(v)\phi_1(x_1)\cdots\phi_n(x_n))$. By expanding this determinant along the bottom row, we see that

$$\left|\tilde{D}_n(\pi_n(v)\phi_1(x_1)\cdots\phi_n(x_n))\right| = \left|\tilde{D}'_{n-1}(v)\right| |x_n + c_n|,$$

where c_n is a rational function of the coordinates v_{ij} and x_j that does not depend on x_n. In this formula, $\tilde{D}_n$ indicates the determinant of the bottom left $n \times n$ submatrix of the $(n+1) \times (n+1)$ matrix, while $\tilde{D}'_{n-1}(v)$ is the determinant of the bottom left $(n-1) \times (n-1)$ submatrix of the $n \times n$ matrix v. Further, $\tilde{D}'_{n-1}(v) \neq 0$, because $v \in V_n^*$. Now, for all positive λ and κ, and any real c,

$$\left|\left\{x \in \mathbb{R} : \frac{1}{\kappa|x+c|} > \lambda\right\}\right| = \frac{2}{\kappa\lambda},$$

and so

$$\left\|\frac{1}{\kappa|\cdot+c|}\right\|_{L^{1,\infty}(\mathbb{R})} = \frac{2}{\kappa}.$$

By applying this to the case at hand, we deduce that

$$\begin{aligned}&\|\Phi_{n+1}(v\phi_1(x_1)\ldots\phi_{n-1}(x_{n-1})\cdot)\|_{L^{1,\infty}(H'_n)} \\ &\quad= \frac{2}{|\tilde{D}'_{n-1}(v)| \prod_{k=1}^{n-1} |\tilde{D}_k(v\phi_1(x_1)\cdots\phi_k(x_k))|}.\end{aligned}$$

It is clear that $\|\Phi_{n+1}(v\phi_1(x_1)\cdots\phi_{n-1}(x_{n-1})\cdot)\|_{L^{1,\infty}(H'_n)}$ depends on x_{n-1} in only one determinant, namely $\tilde{D}_{n-1}(\pi_n(v)\phi_1(x_1)\cdots\phi_{n-1}(x_{n-1}))$. By expanding this determinant along the bottom row much as before, we see that

$$\begin{aligned}&\|\Phi_{n+1}(v\phi_1(x_1)\cdots\phi_{n-2}(x_{n-2})\cdot)\|_{L^{1,\infty}(H'_{n-1};L^{1,\infty}(H'_n))} \\ &\quad= \frac{4}{\prod_{k=n-2}^{n-1} |\tilde{D}'_k(v)| \prod_{k=1}^{n-2} |\tilde{D}_k(v\phi_1(x_1)\cdots\phi_k(x_k))|}.\end{aligned}$$

After n iterations of this argument, we end up with the estimate (3.1). □

Our main convolution theorem is now easy to state and prove.

THEOREM 3.4. *Let V be the group of lower triangular real unipotent $n \times n$ matrices.*

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(i) *Suppose that* λ *is in* $\mathfrak{a}'_{\mathbb{C}}$ *and* I_λ *is bounded from* $L^p(V)$ *to* $L^q(V)$, *where* $1 \le p, q \le \infty$. *Then* $p \le q$ *and the projection* $\mathrm{Proj}_\rho \operatorname{Re}\lambda$ *of* $\operatorname{Re}\lambda$ *onto* ρ *is equal to* $(1/p - 1/q)\rho$.

(ii) *Suppose that* $1 < p < q < \infty$ *and* $\operatorname{Re}\lambda = (1/p - 1/q)\rho$. *Then the convolution operator* I_λ *is bounded from* $L^p(V)$ *to* $L^q(V)$.

PROOF. To prove (i), we first recall a result of Hörmander [4, Theorem 1.1]. It generalises immediately to the context here, and shows that $p \le q$, because V is noncompact.

Now we adapt a classical dilation argument for Riesz potentials on Euclidean spaces. By Lemma 3.1 (iv), we see that

$$r_\lambda(v^a) = A(\bar{w}^* v^a)^{\lambda-\rho} = [a^{-\bar{w}^*} A(\bar{w}^* v) a]^{\lambda-\rho} = [a^{-\bar{w}^*} a]^{\lambda-\rho} r_\lambda(v),$$

and

$$\begin{aligned}
[a^{-\bar{w}^*} a]^{\lambda-\rho} &= \exp\big((\lambda-\rho)\log a - (\lambda-\rho)\operatorname{Ad}(\bar{w}^*)\log(a)\big) \\
&= \exp\big((\lambda-\rho)\log a - \operatorname{Ad}(\bar{w}^*)^t(\lambda-\rho)\log(a)\big) \\
&= a^{\lambda - \operatorname{Ad}(\bar{w}^*)^t\lambda - 2\rho},
\end{aligned}$$

since $\operatorname{Ad}(\bar{w}^*)^t\rho = -\rho$. Given a function f on V and $a \in A$, we write f^a for the function $v \mapsto f(v^a)$. A change of variables shows that

$$\begin{aligned}
f * r_\lambda(v^a) &= \int_V f(v^a v') r_\lambda(v'^{-1})\, \mathrm{d}v' \\
&= a^{2\rho} \int_V f(v^a v'^a) r_\lambda(v'^{-a})\, \mathrm{d}v' \\
&= a^{\lambda - \operatorname{Ad}(\bar{w}^*)^t\lambda} \int_V f^a(vv') r_\lambda(v'^{-1})\, \mathrm{d}v' \\
&= a^{\lambda - \operatorname{Ad}(\bar{w}^*)^t\lambda} f^a * r_\lambda(v).
\end{aligned}$$

Another change of variables shows that $\|f^a\|_q = (a^{-2\rho})^{1/q}\|f\|_q$.

Suppose now that the operator $f \mapsto f * r_\lambda$ is $L^p(V)$–$L^q(V)$ bounded. Then for all f in $L^p(V)$,

$$\begin{aligned}
(a^{-2\rho})^{1/q}\|f * r_\lambda\|_q = \|(f * r_\lambda)^a\|_q &= a^{\operatorname{Re}\lambda - \operatorname{Ad}(\bar{w}^*)^t \operatorname{Re}\lambda}\|f^a * r_\lambda\|_q \\
&\le C\, a^{\operatorname{Re}\lambda - \operatorname{Ad}(\bar{w}^*)^t \operatorname{Re}\lambda}\|f^a\|_p \\
&= C\, a^{\operatorname{Re}\lambda - \operatorname{Ad}(\bar{w}^*)^t \operatorname{Re}\lambda}(a^{-2\rho})^{1/p}\|f\|_p \quad \text{for all } a \in A,
\end{aligned}$$

so that $a^{\operatorname{Re}\lambda - \operatorname{Ad}(\bar{w}^*)^t \operatorname{Re}\lambda} = (a^{2\rho})^{1/p-1/q}$ for all a in A, whence

$$\operatorname{Re}\lambda - \operatorname{Ad}(\bar{w}^*)^t \operatorname{Re}\lambda = 2(1/p - 1/q)\rho.$$

This implies that $\operatorname{Proj}_\rho \operatorname{Re}\lambda = (1/p - 1/q)\rho$, as claimed.

To prove (ii), from Lemma 3.1 (v) and Proposition 3.3 (iii),

$$|r_\lambda| = A(\bar{w}^* \cdot)^{-(1-1/p+1/q)\rho},$$

which is in $L^{s,\infty}(H_1; \dots L^{s,\infty}(H_{n(n-1)/2}) \dots)$, where $1/s = 1 - 1/p + 1/q$. The desired result now follows from Proposition 2.2. $\square$

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School of Mathematics
University of New South Wales
UNSW Sydney
NSW 2052
Australia
e-mail: M.Cowling@unsw.edu.au

Dipartimento di Matematica e Applicazioni
Università di Milano-Bicocca
via Bicocca degli Arcimboldi 8
20126 Milano
Italy
e-mail: stefano.meda@unimib.it