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CHARACTERIZATION OF CERTAIN CLASSES OF SPACES WITH G_δ POINTS AS OPEN IMAGES OF METRIC SPACES

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1. Introduction. The study of metrization has led to the development of a number of new topological spaces, called generalized metric spaces, within the past fifteen years. For a survey of results in metrization theory involving many of these spaces, the reader is referred to [**13**]. Quite a few of these generalized metric spaces have been studied extensively, somewhat independently of their role in metrization theorems. Specifically, we refer here to characterizations of these spaces by various workers as images of metric spaces. Results in this area have been obtained by Alexander [**2**], Arhangel'skii [**3**], Burke [**5**], Heath [**10**], Michael [**15**], Nagata [**16**], and the author [**1**], to mention a few. Later we will recall specifically some of these results.

In this paper we characterize several classes of generalized metric spaces as open weakly continuous images of metric spaces. In Section 2, we give some background results and some definitions. Several classes of spaces defined in terms of open separating covers, including spaces with a G_δ-diagonal, are characterized in Section 3. In Section 4, we use a generalization of the definition of Heath's P-mapping [c.f. [**10**]) to strengthen one implication in his characterization of developable spaces, and derive some corollaries. Stratifiable, semi-stratifiable, and K-semi-stratifiable spaces are all characterized in Section 5. Throughout the paper, $\mathbf{N}$ stands for the natural numbers.

2. Background results and definitions. An open cover of a space is *separating* if given $x \neq y$ in the space, there is an element of the cover, say S, with $x \in S$ and $y \notin S$. In a T_1 space the concept of a separating open cover generalizes the notion of a base. A space with G_δ points can be thought of as a space with a separating open cover $\mathscr{S}$ such that given an element x, there is a countable subcollection $\mathscr{S}_x$ of $\mathscr{S}$ such that if $x \neq y$, there is a $S \in \mathscr{S}_x$ with $x \in S$, $y \notin S$. Of course if we replace y with a closed set H, $x \notin H$, we have the definition of a first countable space. Thus spaces with G_δ points generalize in a natural way the first countable spaces. In the same sort of way, spaces with a point-countable separating open cover generalize spaces with a point-countable base. Recalling the definitions (below), we notice that spaces with a G_δ-diagonal generalize developable spaces in this way also.

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Let X be a space with a sequence of open covers, $\langle \mathscr{G}_n \rangle$, for X. If given $x \in X$ and an open set U containing x, there exists $n \in \mathbf{N}$ with $\mathrm{st}(x, \mathscr{G}_n) \subset U$ (recall $\mathrm{st}(x, \mathscr{G}) = \bigcup \{G \in \mathscr{G} : x \in G\}$), then X is said to be *developable*. If given $x \neq y$ in X, there exists $n \in \mathbf{N}$ with $x \notin \mathrm{st}(y, \mathscr{G}_n)$, X is said to have a G_δ-*diagonal*. Finally, if given $x \neq y$, there exists $n \in \mathbf{N}$ with $x \notin \mathrm{st}^2(y, \mathscr{G}_n)$ then X is said to have a $G_\delta(2)$-*diagonal* (here, $\mathrm{st}^2(y, \mathscr{G}) = \bigcup \{G \in \mathscr{G} : G \cap \mathrm{st}(y, \mathscr{G}) \neq \emptyset.\}$).

Let us now mention some well-known results for reference. The following characterization of first countable spaces was proved independently by Hanai [**8**] and Ponomarev [**17**].

THEOREM 2.1. *A T_1 space Y is first countable if and only if there exists a metric space X and an open continuous mapping from X onto Y.*

In the same paper, Ponomarev described spaces with a point-countable base as follows.

THEOREM 2.2. *A T_1 space Y has a point-countable base if and only if there exists a metric space X and an open continuous S-mapping from X onto Y.*

Arhangel'skii [**3**], Heath [**10**] and Ponomarev [**17**] independently characterized developable spaces with the following theorem.

THEOREM 2.3. *A T_1 space Y is developable if and only if there exists a metric space X and an open continuous P-mapping from X onto Y.*

We will derive analogues to each of these theorems for spaces with G_δ points, spaces with point-countable separating open covers, and spaces with G_δ-diagonals, respectively, in Section 3.

By introducing the idea of a COC-function (COC = countable number of open covers) we can unify the definitions of many spaces defined in terms of sequences of open covers. It should be pointed out that many of the definitions which follow are not the original definitions but are characterizations derived in attempts to unify various concepts.

Let (X, T) be a topological space and let g be a function from $N \times X \to T$. Then g is called a COC-*function for* X if it satisfies these two conditions:

$$(1) \quad x \in \bigcap_{n=1}^{\infty} g(n, x) \quad \text{for all } x \in X;$$

$$(2) \quad g(n+1, x) \subset g(n, x) \quad \text{for all } n \in \mathbf{N} \text{ and } x \in X.$$

Note that if g is a COC-function for X, we obtain countably many open covers of X by taking $\mathscr{G}_n = \{g(n, x) : x \in X\}$ for each $n \in \mathbf{N}$.

Now let X be a space with COC-function g, and consider the following conditions on g:

(A) $y_n \in g(n, x)$ for each $n \in \mathbf{N}$ implies that the sequence $\langle y_n \rangle$ has x as a cluster point;

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(B) $g(n, x) \cap g(n, y_n) \neq \emptyset$ for each $n \in \mathbf{N}$ implies that $\langle y_n \rangle$ has x as a cluster point;

(C) $y_n \in g(n, x)$ and $p_n \in g(n, y_n)$ for each $n \in \mathbf{N}$ implies that $\langle p_n \rangle$ has x as a cluster point;

(D) $\{g(n, x) : n = 1, 2, \ldots\}$ is a fundamental system of neighborhoods for x, for each x, and $x \in g(n, y_n)$ for each $n \in \mathbf{N}$ implies that $\langle y_n \rangle$ has x as a cluster point;

(E) if H is closed and $p \in \overline{\bigcup \{g(n, x) : x \in H\}}$ for each $n \in \mathbf{N}$, then $p \in H$;

(F) $x \in g(n, y_n)$ for each $n \in \mathbf{N}$ implies that $\langle y_n \rangle$ has x as a cluster point;

(G) if H is closed and K is compact with $K \cap (\bigcup \{g(n, x) : x \in H\}) \neq \emptyset$ for all n, then $K \cap H \neq \emptyset$.

If X is a space with COC-function g satisfying (A), X is called a *first countable space* and g a *first countable function for* X; X is called a *Nagata space* and g a *Nagata function for* X if g satisfies (B); if g satisfies (C), X is called a γ-*space* and g a γ-*function for* X; if g satisfies (D), X is called a *semi-metric space* and g a *semi-metric function for* X; X is called a *stratifiable space* and g a *stratifiable function for* X if g satisfies (E); if g satisfies (F), X is called a *semi-stratifiable space* and g a *semi-stratifiable function for* X; and finally, if g satisfies (G), X is called a K-*semi-stratifiable space* and g a K-*semi-stratifiable function for* X.

Ceder [**6**] first studied stratifiable spaces under the name "M_3-spaces". Borges [**4**] renamed them "stratifiable" and investigated them in more detail. Creede [**7**] introduced semi-stratifiable spaces. Our definition of semi-metric spaces is a characterization given by Heath in [**9**] where he studies semi-metric spaces. Hodel introduced γ-spaces in [**12**]. Ceder [**6**] also introduced Nagata spaces, but our definition is a characterization due to Heath [**9**]. Also our definition of stratifiable spaces is a characterization due to Heath [**11**]. For a discussion of K-semi-stratifiable spaces the reader should see Lutzer [**14**].

It is clear from our definitions that a space is a semi-metric space if and only if it is a first countable semi-stratifiable space. Also it is true (c.f. [**4**]) that a space is a Nagata space if and only if it is a first countable stratifiable space.

We now give two characterizations from [**1**].

THEOREM 2.4. *A T_1 space Y is a Nagata space if and only if there exists a metric space X and an open continuous N-mapping from X onto Y.*

THEOREM 2.5. *A T_1 space Y is a semi-metric space if and only if there exists a metric space X and an open continuous SM-mapping from X onto Y.*

In Section 5 we will similarly characterize stratifiable, semi-stratifiable, and K-semi-stratifiable spaces using weakly continuous mappings.

3. Spaces with separating open covers. We begin with several definitions.

Definition 3.1. Let $\psi : X \rightarrow Y$. Then ψ is *weakly continuous* if given $x \in X$

and $y \in Y$ with $\psi(x) \neq y$, there exists an open set V in x with $X \in V$ and $y \notin \psi(V)$.

The next definition is a generalization of Heath's P-mapping to a non-metric domain.

Definition 3.2. Let X be a space with COC-function g. Then $\psi : X \to Y$ is a *P-mapping relative to* g if for any $y \in Y$ and open set W containing y, there exists an $n \in \mathbf{N}$ such that $\bigcup \{g(n, x) : x \in \psi^{-1}(y)\} \subset \psi^{-1}(W)$.

Definition 3.3. Let X be a space with COC-function g. Then $\psi : X \to Y$ is a *weak P-mapping relative to* g if given $x \neq y$ in Y, there is an $n \in \mathbf{N}$ with $y \notin \psi(\bigcup \{g(n, p) : p \in \psi^{-1}(x)\})$; Also ψ is a *sub P-mapping relative to* g if given $x \neq y$ in Y there is $n \in \mathbf{N}$ with

$$\psi(\bigcup \{g(n, p) : p \in \psi^{-1}(x)\}) \cap \psi(g(n, p) : p \in \psi^{-1}(y)\}) = \emptyset.$$

We remark here that if the domain is a metric space (X, d) and we speak of a P-mapping (weak P-mapping, sub P-mapping) without explicit mention of g, we are assuming that g is given by $g(n, x) = \mathrm{Bd}(x, 1/n) =$ open d-ball about x of radius $1/n$.

It is not hard to see that every sub P-mapping is a weak P-mapping, and that every weak P-mapping is weakly continuous. Furthermore if Y is T_1, any P-mapping into Y is a weak P-mapping, and if Y is T_2, any P-mapping into Y is a sub P-mapping.

Michael [**15**] proved a general theorem from which follow the necessity parts of Theorems 2.1, 2.2, and 2.3. We now prove a similar result and derive some corollaries which are analogous to the necessity parts of those theorems.

THEOREM 3.1. *Let Y be a T_1 space, $\mathscr{S}$ an open cover of Y which is closed under finite intersections with the property that each $y \in Y$ is a countable intersection of elements in $\mathscr{S}$. Then there exists a metric space X, a base $\mathscr{B}$ for X, and a weakly continuous mapping $\psi : X \to Y$ such that:*

(1) $\psi(\mathscr{B}) = \mathscr{S}$; *hence ψ is onto and open*

(2) *For any $E \subset Y$,*

$$\mathrm{card}\, \{B \in \mathscr{B} : B \cap \psi^{-1}(E) \neq \emptyset\} \leqq \aleph_0\, \mathrm{card}\, \{S \in \mathscr{S} : S \cap E \neq \emptyset\}.$$

Proof. Let $\mathscr{S} = \{S_\alpha : \alpha \in A\}$, assuming $S_\alpha \neq S_\beta$ when $\alpha \neq \beta$. For each $n \in N$, let $A_n = A$ and let $M = \prod_{n=1}^\infty A_n$. Give M the Baire metric. Define

$$X = \{f \in M : \{S_{f(n)} : n = 1, 2, \ldots\} \text{ is decreasing and } \bigcap_{n=1}^\infty S_{f(n)} = y \text{ for some } y \in Y\}.$$

Then define $\psi(f) = \bigcap_{n=1}^\infty S_{f(n)}$.

Define the base $\mathscr{B}$ as follows: for each $n \in \mathbf{N}$ and $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in \prod_{i=1}^\infty A_i$ for which $S_{\alpha_1} \supset S_{\alpha_2} \supset \ldots \supset S_{\alpha_n}$, let

$$B(\alpha_1, \alpha_2, \ldots, \alpha_n) = \{f \in X : f(i) = \alpha_i \text{ for } i = 1, 2, \ldots, n\}.$$

Notice that $B(\alpha_1, \alpha_2, \ldots, \alpha_n) = \mathrm{Bd}(f, 1/n)$ for any $f \in B(\alpha_1, \ldots \alpha_n)$. It is not difficult to check that the collection of all such sets forms a base for X.

Let us now show that ψ is weakly continuous. Let $x \neq y$, and let $\bigcap_{n=1}^{\infty} S_{\alpha_n} = y$. Then letting n_0 be such that $x \notin S_{\alpha_{n_0}}$, it is easily checked that $y \notin \psi(B(\alpha_1, \ldots, \alpha_{n_0}))$. But letting $f(n) = \alpha_n$ for each n we get $f \in B(\alpha_1, \ldots, \alpha_{n_0})$ and $\psi(f) = x$.

Both (1) and (2) follow from the fact that $\psi(B(\alpha_1, \ldots, \alpha_n)) = S_{\alpha_n}$. To check this fact let $x \in \psi(B(\alpha_1, \ldots, \alpha_n))$. Then there exists an $f \in B(\alpha_1, \ldots, \alpha_n)$ with $x = \psi(f) = \bigcap_{k=1}^{\infty} S_{f(k)} \subset S\alpha_n$. On the other hand suppose $x \in S_{\alpha_n}$. Let $x = \bigcap_{k=1}^{\infty} S_{\beta_k}$ with $\{S_{\beta_k}\}$ decreasing. Define

$$f(i) = \begin{cases} \alpha_i & \text{if } i \leqq n \\ \gamma_i & \text{if } i > n \text{ where } S_{\gamma_i} = S_{\beta_{i-n}} \cap S_{\alpha_n}. \end{cases}$$

Then $\psi(f) = x$ and $f \in B(\alpha_1, \ldots, \alpha_n)$.

COROLLARY 3.1. *Let Y be a T_1 space in which every point is a G_δ. Then there exists a metric space X and a weakly continuous open mapping from X onto Y.*

Proof. For each $y \in Y$, let $y = \bigcap_{n=1}^{\infty} S_{n,y}$, where each $S_{n,y}$ is open. Let $\mathscr{S} = \{S_{n,y} : n \in \mathbf{N}, y \in Y\}$, and let $\mathscr{S}'$ = the set of all finite intersections of elements of $\mathscr{S}$. Applying Theorem 3.1 with the cover $\mathscr{S}'$ proves the corollary.

The proof of the following corollary would be similar to that of Corollary 3.1. Showing that ψ is an S-mapping uses (2) of Theorem 3.1. (Recall that $\psi : X \to Y$ is an *S-mapping* if $\psi^{-1}(y)$ is separable for each $y \in Y$.)

COROLLARY 3.2. *Let X be a T_1 space with a point-countable separating open cover. Then there exists a metric space X and an open weakly continuous S-mapping from X onto Y.*

COROLLARY 3.3. *Let Y be a T_1 space with G_δ-diagonal. Then there exists a metric space X and an open weak P-mapping from X onto Y.*

Proof. Let $\langle \mathscr{G}_n \rangle$ be a G_δ-diagonal sequence for Y. Let $y \in Y$, then $y = \bigcap_{n=1}^{\infty} \mathrm{st}(y, \mathscr{G}_n)$, i.e. each point in Y is a G_δ. Let $\mathscr{S} = \bigcup_{n=1}^{\infty} \mathscr{G}_n$, and let $\mathscr{S}' = \{S_\alpha : \alpha \in A\}$ be the set of all finite intersections of elements of $\mathscr{S}$. Then we apply Theorem 3.1 with a slight alteration in the definition of X. We define

$$X = \{f \in M : \{S_{f(n)} : n\ 1, 2, \ldots\} \text{ is decreasing, } \bigcap_{n=1}^{\infty} S_{f(n)} = y \text{ for some } y \in Y, \text{ and } S_{f(n)} \subset G \text{ for some } G \in \mathscr{G}_n \text{ for each } n \in \mathbf{N}\}.$$

Define ψ as before. To see that ψ is onto, let $y \in Y$. Choose $S_{f(1)}$ to contain y and be an element of $\mathscr{G}_1$. Choose $S_{f(2)}'$ to contain y and be an element of $\mathscr{G}_2$; then let $S_{f(2)} = S_{f(1)} \cap S_{f(2)}'$. In general, choose $S_{f(n)}'$ to contain y and be an element of $\mathscr{G}_n$ and then let $S_{f(n)} = S_{f(n-1)} \cap S_{f(n)}'$. Then we have

$$y \in \bigcap_{n=1}^{\infty} S_{f(n)} \subset \bigcap_{n=1}^{\infty} \mathrm{st}(y, \mathscr{G}_n) = y,$$

i.e. $f \in X$ and $\psi(f) = y$. It remains to show that ψ is a weak P-mapping.

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Suppose $x \neq y$. Then there exists $n_0 \in \mathbf{N}$ with $y \notin \operatorname{st}(x, \mathscr{G}_{n_0})$. It is not hard to show that $y \notin \psi(Bd(\psi^{-1}(x), 1/n_0))$. For suppose $f \in \psi^{-1}(x)$ and $f(i) = \alpha_i$ for each $i \in \mathbf{N}$. Then if $g \in Bd(f, 1/n_0) = B(\alpha_1, \ldots, \alpha_{n_0})$, we have $\psi(g) \in \psi(B(\alpha_1, \ldots, \alpha_{n_0})) = S_{\alpha n_0}$. But $S_{\alpha n_0} \subset \operatorname{st}(x, \mathscr{G}_{n_0})$ and so $\psi(g) \neq y$ since $y \notin \operatorname{st}(x, \mathscr{G}_{n_0})$.

Proof of the next corollary is similar to that of Corollary 3.3 and so we omit it here.

COROLLARY 3.4. *Let Y be a T_1 space with $G_\delta(2)$-diagonal. Then there exists a metric space X and an open sub P-mapping from X onto Y.*

We are now in a position to provide analogues to Theorems 2.1, 2.2, and 2.3. The necessity parts of the next four theorems are Corollaries 3.1, 3.2, 3.3, and 3.4 respectively, and so we mention only the sufficiency in the proofs.

THEOREM 3.1. *A T_1 space is such that every point is a G_δ if and only if there exists a metric space X and an open weakly continuous mapping from X onto Y.*

Proof. For sufficiency we actually only need a first countable domain. Let $y \in Y$, $x \in X$ with $\psi(x) \neq y$. Now let $\{V_n : n = 1, \ldots\}$ be a fundamental system of neighborhoods for X. It is then easy to check that $y = \bigcap_{n=1}^{\infty} \psi(V_n)$ using the fact that ψ is weakly continuous.

THEOREM 3.3. *A T_1 space Y has a point-countable separating open cover if and only if there exists a metric space X and an open weakly continuous S-mapping from X onto Y.*

Proof. For the sufficiency we need only that the domain have a point-countable base $\{B_\alpha : \alpha \in A\}$. Then using the weak continuity of ψ, it is not hard to show that $\mathscr{S} = \{\psi(B_\alpha) : \alpha \in A\}$ is an open separating cover of Y. It remains to show that $\mathscr{S}$ is point-countable. Let $y \in Y$; then $\psi^{-1}(y)$ is separable. Hence the collection $F = \{B_\alpha : B_\alpha \cap \psi^{-1}(y) \neq \emptyset\}$ is countable. Consequently, $y \in \psi(B_\alpha)$ for only a countable number of α.

PROPOSITION 3.1. *Let X have a sequence of open covers $\langle \mathscr{G}_n \rangle$. For each $x \in X$ and $n \in \mathbf{N}$ let $g(n, x) = \operatorname{st}(x, \mathscr{G}_n)$, and let $\psi : X \rightarrow Y$ be an open, onto, weak P-mapping relative to g. Then Y has a G_δ-diagonal.*

Proof. Let $\mathscr{H}_n = \{\psi(G) : G \in \mathscr{G}_n\}$ for each n. Then $\langle \mathscr{H}_n \rangle$ can be shown to be a G_δ-diagonal sequence for Y, using the fact that ψ is a weak P-mapping.

THEOREM 3.4. *A T_1 space Y has a G_δ-diagonal if and only if there exists a metric space and an open weak P-mapping from X onto Y.*

Proof. Set $\mathscr{G}_n = \{\operatorname{Bd}(x, 1/2n) : x \in X\}$. Then $\operatorname{st}(x, \mathscr{G}_n) = \operatorname{Bd}(x, 1/n)$. Consequently an application of Proposition 3.1 completes the proof.

We can prove a proposition similar to Proposition 3.1 for spaces with a $G_\delta(2)$-diagonal and sub P-mappings and then derive the following characterization.

THEOREM 3.5. *A T_1 space Y has a $G_\delta(2)$-diagonal if and only if there exists a metric space X and an open sub P-mapping from X onto Y.*

Each of the kinds of mappings we have introduced is countably productive. Consequently we can use the characterizations we have proved to prove that (for T_1 spaces) spaces with G_δ points, spaces with point-countable separating open covers, spaces with G_δ-diagonal, and spaces with $G_\delta(2)$-diagonal are all countably productive (c.f. [**1**]).

4. Generalized P-mappings. In this section we study the question of when finite-to-one and compact mappings are P-mappings or weak P-mappings and derive some corollaries concerning images of spaces under such mappings. First, we give a generalization of one implication of Theorem 2.3.

THEOREM 4.1. *Let X be a space with a sequence of open covers $\langle \mathscr{G}_n \rangle$ and let $g(n, x) = \mathrm{st}(x, \mathscr{G}_n)$ for each $n \in \mathbf{N}$, $x \in X$. If $\psi : X \to Y$ is an almost open P-mapping relative to g, then Y is developable.*

Proof. Let $y \in Y$, $n \in \mathbf{N}$. Now there is a $p_y \in \psi^{-1}(y)$ which has an open system of neighborhoods such that $\psi(B)$ is open for every $B \in \mathscr{B}$. Let $B_{n,y} \in \mathscr{B}$ be such that $\psi(B_{n,y}) \subset G$ for some element $G \in \mathscr{G}_n$. Then define $\mathscr{H}_n = \{\psi(B_{m,y}) : y \in Y \text{ and } m \geqq n\}$. We will show that $\langle \mathscr{H}_n \rangle$ is a development for Y.

Let $y \in Y$, W an open set containing y. Then there is an n_0 such that $\bigcup \{g(n_0, x) : x \in \psi^{-1}(y)\} \subset \psi^{-1}(W)$. Now let $y \in H \in \mathscr{H}_{n_0}$. Then $H = \psi(B_{m,p})$ for some $m \geqq n_0$ and $p \in X$. Let $z \in H$; then there is a $t \in \psi^{-1}(z) \cap B_{m,p} \subset G$ for some $G \in \mathscr{G}_m$. Also there is a $x \in \psi^{-1}(y) \cap B_{m,p} \subset G$. Thus $t \in \mathrm{st}(x, \mathscr{G}_m) \subset \mathrm{st}(x, \mathscr{G}_{n_0})$, (we may assume that $\mathscr{G}_{n+1}$ refines $\mathscr{G}_n$ for all n). Consequently $\psi(t) = z$ is an element of

$$\psi(\mathrm{st}(x, \mathscr{G}_{n_0})) \subset \psi(\bigcup \{g(n_0, x) : x \in \psi^{-1}(y)\}) \subset W,$$

i.e. $H \subset W$ or $\mathrm{st}(y, \mathscr{H}_{n_0}) \subset W$.

THEOREM 4.2. *Let X be a space with γ-function g, let $\psi : X \to Y$ be a continuous, compact, onto mapping. Then ψ is a P-mapping relative to g.*

Proof. Let $y \in Y$, W an open set containing y, and let $\psi^{-1}(y) = K$. Now suppose ψ is not a P-mapping relative to g, i.e. that $\bigcup \{g(n, x) : x \in K\} \not\subset \psi^{-1}(W)$ for any $n \in \mathbf{N}$. Then we can choose sequences $\langle x_n \rangle \subset K$ and $\langle y_n \rangle$ with $y_n \in g(n, x_n) \backslash \psi^{-1}(W)$ for each n. Now since K is compact $\langle x_n \rangle$ has a cluster point $p \in K$. Then we can choose subsequences so that for every k, $x_{n_k} \in g(k, p)$, $y_{n_k} \in g(k, x_{n_k})$; hence p is a cluster point of $\langle y_{n_k} \rangle$ since g is a γ-function. But $\psi^{-1}(W)$ is an open neighborhood of p such that $y_{n_k} \notin \psi^{-1}(W)$ for any k, a contradiction.

COROLLARY 4.1. *An almost-open, compact, continuous image of a metric space is developable.*

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Proof. Let $\mathcal{G}_n = \{\mathrm{Bd}(x, 1/2n) : x \in X\}$, and $g(n, x) = \mathrm{st}(x, \mathcal{G}_n) = \mathrm{Bd}(x, 1/n)$. Then g is a γ-function. Now apply Theorems 4.2 and 4.1.

The proof of the following theorem is not hard, and so we omit it.

THEOREM 4.3. *Let X be a space with first countable function g; let $\psi : X \to Y$ be continuous, finite-to-one and onto. Then ψ is a P-mapping relative to g.*

COROLLARY 4.2. *An almost-open, finite-to-one, continuous image of a developable space is developable.*

Proof. Let $\langle \mathcal{G}_n \rangle$ be a development for X, the domain. Then $g(n, x) = \mathrm{st}(x, \mathcal{G}_n)$ is a first countable function. Apply Theorems 4.3 and 4.1.

A COC-function for a space X is called *separating* if $\bigcap_{n=1}^{\infty} g(n, x) = x$, for all $x \in X$. The following theorem is easily established.

THEOREM 4.4. *Let X be a space with separating* COC-*function g, and let $\psi : X \to Y$ be finite-to-one and onto. Then ψ is a weak P-mapping relative to g.*

COROLLARY 4.3. *An almost-open, finite-to-one image of a space with a G_δ-diagonal also has a G_δ-diagonal.*

Proof. Use Theorem 4.4 together with Proposition 3.1 modified slightly to get "almost-open" rather than "open", using a proof almost identical to the proof of Theorem 4.1. Notice that if $\langle \mathcal{G}_n \rangle$ is a G_δ-diagonal sequence, $g(n, x) = \mathrm{st}(x, \mathcal{G}_n)$ is a separating COC-function for X.

5. Stratifiable and semi-stratifiable spaces. For reference we reproduce several definitions from [**1**].

Definition 5.1. Let X and Y be topological spaces, let $\psi : X \to Y$ be an onto mapping, and let g be a COC-function for X. Then ψ is an *N-mapping relative to* g if given any $y \in Y$ and a neighborhood W of y, there is a neighborhood V of y and $n \in \mathbf{N}$ such that $g(n, x) \cap \psi^{-1}(V) \neq \emptyset$, then $\psi(x) \in W$. More simply, ψ will be called an *N-mapping* if there is a COC-function g such that ψ is an N-mapping relative to g. Also ψ is a *SM-mapping relative to g* if given any $y \in Y$ and a neighborhood W, there is an n such that $g(n, x) \cap \psi^{-1}(y) \neq \emptyset$ implies that $\psi(x) \in W$. More simply, ψ is an *SM-mapping* if there is a COC-function g for X such that ψ is an SM-mapping relative to g.

We now prove two theorems very similar to Theorems 2.4 and 2.5. In addition, the reader should compare Theorems 5.1 and 5.2 with results in Nagata [**16**] (Theorems 3 and 1, respectively).

LEMMA 5.1. *Let $\psi : X \to Y$, let h be a stratifiable function for Y, and define $g(n, x) = \psi^{-1}(h[n, \psi(x)])$. Then ψ is an N-mapping relative to g.*

Proof. Let $y \in Y$, W a neighborhood of y. Then there is an $n_0 \in \mathbf{N}$ such that

$y \notin \overline{\bigcup \{h(n_0, p) : p \in Y \backslash W\}}$. Let

$$V = Y - \overline{\bigcup \{h(n_0, p) : p \in Y \backslash W\}}.$$

Then if $g(n_0, x) \cap \psi^{-1}(V) \neq \emptyset$, $h(n_0, \psi(x)) \cap V \neq \emptyset$, i.e. $\psi(x) \in W$.

PROPOSITION 5.1. *Let $\psi : X \to Y$ be open. Then Y is stratifiable if and only if ψ is an N-mapping.*

Proof. The necessity follows from Lemma 5.1. For sufficiency, let $y \in Y$ and $n \in \mathbf{N}$. Choose an $s \in \psi^{-1}(y)$ and define $h(n, y) = \psi(g(n, s))$. Now let H be closed in Y, and suppose that $p \in \overline{\bigcup \{h(n, z) : z \in H\}}$, for each $n \in \mathbf{N}$. If $p \notin H$ then there exists a neighborhood V of p and an n_0 such that if $g(n_0, x) \cap \psi^{-1}(V) \neq \emptyset$ then $\psi(x) \in W = Y \backslash H$. Now since V is a neighborhood of p, $V \cap (\bigcup(\{h(n, z) : z \in H\}) \neq \emptyset$ for each $n \in \mathbf{N}$. Thus there is a $z \in H$ such that $h(n_0, z) \cap V \neq \emptyset$. Therefore if t is such that $h(n_0, z) = \psi(g(n_0, t))$, we have $g(n_0, t) \cap \psi^{-1}(V) \neq \emptyset$. But this implies that $\psi(t) = z \in W$, a contradiction.

THEOREM 5.1. *A T_1 space Y is stratifiable if and only if there exists a metric space X and an open N-mapping from X onto Y.*

Proof. Sufficiency is immediate from Proposition 5.1. For the necessity apply Corollary 3.1 and then Proposition 5.1.

We could state and prove analogues to Lemma 5.1 and Proposition 5.1 for semi-stratifiable spaces and SM-mappings, thereby deriving the following theorem.

THEOREM 5.2. *A T_1 space Y is semi-stratifiable if and only there exists a metric space X and an open SM-mapping from X onto Y.*

We introduce another kind of mapping and use it to characterize K-semi-stratifiable spaces similarly.

Definition 5.2. Let $\psi : X \to Y$, and let g be a COC-function for X. Then ψ is a *KS-mapping relative to* g if given a compact set K in Y and an open set W with $K \subset W$, there exists an $n \in \mathbf{N}$ such that $g(n, x) \cap \psi^{-1}(K) \neq \emptyset$ implies $\psi(x) \in W$. More simply, ψ is called a *KS-mapping* if there exists a COC-function on X such that ψ is a KS-mapping relative to g.

Proceeding as for stratifiable spaces, we can prove the following.

THEOREM 5.3. *A T_1 space Y is a K-semi-stratifiable space if and only if there exists a metric space X and an open KS-mapping from X onto Y.*

We remark that N-mappings, SM-mappings, and KS-mappings are all countably productive. Hence we could use our characterizations to derive countable product theorems, as mentioned in Section 3.

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