19

Quantum fields on space-time

In this chapter we describe the most important examples of (non-interacting) relativistic quantum fields. We will use extensively the formalism developed in Chap. 18.

Most textbook presentations of this subject start from the discussion of representations of the *Poincaré group*. They stress that the most fundamental quantum fields are covariant with respect to this group. In our presentation the Poincaré covariance is a secondary property. The property that we emphasize more is the *Einstein causality* of fields. In the mathematical language this is expressed by the fact that observables belonging to *causally separated* subsets of space-time commute with one another. This property can be true even when there is no Poincaré covariance, e.g. due to the presence of an external (vector) potential in a curved space-time.

The chapter is naturally divided in two parts. In the first part we consider the flat *Minkowski space* and in the second an arbitrary *globally hyperbolic manifold*. In both cases we discuss the influence of an external (classical) potential and a variable mass. In the Minkowski case, we discuss separately the Poincaré covariance, which holds if the potential is zero and mass is constant.

The quantization consists of two stages. In the first stage one introduces the CCR or CAR algebra describing the observables of the system. The underlying phase space is the space of solutions of the corresponding equation defined on the space-time. This space is equipped with a bilinear or sesquilinear form, which leads to the appropriate CCR or CAR.

In the second stage one chooses a representation of the algebra of observables on a Hilbert space. In order to determine this representation one usually assumes that the generator of the time evolution of the classical system is time-independent. Then one can apply the formalism described in Chap. 18. In the case of the Klein–Gordon and Dirac equations on Minkowski space this means that the external potential and the mass do not depend on time.

Sects. 19.5, resp. 19.6 are generalizations of Sects. 19.2, resp. 19.3 to a curved space-time. We limit our discussion to the algebraic quantization of Klein–Gordon and Dirac equations on a globally hyperbolic manifold. As a result we obtain a net of CCR, resp. CAR algebras satisfying the Einstein causality.

Our presentation is limited to the most basic elements of the theory of quantum fields on curved space-time. One of the topics that we leave out, which however is easy to figure out mimicking the discussion in Sects. 19.2, resp. 19.3, is the positive energy quantization of time-independent equations on a stationary space-time; see e.g. Kay (1978). The subject that is more difficult is how to choose representations of quantum fields in the case of non-stationary curved space-times, where there is no preferred vacuum state. There has been significant progress in our understanding of this question. It is believed that one should choose representations generated by states whose correlation functions satisfy a certain natural microlocal condition, the so-called *Hadamard states*. One can find more about this subject in the literature; see Brunetti–Fredenhagen–Köhler (1996) and Brunetti–Fredenhagen–Verch (2003).

19.1 Minkowski space and the Poincaré group

19.1.1 Minkowski space

Consider the Minkowski space $\mathbb{R}^{1,d}$. Recall that it is the vector space $\mathbb{R}^{1+d}$ equipped with a pseudo-Euclidean form of signature $(1,d)$; see Sect. 15.3. In the coordinates $x=(x^\mu)$, $\mu=0,1,\dots,d$, the pseudo-Euclidean quadratic form will be denoted

$$\langle x|x\rangle = -(x^0)^2 + \sum_{i=1}^{d}(x^i)^2.$$

Definition 19.1 *A non-zero vector $x\in\mathbb{R}^{1,d}$ is called*

$$\begin{aligned} &\text{time-like} && \text{if } \langle x|x\rangle < 0,\\ &\text{causal} && \text{if } \langle x|x\rangle \le 0,\\ &\text{light-like} && \text{if } \langle x|x\rangle = 0,\\ &\text{space-like} && \text{if } \langle x|x\rangle > 0. \end{aligned}$$

The set of causal, resp. time-like vectors is denoted J, resp. I. A causal vector x is called

$$\begin{aligned} &\text{future oriented} && \text{if } x^0 > 0,\\ &\text{past oriented} && \text{if } x^0 < 0. \end{aligned}$$

The set of future, resp. past oriented causal vectors is denoted $J^\pm$. The set of future, resp. past oriented time-like vectors is denoted $I^\pm$.

Clearly, $I^\pm$ is the interior of $J^\pm$.

Definition 19.2 *For $\mathcal{U}\subset\mathbb{R}^{1,d}$, we set $J(\mathcal{U}) := J+\mathcal{U}$ and $J^\pm(\mathcal{U}) := J^\pm+\mathcal{U}$. $J(\mathcal{U})$ is called the* causal shadow of *$\mathcal{U}$ and $J^\pm(\mathcal{U})$ the* causal future, *resp.* past of *$\mathcal{U}$. A function on $\mathbb{R}^{1,d}$ is called* space-compact *if there exists a compact $\mathcal{U}\subset\mathbb{R}^{1,d}$ such that* $\operatorname{supp} f\subset J(\mathcal{U})$. *It is called* future, *resp.* past space-compact *if there exists a compact $\mathcal{U}\subset\mathbb{R}^{1,d}$ such that* $\operatorname{supp} f\subset J^\pm(\mathcal{U})$.

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The set of space-compact smooth functions will be denoted $C^\infty_{\rm sc}(\mathbb{R}^{1,d})$. *The set of future, resp. past space-compact smooth functions will be denoted* $C^\infty_{\pm{\rm sc}}(\mathbb{R}^{1,d})$.

Definition 19.3 *Let* $\mathcal{U}_1, \mathcal{U}_2 \subset \mathbb{R}^{1,d}$. *We say that* $\mathcal{U}_1$ *and* $\mathcal{U}_2$ *are* causally separated *if* $J(\mathcal{U}_1) \cap \mathcal{U}_2 = \emptyset$, *or equivalently if* $\mathcal{U}_1 \cap J(\mathcal{U}_2) = \emptyset$.

Definition 19.4 *The operator*

$$\Box := \partial_\mu \partial^\mu = -(\partial^0)^2 + \sum_{i=1}^{d} (\partial^i)^2$$

is called the d'Alembertian.

19.1.2 The Lorentz group

Definition 19.5 *The pseudo-Euclidean group* $O(\mathbb{R}^{1,d}) \simeq O(\mathbb{R}^{d,1})$ *is called the* Lorentz group in $1+d$ dimensions.

Let $r \in O(\mathbb{R}^{1,d})$. *We will say that* r *is* space-time even *if* $\det r = 1$ *and* space-time odd *if* $\det r = -1$.

Note that $r(J^+) = J^+$ *or* $r(J^+) = J^-$. *In the former case we say that* r *is* orthochronous *and in the latter case we say that* r *is* anti-orthochronous.

Thus $O(\mathbb{R}^{1,d})$ has four connected components:

(1) the space-time even orthochronous component $O_+^\uparrow(\mathbb{R}^{1,d})$,
(2) the space-time even anti-orthochronous component $O_+^\downarrow(\mathbb{R}^{1,d})$,
(3) the space-time odd orthochronous component $O_-^\uparrow(\mathbb{R}^{1,d})$,
(4) the space-time odd anti-orthochronous component $O_-^\downarrow(\mathbb{R}^{1,d})$.

Clearly, $O_+^\uparrow(\mathbb{R}^{1,d})$, also denoted $SO^\uparrow(\mathbb{R}^{1,d})$, is a normal subgroup and we have an exact sequence

$$1 \to SO^\uparrow(\mathbb{R}^{1,d}) \to O(\mathbb{R}^{1,d}) \to \mathbb{Z}_2 \times \mathbb{Z}_2 \to 1. \tag{19.1}$$

We have three subgroups of $O(\mathbb{R}^{1,d})$ of index 2:

$$\begin{aligned} O^\uparrow(\mathbb{R}^{1,d}) &= O_+^\uparrow(\mathbb{R}^{1,d}) \cup O_-^\uparrow(\mathbb{R}^{1,d}), \\ O_+(\mathbb{R}^{1,d}) &= O_+^\uparrow(\mathbb{R}^{1,d}) \cup O_+^\downarrow(\mathbb{R}^{1,d}), \\ SO(\mathbb{R}^{1,d}) &= O_+^\uparrow(\mathbb{R}^{1,d}) \cup O_-^\downarrow(\mathbb{R}^{1,d}). \end{aligned}$$

Definition 19.6 *The* temporal parity *is the homomorphism*

$$O(\mathbb{R}^{1,d}) \ni L \mapsto \rho_L \in \{1, -1\}$$

that equals 1 *on an orthochronous and* -1 *on an anti-orthochronous* L.

Definition 19.7 *The affine extension of the Lorentz group*

$$AO(\mathbb{R}^{1,d}) = \mathbb{R}^{1+d} \rtimes O(\mathbb{R}^{1,d})$$

is called the Poincaré group in $1+d$ dimensions.

We refer to Def. 1.100 for the definition of the affine extension.

19.1.3 Pin groups for the Lorentzian signature

In the case of the Lorentz group we have the following refinement of Diagram (14.18):

$$\begin{array}{ccccccccc}
 & & 1 & & 1 & & & & \\
 & & \downarrow & & \downarrow & & & & \\
1 & \to & \mathbb{Z}_2 & \to & \mathbb{Z}_2 & \to & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & Spin^{\uparrow}(\mathbb{R}^{1,d}) & \to & Pin(\mathbb{R}^{1,d}) & \to & \mathbb{Z}_2 \times \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & SO^{\uparrow}(\mathbb{R}^{1,d}) & \to & O(\mathbb{R}^{1,d}) & \to & \mathbb{Z}_2 \times \mathbb{Z}_2 & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 1 & & 1 & & 1 & &
\end{array} \tag{19.2}$$

$Pin(\mathbb{R}^{1,d})$ has four connected components

$$\begin{aligned}
&Pin_+^{\uparrow}(\mathbb{R}^{1,d}),\\
&Pin_+^{\downarrow}(\mathbb{R}^{1,d}),\\
&Pin_-^{\uparrow}(\mathbb{R}^{1,d}),\\
&Pin_-^{\downarrow}(\mathbb{R}^{1,d}),
\end{aligned}$$

which cover the corresponding connected components of $O(\mathbb{R}^{1,d})$ listed in Subsect. 19.1.2.

If we replace $\mathbb{R}^{1,d}$ with $\mathbb{R}^{d,1}$, then all the entries of the above diagram remain the same except for $Pin(\mathbb{R}^{1,d})$ replaced with $Pin(\mathbb{R}^{d,1})$, which are not isomorphic to one another. Both have four connected components, with the obvious notation. Note that we can identify $Spin(\mathbb{R}^{1,d}) \simeq Spin(\mathbb{R}^{d,1})$, hence we can identify $Pin_+^{\uparrow}(\mathbb{R}^{1,d})$, resp. $Pin_-^{\downarrow}(\mathbb{R}^{1,d})$, with $Pin_+^{\uparrow}(\mathbb{R}^{d,1})$, resp. $Pin_-^{\downarrow}(\mathbb{R}^{d,1})$.

We will also use the affine extensions of the Pin groups

$$\begin{aligned}
APin(\mathbb{R}^{1,d}) &= \mathbb{R}^{1,d} \rtimes Pin(\mathbb{R}^{1,d}),\\
APin(\mathbb{R}^{d,1}) &= \mathbb{R}^{d,1} \rtimes Pin(\mathbb{R}^{d,1}),
\end{aligned}$$

which are two-fold coverings of the Poincaré group.

19.1.4 Positive energy representations of Clifford relations

Let $\mathcal{V}$ be a pseudo-unitary vector space equipped with a Hermitian form β. As in Subsect. 15.3.5, we denote by $A^{\dagger}$ the adjoint of A w.r.t. this form.

Definition 19.8 *We say that a representation of Clifford relations*

$$\mathbb{R}^{1,d} \ni y \mapsto \gamma(y) \in L(\mathcal{V}) \tag{19.3}$$

is a positive energy representation *if* $\gamma(y) = -\gamma(y)^\dagger$, $y \in \mathcal{Y}$, *and*

$$\mathrm{i}\overline{v} \cdot \beta\gamma(y_0)v > 0, \quad v \in \mathcal{V}, \ v \neq 0,$$

for some time-like future oriented $y_0 \in \mathbb{R}^{1,d}$.

Lemma 19.9 $\mathrm{i}\beta\gamma(y_0)$ *is positive definite for some time-like future oriented* y_0 *iff* $\mathrm{i}\beta\gamma(y)$ *is positive definite for all time-like future oriented* y.

Proof Let y_1, y_2 be two future oriented vectors. We may assume that $\langle y_1|y_1\rangle = \langle y_2|y_2\rangle$. There exists $r \in SO^\uparrow(\mathbb{R}^{1,d})$ such that $y_2 = ry_1$. By Thm. 15.28, there exists $U \in Spin^\uparrow(\mathbb{R}^{1,d})$ implementing r and such that $UU^\dagger = \mathbb{1}$. $Spin^\uparrow$ is connected, hence $UU^\dagger = \mathbb{1}$, i.e. U is pseudo-unitary. Therefore,

$$\overline{v}\cdot\beta\gamma(y_2)v = \overline{v}\cdot\beta U\gamma(y_1)U^\dagger v = \overline{U^\dagger v}\cdot\beta\gamma(y_1)U^\dagger v.$$

Hence, $\mathrm{i}\beta\gamma(y_1)$ is positive definite iff $\mathrm{i}\beta\gamma(y_2)$ is positive definite. □

Positive energy representations act on a pseudo-unitary space. After fixing a future oriented time-like vector y_0, their representation space can be equipped with the positive definite scalar product

$$\mathrm{i}v_1 \cdot \beta\gamma(y_0)v_2, \quad v_1, v_2 \in \mathcal{V}.$$

19.2 Quantization of the Klein–Gordon equation

The *homogeneous Klein–Gordon equation* has the form

$$(-\Box + m^2)\zeta(x) = 0, \tag{19.4}$$

where $\mathbb{R}^{1,d} \ni x \mapsto \zeta(x)$ is a function on the Minkowski space. This equation is *Poincaré invariant.* That means, elements of the Poincaré group transform solutions of (19.4) into solutions of (19.4).

Besides (19.4), we will consider the Klein–Gordon equation with an external potential and a variable mass,

$$\big(-\left(\partial_\mu + \mathrm{i}A_\mu(x)\right)\left(\partial^\mu + \mathrm{i}A^\mu(x)\right) + m^2(x)\big)\,\zeta(x) = 0. \tag{19.5}$$

It is *Poincaré covariant.* That means solutions of (19.5) are transformed by elements of the Poincaré group into solutions of (19.5) with the transformed external potential and mass.

The equation (19.5) has several interesting properties. First of all, its solutions do not propagate faster than the speed of light. In other words, solutions of the Cauchy problem are supported in the causal shadow of the support of its initial conditions. Secondly, the space of real, resp. complex, space-compact solutions (19.5), denoted $\mathcal{Y}$, has a natural symplectic, resp. charged symplectic form given by a local expression. As a consequence, two solutions of (19.5) with the Cauchy

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data supported in disjoint regions are orthogonal w.r.t. this symplectic, resp. charged symplectic form.

Let us associate with $\mathcal{Y}$ the corresponding CCR algebra. It will satisfy the *Einstein causality* (a property also known by the name *locality*). This means that observables associated with the Cauchy data with disjoint supports will commute. This property is one of the basic postulates of quantum field theory. It is incorporated in the standard sets of axioms of quantum field theory: the *Wightman axioms* (see Streater–Wightman (1964)) and the *Haag–Kastler axioms* (see Haag–Kastler (1964) and Haag (1992)).

The space $\mathcal{Y}$ is equipped with a natural symplectic, resp. charged symplectic dynamics. If the external potential and the mass do not depend on time, this dynamics is generated by a time-independent classical Hamiltonian. If this Hamiltonian is positive, we can apply the positive energy quantization, as described in Subsects. 18.1.1, resp. 18.2.1. We obtain a one-particle Hilbert space $\mathcal{Z}$ and a positive Hamiltonian implementing the dynamics acting on the bosonic Fock space $\Gamma_{\mathrm{s}}(\mathcal{Z})$.

The discussion of the quantization of the Klein–Gordon equation with an external potential and a variable mass gives a good illustration of the difference between the dual phase space $\mathcal{Y}$ and the one-particle space $\mathcal{Z}$. This difference is visible in particular in our discussion of the Poincaré covariance. In particular, we will describe the charge, parity and time reversal covariance on the level of the classical equation, its algebraic quantization and its Hilbert space quantization.

19.2.1 Klein–Gordon operator

Definition 19.10 *Let $m^2 \in \mathbb{R}$. The* Klein–Gordon operator with squared mass *m^2 is the operator on $\mathbb{R}^{1,d}$ given by*

$$\square(m^2) := -\partial_\mu \partial^\mu + m^2. \tag{19.6}$$

Definition 19.11 *Let*

$$\mathbb{R}^{1,d} \ni x \mapsto m^2(x) \in \mathbb{R},$$
$$\mathbb{R}^{1,d} \ni x \mapsto A(x) = \big(A^\mu(x)\big) \in \mathbb{R}^{1,d}$$

be smooth functions. The Klein–Gordon operator with squared mass m^2 and external potential A *is defined as*

$$\square(m^2, A) := -\big(\partial_\mu + \mathrm{i}A_\mu(x)\big)\big(\partial^\mu + \mathrm{i}A^\mu(x)\big) + m^2(x). \tag{19.7}$$

Note that in the real case the external potential has to be zero, because of the imaginary unit in front of it. In what follows, for definiteness, we will consider mostly the complex-valued case.

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19.2.2 Lagrangian of the Klein–Gordon equation

In our presentation we avoid using the Lagrangian formalism. Nevertheless, it is worth mentioning that (19.5) can be obtained as the Euler–Lagrange equations of a variational problem. The Lagrangian can be taken as

$$L(\zeta, \partial\zeta) = -\frac{1}{2}\partial_\mu\zeta\partial^\mu\zeta - \frac{1}{2}m^2\zeta^2, \qquad \text{in the real case,}$$
$$L(\zeta, \overline{\zeta}, \partial\zeta, \partial\overline{\zeta}) = -\overline{(\partial_\mu + \mathrm{i}A_\mu)\zeta}(\partial_\mu + \mathrm{i}A_\mu)\zeta - m^2\overline{\zeta}\zeta, \text{ in the complex case.}$$

In the real case the Euler–Lagrange equations

$$\partial_\zeta L - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\zeta)} = 0 \tag{19.8}$$

yield $\Box(m^2, 0)\zeta = 0$. (Recall that in the real case we do not consider the external potential.)

In the complex case we have two sets of Euler–Lagrange equations. (19.8) yields $\Box(m^2, A)\zeta = 0$. It should be supplemented by

$$\partial_{\overline{\zeta}} L - \partial_\mu \frac{\partial L}{\partial(\partial_\mu\overline{\zeta})} = 0, \tag{19.9}$$

which yields $\Box(m^2, -A)\overline{\zeta}(x) = 0$.

19.2.3 Green's functions

The following theorem describes advanced and retarded Green's functions of the inhomogeneous Klein–Gordon equation.

Theorem 19.12 *Write $\Box = \Box(m^2, A)$. For any $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^{1,d})$ there exist unique functions $\zeta^\pm \in C_{\pm\mathrm{sc}}^\infty(\mathbb{R}^{1,d})$, solutions of*

$$\Box\zeta^\pm = f.$$

Moreover,

$$\zeta^\pm(x) = (G^\pm f)(x) = \int_{\mathbb{R}^{1,d}} G^\pm(x,y)f(y)\mathrm{d}y,$$

where $G^\pm = G^\pm(m^2, A) \in \mathcal{D}'(\mathbb{R}^{1,d} \times \mathbb{R}^{1,d})$ satisfy

$$\Box G^\pm = G^\pm\Box = \mathbb{1},$$
$$\operatorname{supp} G^\pm \subset \big\{(x,y) \ : \ x \in J^\pm(y)\big\},$$
$$\overline{G^\pm(x,y)} = G^\mp(y,x).$$

The proof of Thm. 19.12 can be found e.g. in Bär–Ginoux–Pfäffle (2007).

Definition 19.13 *$G^\pm$ is called the* retarded*, resp.* advanced Green's function*.*

Note that by duality $G^\pm$ can be applied to distributions of compact support.

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Definition 19.14 *The* Pauli–Jordan *or* commutator function *is defined as*

$$G(x,y) := G^+(x,y) - G^-(x,y).$$

Note that

$$\begin{aligned}\Box G = G\Box &= 0,\\ \operatorname{supp} G &\subset \big\{(x,y)\ :\ x \in J(y)\big\},\\ \overline{G(x,y)} &= -G(y,x).\end{aligned}$$

19.2.4 Cauchy problem

Let us introduce a non-covariant notation. We will write t for x^0 and x for $(x^1,\dots,x^d)$. The dot will denote the derivative w.r.t. t. We will also write V for A^0 and A for $(A^1,\dots,A^d)$. Thus, the free Klein–Gordon operator becomes

$$\Box(m^2) = \partial_t^2 - \Delta_{\mathrm{x}} + m^2, \tag{19.10}$$

and the Klein–Gordon operator with an external potential and a variable squared mass becomes

$$\Box(m^2,V,\mathrm{A}) = \big(\partial_t - \mathrm{i}V(t,\mathrm{x})\big)^2 - \sum_{i=1}^{d}\big(\partial_{\mathrm{x}^i} + \mathrm{i}\mathrm{A}_i(t,\mathrm{x})\big)^2 + m^2(t,\mathrm{x}). \tag{19.11}$$

The Pauli–Jordan function G can be used to describe the solution of the Cauchy problem of the Klein–Gordon equation.

Theorem 19.15 *Let $\vartheta,\varsigma \in C_{\mathrm{c}}^\infty(\mathbb{R}^d)$. Then there exists a unique $\zeta \in C_{\mathrm{sc}}^\infty(\mathbb{R}^{1,d})$ that solves*

$$\Box(m,V,\mathrm{A})\zeta = 0 \tag{19.12}$$

with initial conditions

$$\zeta(0,\mathrm{x}) = \varsigma(\mathrm{x}),\quad \dot\zeta(0,\mathrm{x}) = \vartheta(\mathrm{x}) - \mathrm{i}V(0,\mathrm{x})\varsigma(\mathrm{x}).$$

It satisfies $\operatorname{supp}\zeta \subset J(\operatorname{supp}\varsigma \cup \operatorname{supp}\vartheta)$ *and is given by*

$$\begin{aligned}\zeta(t,\mathrm{x}) = &-\int_{\mathbb{R}^d}\big(\partial_s G(t,\mathrm{x};0,\mathrm{y}) - \mathrm{i}G(t,\mathrm{x};0,\mathrm{y})V(0,\mathrm{y})\big)\varsigma(\mathrm{y})\mathrm{d}\mathrm{y}\\ &+\int_{\mathbb{R}^d} G(t,\mathrm{x};0,\mathrm{y})\vartheta(\mathrm{y})\mathrm{d}\mathrm{y}.\end{aligned}$$

19.2.5 Symplectic form on the space of solutions

Definition 19.16 *Let $\mathcal{Y}(m^2,A)$ (also denoted for brevity $\mathcal{Y}$) be the space of smooth space-compact solutions of the Klein–Gordon equation, that is, $\zeta \in C_{\mathrm{sc}}^\infty(\mathbb{R}^{1,d})$ satisfying (19.5).*

Definition 19.17 *Let $\zeta_1, \zeta_2 \in C^\infty(\mathcal{X})$. We define*

$$j^\mu(\zeta_1, \zeta_2, x) := \overline{\partial^\mu \zeta_1(x)}\zeta_2(x) - \overline{\zeta_1(x)}\partial^\mu \zeta_2(x) - 2\mathrm{i}A^\mu(x)\overline{\zeta_1(x)}\zeta_2(x).$$

(In the real case the definition is the same except that we do not need the complex conjugation and the term involving the potential is absent.) We easily check that

$$\partial_\mu j^\mu(x) = -\overline{\Box\zeta_1(x)}\zeta_2(x) + \overline{\zeta_1(x)}\Box\zeta_2(x),$$

where $\Box = \Box(m^2, A)$. Therefore, if $\zeta_1, \zeta_2 \in \mathcal{Y}(m^2, A)$, then

$$\partial_\mu j^\mu(x) = 0,$$

and in such a case the flux of j^μ across a space-like subspace $\mathcal{S}$ of co-dimension 1 does not depend on its choice.

If the space-like hyper-subspace $\mathcal{S}$ is given by the parametrization

$$\mathcal{S} = \{(a + \mathrm{b}\cdot\mathrm{x}, \mathrm{x}) \ : \ \mathrm{x} \in \mathbb{R}^d\}, \tag{19.13}$$

for some $a \in \mathbb{R}$ and $\mathrm{b} \in \mathbb{R}^d$ with $|\mathrm{b}| < 1$, then the flux of j^μ across $\mathcal{S}$

$$\overline{\zeta_1}\cdot\omega\zeta_2 = \int_{\mathbb{R}^d} (1 - |b|^2)\left(j^0 - \mathrm{b}\cdot\mathrm{j}\right)(\zeta_1, \zeta_2, a + \mathrm{b}\cdot\mathrm{x}, \mathrm{x})\mathrm{dx}$$

defines a (charged) symplectic form on $\mathcal{Y}(m^2, A)$.

Note that the (charged) symplectic form ω is defined covariantly under the group $AO^\uparrow(\mathbb{R}^{1,d})$ (it does not depend on the choice of coordinates that preserves the time direction). This is true even if A and m^2 are variable. Under the change of coordinates in $AO^\downarrow(\mathbb{R}^{1,d})$ the (charged) symplectic form changes its sign.

For $\zeta_1, \zeta_2 \in \mathcal{Y}(m^2, V, \mathrm{A})$, the (charged) symplectic form is

$$\begin{aligned}\overline{\zeta_1}\cdot\omega\zeta_2 &= \int_{\mathbb{R}^d} \left(\overline{\dot\zeta_1(0,\mathrm{x})}\zeta_2(0,\mathrm{x}) - \overline{\zeta_1(0,\mathrm{x})}\dot\zeta_2(0,\mathrm{x}) - 2\mathrm{i}V(0,\mathrm{x})\overline{\zeta_1(0,\mathrm{x})}\zeta_2(0,\mathrm{x})\right)\mathrm{dx}\\ &= \int_{\mathbb{R}^d} \left(\overline{\vartheta_1(\mathrm{x})}\varsigma_2(\mathrm{x}) - \overline{\varsigma_1(\mathrm{x})}\vartheta_2(\mathrm{x})\right)\mathrm{dx}.\end{aligned} \tag{19.14}$$

19.2.6 Solutions parametrized by test functions

The Pauli–Jordan function G can be used to construct solutions of the Klein–Gordon equation, which are especially useful in the axiomatic formulation of quantum field theory.

Theorem 19.18 (1) *For any $f \in C_\mathrm{c}^\infty(\mathbb{R}^{1,d})$, $Gf \in \mathcal{Y}$.*
(2) *Every element of $\mathcal{Y}$ is of this form.*
(3) $\overline{Gf_1}\cdot\omega Gf_2 = \int \overline{f_1(x)}G(x,y)f_2(y)\mathrm{d}x\mathrm{d}y$.

For an open set $\mathcal{O} \subset \mathbb{R}^{1,d}$ we set

$$\mathcal{Y}(\mathcal{O}) = \mathcal{Y}(\mathcal{O}, m^2, A) := \left\{Gf \ : \ f \in C_\mathrm{c}^\infty(\mathcal{O})\right\}.$$

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Theorem 19.19 (1) *$\mathcal{Y} = \mathcal{Y}(\mathbb{R}^{1,d})$.*
(2) *$\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{Y}(\mathcal{O}_1) \subset \mathcal{Y}(\mathcal{O}_2)$.*
(3) *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated and $\zeta_i \in \mathcal{Y}(\mathcal{O}_i)$, $i = 1, 2$, then*

$$\overline{\zeta}_1 \cdot \omega \zeta_2 = 0.$$

19.2.7 Algebraic quantization

In the neutral case, starting from symplectic space $(\mathcal{Y}, \omega)$, we define the C^*-algebra

$$\mathfrak{A} = \mathfrak{A}(m^2, A) := \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}).$$

More generally, we have a similar definition for any open set $\mathcal{O} \subset \mathbb{R}^{1,d}$:

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}, m^2, A) := \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y}(\mathcal{O})).$$

In the charged case, we replace the algebra $\mathrm{CCR}^{\mathrm{Weyl}}$ with $\mathrm{CCR}^{\mathrm{reg}}_{\mathrm{gi}}$, as explained in Subsect. 18.2.1.

Theorem 19.20 (1) *$\mathfrak{A} = \mathfrak{A}(\mathbb{R}^{1,d})$.*
(2) *$\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.*
(3) *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated, and $B_i \in \mathfrak{A}(\mathcal{O}_i)$, then*

$$B_1 B_2 = B_2 B_1.$$

19.2.8 Fock quantization

Assume that A and m^2 do not depend on t. For $\zeta \in \mathcal{Y}(m^2, A)$, set

$$r_t \zeta(s, \mathrm{x}) := \zeta(s - t, \mathrm{x}).$$

Clearly,

$$\begin{aligned} r_t &: \mathcal{Y}(m^2, A) \to \mathcal{Y}(m^2, A), \\ \hat{r}_t &: \mathfrak{A}(m^2, A) \to \mathfrak{A}(m^2, A) \end{aligned}$$

are one-parameter groups. (Recall that $\hat{r}_t$ denotes the Bogoliubov automorphism associated with r_t.)

Assume in addition that $m^2(\mathrm{x}) \geq 0$. The Klein–Gordon equation is then a special case of what we called the abstract Klein–Gordon equation in an external potential considered in Subsect. 18.3.6 with

$$\epsilon = \Big(-\sum_{i=1}^{d} \big(\partial_{\mathrm{x}^i} + \mathrm{i}\mathrm{A}_i(\mathrm{x})\big)^2 + m^2(\mathrm{x})\Big)^{\frac{1}{2}}.$$

Assuming that $V^2(\mathrm{x}) < \epsilon^2$, we can apply the formalism of the positive energy quantization to the space $\mathcal{Y}(m^2, A)$, described in Subsect. 18.3.6. If $A(\mathrm{x}) \equiv 0$, we can apply Subsect. 18.3.2 in the neutral case or Subsect. 18.3.5 in the charged case.

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First we obtain the one-particle Hilbert space $\mathcal{Z}(m^2, A)$ together with a self-adjoint operator h. Then we obtain a (neutral or charged) CCR representation over $\mathcal{Y}(m^2, A)$ in $\Gamma_{\mathrm{s}}(\mathcal{Z}(m^2, A))$. We also obtain a positive Hamiltonian $H = \mathrm{d}\Gamma(h)$ that acts on $\Gamma_{\mathrm{s}}(\mathcal{Z}(m^2, A))$, so that

$$\hat{r}_t(B) = \mathrm{e}^{\mathrm{i}tH} B \mathrm{e}^{-\mathrm{i}tH}, \quad B \in \mathfrak{A}.$$

19.2.9 Charge symmetry and charge reversal

Let us consider the complex case. We then have the action of the $U(1)$ group on each $\mathcal{Y}(\mathcal{O}, m^2, A)$, and hence the corresponding group of automorphisms $\{\widehat{\mathrm{e}^{\mathrm{i}\theta}}\}_{\theta \in U(1)}$ on each $\mathfrak{A}(\mathcal{O}, m^2, A)$.

We also have the charge reversal operator χ, defined as $\chi\zeta = \overline{\zeta}$. Clearly, we obtain the isomorphisms

$$\begin{aligned}\chi &: \mathcal{Y}(\mathcal{O}, m^2, A) \to \mathcal{Y}(\mathcal{O}, m^2, -A),\\ \hat{\chi} &: \mathfrak{A}(\mathcal{O}, m^2, A) \to \mathfrak{A}(\mathcal{O}, m^2, -A).\end{aligned}$$

If m^2, A do not depend on time, then we obtain the unitary operator $\chi_{\mathcal{Z}} : \mathcal{Z}(m^2, A) \to \mathcal{Z}(m^2, -A)$ such that

$$\hat{\chi}(B) = \Gamma(\chi_{\mathcal{Z}}) B \Gamma(\chi_{\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}.$$

19.2.10 Covariance under the Poincaré group

An element $\Lambda = (a, L) \in AO(\mathbb{R}^{1,d})$ acts on $\mathbb{R}^{1,d}$ by $\Lambda x = Lx + a$. It also acts on $\zeta \in C^\infty(\mathbb{R}^{1,d})$ by

$$u_\Lambda \zeta(x) := \zeta(\Lambda^{-1} x),$$

or $u_\Lambda \zeta = \zeta \circ \Lambda^{-1}$, and on $A \in C^\infty(\mathbb{R}^{1,d}, \mathbb{R}^{1,d})$ by

$$u_\Lambda A(x) = L A(\Lambda^{-1} x),$$

or $u_\Lambda A = LA \circ \Lambda^{-1}$. Clearly, u_Λ preserves $C^\infty_{\mathrm{sc}}(\mathbb{R}^{1,d})$. We obtain the isomorphisms

$$u_\Lambda : \mathcal{Y}(\mathcal{O}, m^2, A) \to \mathcal{Y}(\Lambda\mathcal{O}, u_\Lambda m^2, u_\Lambda A). \tag{19.15}$$

(19.15) preserves the (charged) symplectic form ω for $\Lambda \in AO^\uparrow(\mathbb{R}^{1,d})$ and changes its sign for $\Lambda \in AO^\downarrow(\mathbb{R}^{1,d})$.

Actually, the standard choice for the action of the Poincaré group is

$$w_\Lambda := \begin{cases} u_\Lambda, & \text{for orthochronous } \Lambda,\\ \chi u_\Lambda, & \text{for anti-orthochronous } \Lambda.\end{cases}$$

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We have

$$w_\Lambda : \mathcal{Y}(\mathcal{O}, m^2, A) \to \mathcal{Y}(\Lambda\mathcal{O}, u_\Lambda m^2, \rho_\Lambda u_\Lambda A), \tag{19.16}$$

$$\hat{w}_\Lambda : \mathfrak{A}(\mathcal{O}, m^2, A) \to \mathfrak{A}(\Lambda\mathcal{O}, u_\Lambda m^2, \rho_\Lambda u_\Lambda A), \tag{19.17}$$

where we recall that ρ_Λ denotes the temporal parity of Λ. Equations (19.16) and (19.17) are linear for Λ orthochronous, and anti-linear otherwise.

Assume now that $m^2(x)$ and $A(x)$ do not depend on time. Consider an element of the Poincaré group $\Lambda = (a, L)$ such that $w_\Lambda m^2$ and $w_\Lambda A$ do not depend on time as well. In particular, this is the case when Λ or $\mathrm{T}\Lambda$ belongs to $\mathbb{R}^{1,d} \rtimes O(\mathbb{R}^d)$, where T denotes the time reversal. Under this assumption, we can introduce

$$w_{\Lambda,\mathcal{Z}} : \mathcal{Z}(m^2, V, \mathrm{A}) \to \mathcal{Z}(m^2 \circ \Lambda^{-1}, V \circ \Lambda^{-1}, L\mathrm{A} \circ \Lambda^{-1}),$$

so that the automorphism $\hat{w}_\Lambda$ is implemented by a unitary or anti-unitary operator:

$$\hat{w}_\Lambda(B) = \Gamma(w_{\Lambda,\mathcal{Z}}) B \Gamma(w_{\Lambda,\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}.$$

If $V = 0$, $\mathrm{A} = 0$, then the whole group $AO^\uparrow(\mathbb{R}^{1,d})$ acts by unitary transformations on the quantum level, and $AO^\downarrow(\mathbb{R}^{1,d})$ acts by anti-unitary transformations. Thus we obtain the action of the whole Poincaré group. In the complex case it should be supplemented by the action of the charge symmetry and the charge reversal.

19.2.11 Parity reversal

An important special element of the Poincaré group is the parity reversal. It is defined as $\mathrm{P} = \mathrm{diag}(1, -1, \dots, -1)$. For $\pi := w_\mathrm{P}$ we have

$$\pi : \mathcal{Y}(\mathcal{O}, m^2, V, \mathrm{A}) \to \mathcal{Y}(\mathrm{P}\mathcal{O}, m^2 \circ \mathrm{P}, V \circ \mathrm{P}, -\mathrm{A} \circ \mathrm{P}),$$
$$\hat{\pi} : \mathfrak{A}(\mathcal{O}, m^2, V, \mathrm{A}) \to \mathfrak{A}(\mathrm{P}\mathcal{O}, m^2 \circ \mathrm{P}, V \circ \mathrm{P}, -\mathrm{A} \circ \mathrm{P}).$$

If in addition m^2, A do not depend on time, then we have the unitary operator

$$\pi_\mathcal{Z} : \mathcal{Z}(m^2, V, \mathrm{A}) \to \mathcal{Z}(m^2 \circ \mathrm{P}, V \circ \mathrm{P}, -\mathrm{A} \circ \mathrm{P})$$

such that

$$\hat{\pi}(B) = \Gamma(\pi_\mathcal{Z}) B \Gamma(\pi_\mathcal{Z})^{-1}, \quad B \in \mathfrak{A}.$$

19.2.12 Time reversal

Let $\mathrm{T} = \mathrm{diag}(-1, 1, \dots, 1)$ be the time reversal as an element of the Poincaré group. The standard choice for the time reversal (Wigner's time reversal) is $\tau := w_\mathrm{T} = \chi u_\mathrm{T}$. We have

$$\tau : \mathcal{Y}(\mathcal{O}, m^2, V, \mathrm{A}) \to \mathcal{Y}(\mathrm{T}\mathcal{O}, m^2 \circ \mathrm{T}, V \circ \mathrm{T}, -\mathrm{A} \circ \mathrm{T}),$$
$$\hat{\tau} : \mathfrak{A}(\mathcal{O}, m^2, V, \mathrm{A}) \to \mathfrak{A}(\mathrm{T}\mathcal{O}, m^2 \circ \mathrm{T}, V \circ \mathrm{T}, -\mathrm{A} \circ \mathrm{T}).$$

If in addition m^2, A do not depend on time, then we obtain the anti-unitary operator

$$\tau_{\mathcal{Z}} : \mathcal{Z}(m^2, V, \mathrm{A}) \to \mathcal{Z}(m^2, V, -\mathrm{A})$$

such that

$$\hat{\tau}(B) = \Gamma(\tau_{\mathcal{Z}}) B \Gamma(\tau_{\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}.$$

19.2.13 Klein–Gordon equation in the momentum representation

In this subsection we assume that $A = (V, \mathrm{A}) = 0$ (the external potential vanishes).

We denote by $\xi = (\tau, \mathrm{k}) \in \mathbb{R}^{1,d}$ the variables dual to $(t, \mathrm{x}) \in \mathbb{R}^{1,d}$, paired by the Lorentz metric:

$$\langle \xi | x \rangle = -\tau \cdot t + \mathrm{k} \cdot \mathrm{x}.$$

If $\varsigma \in \mathcal{S}(\mathbb{R}^d)$, $\hat{\varsigma}$ will denote the usual unitary Fourier transform of ς, and if $f \in \mathcal{S}(\mathbb{R}^{1+d})$, we set

$$\hat{f}(\xi) = (2\pi)^{-\frac{1}{2}(1+d)} \int \mathrm{e}^{-\mathrm{i}\langle \xi | x \rangle} f(x) \mathrm{d}x.$$

Note that

$$\widehat{u_\Lambda \zeta}(\xi) = \mathrm{e}^{-\mathrm{i}\langle a | \xi \rangle} \hat{\zeta}(L^{-1}\xi), \quad \Lambda = (a, L). \tag{19.18}$$

Definition 19.21 *Define the* mass hyperboloid

$$C_m := \{\xi \in \mathbb{R}^{1,d} \ : \ \langle \xi | \xi \rangle + m^2 = 0\},$$

which splits into the two connected components $C_m^\pm = C_m \cap \{\pm\tau > 0\}$.

Note that

$$\delta\big(\langle \xi | \xi \rangle + m^2\big) \tag{19.19}$$

is a measure supported on C_m invariant w.r.t. $O(\mathbb{R}^{1,d})$. (We refer to Subsect. 4.1.2 for the notation used in (19.19).)

Set $\epsilon(\mathrm{k}) := (\mathrm{k}^2 + m^2)^{\frac{1}{2}}$. (19.19) has a decomposition

$$\delta\big(\langle \xi | \xi \rangle + m^2\big) = \frac{\delta\big(\tau - \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} + \frac{\delta\big(\tau + \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})}$$

into the sum of measures supported on C_m^+ and C_m^-, invariant w.r.t. $O^\uparrow(\mathbb{R}^{1,d})$.

Proposition 19.22 *Let ζ be the solution of (19.4) with initial data (ς,ϑ) at $t=0$, that is,*

$$\begin{aligned}\Box\zeta &= 0,\\ \zeta(0,\mathrm{x}) = \varsigma(\mathrm{x}),\quad \dot\zeta(0,\mathrm{x}) &= \vartheta(\mathrm{x}).\end{aligned}$$

Define a function f on C_m by

$$f(\tau,\mathrm{k}) = \epsilon(\mathrm{k})\hat\varsigma(\mathrm{k}) + \mathrm{i}\,\mathrm{sgn}(\tau)\hat\vartheta(\mathrm{k}),\quad (\tau,k)\in C_m.$$

Then,

$$\hat\zeta(\xi) = (2\pi)^{\frac12} f(\xi)\delta\big(\langle\xi|\xi\rangle + m^2\big), \tag{19.20}$$

$$\overline{\zeta}_1\cdot\omega\zeta_2 = \frac{1}{2\pi}\mathrm{Im}\int\overline{f_2(\xi)}\,\mathrm{sgn}(\tau)f_1(\xi)\delta\big(\langle\xi|\xi\rangle + m^2\big)\mathrm{d}\xi, \tag{19.21}$$

$$\widehat{\mathrm{j}\zeta}(\xi) = -\mathrm{i}\,\mathrm{sgn}(\tau)\hat\zeta(\xi).$$

Proof Using (18.26), we get that

$$\begin{aligned}&\hat\zeta(\tau,\mathrm{k})\\ &= 2^{-1}(2\pi)^{\frac12}\left(\Big(\delta(\tau-\epsilon(\mathrm{k})\Big)\Big(\hat\varsigma(\mathrm{k}) + \mathrm{i}\frac{\hat\vartheta(\mathrm{k})}{\epsilon(\mathrm{k})}\Big) + \delta\Big(\tau+\epsilon(\mathrm{k})\Big)\Big(\hat\varsigma(\mathrm{k}) - \mathrm{i}\frac{\hat\vartheta(\mathrm{k})}{\epsilon(\mathrm{k})}\Big)\right)\\ &= (2\pi)^{\frac12}\Big(\mathrm{i}\,\mathrm{sgn}(\tau)\hat\vartheta(\mathrm{k}) + \epsilon(\mathrm{k})\hat\varsigma(\mathrm{k})\Big)\,\delta\big(\langle\xi|\xi\rangle + m^2\big),\end{aligned}$$

which yields (19.20). To see (19.21) we use the expression of ω in terms of the Cauchy data given in Subsect. 18.3.5 and

$$\begin{aligned}\hat\varsigma(\mathrm{k}) &= \frac{1}{2\epsilon(\mathrm{k})}\big(f(\epsilon(\mathrm{k}),\mathrm{k}) + f(-\epsilon(\mathrm{k}),\mathrm{k})\big),\\ \hat\vartheta(\mathrm{k}) &= \frac{1}{2\mathrm{i}}\big(f(\epsilon(\mathrm{k}),\mathrm{k}) - f(-\epsilon(\mathrm{k}),\mathrm{k})\big).\end{aligned}$$

$\Box$

Using that $\delta\big(\langle\xi|\xi\rangle + m^2\big)$ is invariant under the Lorentz group, we see that the action of the Poincaré group becomes

$$w_\Lambda f(\xi) = \begin{cases}\mathrm{e}^{-\mathrm{i}\langle a|\xi\rangle}f(L^{-1}\xi), & (a,L)\in AO^\uparrow(\mathbb{R}^{1,d}),\\ \mathrm{e}^{-\mathrm{i}\langle a|\xi\rangle}\overline{f(-L^{-1}\xi)}, & (a,L)\in AO^\downarrow(\mathbb{R}^{1,d}).\end{cases}$$

This is another way to see that w_Λ is symplectic and commutes with j, resp. anti-symplectic and anti-commutes with j, for all $\Lambda\in AO^\uparrow(\mathbb{R}^{1,d})$, resp. $\Lambda\in AO^\downarrow(\mathbb{R}^{1,d})$. Therefore, all elements of $AO^\uparrow(\mathbb{R}^{1,d})$, resp. $AO^\downarrow(\mathbb{R}^{1,d})$ can be implemented by the unitaries, resp. anti-unitaries $\Gamma(w_\Lambda)$ in the Fock representation.

19.3 Quantization of the Dirac equation

The *homogeneous Dirac equation* has the form

$$(\gamma^\mu\partial_\mu - m)\zeta(x) = 0. \tag{19.22}$$

Here, $\mathbb{R}^{1,d} \ni x \mapsto \zeta(x)$ is a function on the Minkowski space with values in Dirac spinors and γ^μ are the Dirac matrices. Equation (19.22) is invariant w.r.t. $APin(\mathbb{R}^{1,d})$, the covering of the Poincaré group.

Besides (19.22) we will consider the Dirac equation with an external potential and a variable mass,

$$\big(\gamma^\mu(\partial_\mu + \mathrm{i}A_\mu(x)) - m(x)\big)\zeta(x) = 0. \tag{19.23}$$

It is covariant w.r.t. the group $APin(\mathbb{R}^{1,d})$.

The Dirac equation has a number of similarities with the Klein–Gordon equation. First of all, solutions of (19.23) do not propagate faster than the speed of light. Secondly, on the space of space-compact solutions of (19.23), denoted by $\mathcal{Y}$, there exists a locally defined sesquilinear form. Recall that in the case of the Klein–Gordon equation this form was symplectic or charged symplectic. In the case of the Dirac equation, this form is a positive definite scalar product given by a local expression. As a consequence, two solutions of (19.23) with the Cauchy data supported in disjoint regions are orthogonal.

In the case of a positive definite scalar product, it is natural to use the fermionic quantization. Thus, let us associate with $\mathcal{Y}$ the corresponding CAR algebra. It will satisfy the fermionic version of the Einstein causality. This means that fields associated with the Cauchy data with disjoint supports will anti-commute. Consequently, even observables associated with data with disjoint supports will commute.

The space $\mathcal{Y}$ has a natural unitary dynamics. If the external potential and the mass do not depend on time, this dynamics has a time invariant generator. If the dynamics is non-degenerate, we can apply the positive energy quantization, as described in Subsect. 18.2.2. We obtain a one-particle Hilbert space $\mathcal{Z}$ and a positive Hamiltonian implementing the dynamics acting on the fermionic Fock space $\Gamma_{\mathrm{a}}(\mathcal{Z})$.

The discussion of the Poincaré invariance and covariance of the Dirac equation, which we give at the end of this section, is more complicated than in the case of the Klein–Gordon equation. In particular, to discuss the charge and time reversal we need some properties of Clifford algebras obtained in Subsect. 15.3.2.

19.3.1 Dirac operator

Let $(\mathcal{V}, \beta)$ be a finite-dimensional pseudo-unitary space. Let

$$\mathbb{R}^{1,d} \ni y \mapsto \gamma(y) \in L(\mathcal{V})$$

be a positive energy representation of $\mathrm{Cliff}(\mathbb{R}^{1,d})$. We fix a future oriented unit vector e in $\mathbb{R}^{1,d}$. Without loss of generality we may assume that $e = e^0$, where $e^0, \dots, e^d$ is the canonical basis of $\mathbb{R}^{1,d}$. We set $\gamma^\mu = \gamma(e^\mu)$, $\mu = 0, \dots, d$, and equip $\mathcal{V}$ with the scalar product

$$\overline{v}_1 \cdot v_2 := \mathrm{i}\overline{v}_1 \cdot \beta\gamma^0 v_2.$$

Using this scalar product, we can identify the Hermitian form β with $\mathrm{i}\gamma^0$. Therefore in the rest of this section, β will denote the operator $\mathrm{i}\gamma^0$. The adjoint of A w.r.t. the above scalar product will be denoted as usual by A^*.

As seen in Subsect. 19.1.4, we have

$$\begin{aligned}(\gamma^0)^2 = -\mathbb{1}, \quad (\gamma^i)^2 = \mathbb{1}, \quad & i = 1, \dots, d;\\ \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0, \quad & 0 \le \mu < \nu \le d;\\ (\gamma^0)^* = -\gamma^0, \quad (\gamma^i)^* = \gamma^i, \quad & i = 1, \dots, d.\end{aligned}$$

Definition 19.23 *Let $m \in \mathbb{R}$. The* Dirac operator of mass m *is the operator on $C^\infty(\mathbb{R}^{1,d}, \mathcal{V})$ given by*

$$\mathbb{D}(m) := \gamma^\mu \partial_\mu - m.$$

Definition 19.24 *Let*

$$\begin{aligned}&\mathbb{R}^{1,d} \ni x \mapsto m(x) \in \mathbb{R},\\ &\mathbb{R}^{1,d} \ni x \mapsto A(x) = \big(A^\mu(x)\big) \in \mathbb{R}^{1,d}\end{aligned}$$

be smooth functions. The Dirac operator with mass m and external potential A *is defined as*

$$\mathbb{D}(m, A) := \gamma^\mu\big(\partial_\mu + \mathrm{i}A_\mu(x)\big) - m(x).$$

19.3.2 Lagrangian of the Dirac equation

The Dirac equation $\mathbb{D}(m, A)\zeta = 0$ can be obtained as the Euler–Lagrange equation of the following Lagrangian:

$$L_1(\zeta, \overline{\zeta}, \partial\zeta, \partial\overline{\zeta}) := \overline{\zeta} \cdot \beta\big(\gamma^\mu(\partial_\mu + \mathrm{i}A_\mu) - m\big)\zeta. \tag{19.24}$$

It is also the Euler–Lagrange equation of the following more symmetric Lagrangian:

$$\begin{aligned}&L(\zeta, \overline{\zeta}, \partial\zeta, \partial\overline{\zeta})\\ &:= \frac{1}{2}(\overline{\zeta} \cdot \beta\gamma^\mu\partial_\mu\zeta - \overline{\gamma^\mu\partial_\mu\zeta} \cdot \beta\zeta) + \mathrm{i}\overline{\zeta} \cdot \beta\gamma^\mu A_\mu\zeta - m\overline{\zeta} \cdot \beta\zeta.\end{aligned} \tag{19.25}$$

It is easy to see that (19.24) and (19.25) differ by a full derivative.

Remark 19.25 *In our notation $\overline{\zeta}$ will always denote the complex conjugate of ζ (to be consistent with the usage of $\overline{\zeta}$ elsewhere in our work and in most of the literature). In a large part of the physics literature, in the context of the Dirac equation, $\overline{\zeta}$ has a special meaning: in our notation it means $\overline{\beta\zeta}$.*

19.3.3 Green's functions

Note the identity

$$\begin{aligned}-\mathbb{D}(-m, A)\mathbb{D}(m, A) = &-\big(\partial_\mu + \mathrm{i}A_\mu(x)\big)\big(\partial^\mu + \mathrm{i}A^\mu(x)\big)\\ &+\gamma^{\mu\nu}F_{\mu\nu}(x) + \gamma^\mu G_\mu(x) + m(x)^2,\end{aligned} \tag{19.26}$$

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where

$$\begin{aligned}\gamma^{\mu\nu} &:= \frac{1}{2\mathrm{i}}[\gamma^\mu, \gamma^\nu],\\ F_{\mu\nu}(x) &:= \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x),\\ G_\mu(x) &:= \partial_\mu m(x).\end{aligned}$$

(19.26) is a Klein–Gordon operator with a matrix-valued mass. It has a Green's function, which can be used to express the Green's function of $\mathbb{D}(m, A)$. In fact, let $G^\pm(x, y)$ be the retarded, resp. advanced Green's function of (19.26).

Definition 19.26 *$S^\pm(x, y) := -\mathbb{D}(-m, A)G^\pm(x, y)$ is called the* retarded*, resp.* advanced Green's function of the Dirac equation.

Theorem 19.27 *Write $\mathbb{D}$ for $\mathbb{D}(m, A)$. For any $f \in C_{\mathrm{c}}^\infty(\mathbb{R}^{1,d}, \mathcal{V})$ there exist unique functions $\zeta^\pm \in C_{\pm\mathrm{sc}}^\infty(\mathbb{R}^{1,d}, \mathcal{V})$ that solve*

$$\mathbb{D}\zeta^\pm = f. \tag{19.27}$$

Moreover,

$$\zeta^\pm(x) = (S^\pm f)(x) = \int_{\mathbb{R}^{1,d}} S^\pm(x, y) f(y)\mathrm{d}y,$$

where $S^\pm$ is a distribution in $\mathcal{D}'\big(\mathbb{R}^{1,d} \times \mathbb{R}^{1,d}, L(\mathcal{V})\big)$, which satisfies

$$\begin{aligned}\mathbb{D}S^\pm &= S^\pm\mathbb{D} = \mathbb{1},\\ \operatorname{supp} S^\pm &\subset \big\{(x, y) \ : \ x \in J^\pm(y)\big\},\\ S^\pm(x, y)^* &= S^\mp(y, x).\end{aligned}$$

Definition 19.28 *We set $S(x, y) = S^+(x, y) - S^-(x, y)$.*

We have

$$\begin{aligned}\mathbb{D}S &= 0,\\ \operatorname{supp} S &\subset \big\{(x, y) \ : \ x \in J(y)\big\},\\ S(x, y)^* &= -S(y, x).\end{aligned}$$

19.3.4 Cauchy problem

We will use the non-covariant notation introduced in Subsect. 19.2.4. We also set $\alpha^i := -\gamma^0\gamma^i$, $i = 1, \dots, d$, obtaining a representation of CAR $\beta, \alpha^1, \dots, \alpha^d$, that is,

$$\begin{aligned}\beta^2 = \mathbb{1}, \quad (\alpha_i)^2 = \mathbb{1}, &\quad i = 1, \dots, d;\\ \beta\alpha_i + \alpha_i\beta = 0, \quad \alpha_i\alpha_j + \alpha_j\alpha_i = 0, &\quad 1 \le i < j \le d;\\ \beta^* = \beta, \quad \alpha_i^* = \alpha_i, &\quad i = 1, \dots, d.\end{aligned}$$

We can rewrite the Dirac equation in the Hamiltonian form

$$\begin{aligned}\partial_t \vartheta_t &= \mathrm{i} b(t)\vartheta_t, \\ b(t) &:= -\alpha^i \big(D_i - \mathrm{A}_i(t,\mathrm{x})\big) + V(t,\mathrm{x}) + m(t,\mathrm{x})\beta,\end{aligned} \tag{19.28}$$

where $\vartheta_t(\mathrm{x}) = \zeta(t,\mathrm{x})$.

Theorem 19.29 *Let $\vartheta \in C_{\mathrm{c}}^\infty(\mathbb{R}^d, \mathcal{V})$. Then there exists a unique $\zeta \in C_{\mathrm{sc}}^\infty(\mathbb{R}^{1,d}, \mathcal{V})$ that solves*

$$\begin{cases} \mathbb{D}(m,A)\zeta(x) = 0, \\ \qquad\quad \zeta(0,\mathrm{x}) = \vartheta(\mathrm{x}). \end{cases}$$

It satisfies $\mathrm{supp}\zeta \subset J(\mathrm{supp}\,\vartheta)$ *and is given by*

$$\zeta(t,\mathrm{x}) = -\int_{\mathbb{R}^d} S(t,\mathrm{x};0,\mathrm{y})\gamma^0\vartheta(\mathrm{y})\mathrm{d}\mathrm{y}.$$

Moreover, $S(t,\mathrm{x};0,y)\gamma^0$ is the integral kernel of the operator

$$\mathrm{T}\exp\left(\mathrm{i}\int_0^t b(s)\mathrm{d}s\right).$$

19.3.5 Scalar product in the space of solutions

Definition 19.30 *Let $\mathcal{Y}(m,A)$ be the space of smooth space-compact solutions of the Dirac equation, that is, $\zeta \in C_{\mathrm{sc}}^\infty(\mathbb{R}^{1,d}, \mathcal{V})$ satisfying (19.23).*

Definition 19.31 *Let $\zeta_1, \zeta_2 \in C^\infty(\mathbb{R}^{1,d}, \mathcal{V})$. Set*

$$j^\mu(\zeta_1,\zeta_2,x) := \mathrm{i}\overline{\zeta_1(x)}\cdot\beta\gamma^\mu\zeta_2(x).$$

We easily check that

$$\partial_\mu j^\mu(x) = \mathrm{i}\overline{\mathbb{D}\zeta_1(x)}\cdot\beta\zeta_2(x) + \mathrm{i}\overline{\zeta_1(x)}\cdot\beta\mathbb{D}\zeta_2(x),$$

where $\mathbb{D} = \mathbb{D}(m,A)$. Therefore, if $\zeta_1,\zeta_2 \in \mathcal{Y}(m,A)$, then

$$\partial_\mu j^\mu(x) = 0,$$

and in such a case the flux of j^μ across a space-like hyper-subspace does not depend on its choice. This choice defines a scalar product on $\mathcal{Y}(m,A)$. For instance, if we consider the hyper-subspace (19.13), then we obtain the following expression for this scalar product:

$$\overline{\zeta}_1\cdot\zeta_2 = \int_{\mathbb{R}^d}(1-|b|^2)\left(j^0 - \mathrm{b}\cdot\mathrm{j}\right)(\zeta_1,\zeta_2,a+\mathrm{b}\cdot\mathrm{x},\mathrm{x})\mathrm{d}\mathrm{x}.$$

In terms of the Cauchy data we have

$$\overline{\zeta}_1\cdot\zeta_2 = \int_{\mathbb{R}^d}\overline{\zeta_1(0,\mathrm{x})}\cdot\zeta_2(0,\mathrm{x})\mathrm{d}\mathrm{x}.$$

19.3.6 Solutions parametrized by test functions

Similarly as in the case of the Klein–Gordon equation, solutions of the Dirac equation can be parametrized by space-time functions:

Theorem 19.32 (1) *For any* $f \in C_{\mathrm{c}}^{\infty}(\mathbb{R}^{1,d}, \mathcal{V})$, $Sf \in \mathcal{Y}$.
(2) *Every element of* $\mathcal{Y}$ *is of this form.*
(3) $\overline{Sf_1} \cdot Sf_2 = \mathrm{i} \int_{\mathbb{R}^{1,d}} \int_{\mathbb{R}^{1,d}} \overline{f_1(x)} \beta S(x,y) f_2(y) \mathrm{d}x \mathrm{d}y$.

For an open set $\mathcal{O} \subset \mathbb{R}^{1,d}$ we put

$$\mathcal{Y}(\mathcal{O}) = \mathcal{Y}(\mathcal{O}, m, A) := \big\{ Sf \ : \ f \in C_{\mathrm{c}}^{\infty}(\mathcal{O}, \mathcal{V}) \big\}.$$

Theorem 19.33 (1) $\mathcal{Y} = \mathcal{Y}(\mathbb{R}^{1,d})$.
(2) $\mathcal{O}_1 \subset \mathcal{O}_2$ *implies* $\mathcal{Y}(\mathcal{O}_1) \subset \mathcal{Y}(\mathcal{O}_2)$.
(3) *If* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are causally separated and* $\zeta_i \in \mathcal{Y}(\mathcal{O}_i)$, $i = 1, 2$, *then*

$$\overline{\zeta}_1 \cdot \zeta_2 = 0.$$

19.3.7 Algebraic quantization

Let

$$\zeta_1 \cdot \nu \zeta_2 := \frac{1}{2} \mathrm{Re}\, \overline{\zeta}_1 \cdot \zeta_2, \quad \zeta_1, \zeta_2 \in \mathcal{Y}.$$

As explained in Subsect. 18.2.2, the Euclidean space $(\mathcal{Y}_{\mathbb{R}}, \nu)$ is used to define the field algebra of the fermionic system, the C^*-algebra

$$\mathfrak{A} = \mathfrak{A}(m, A) := \mathrm{CAR}^{C^*}(\mathcal{Y}_{\mathbb{R}}).$$

More generally, we have a similar definition for any open set $\mathcal{O} \subset \mathbb{R}^{1,d}$:

$$\mathfrak{A}(\mathcal{O}) = \mathfrak{A}(\mathcal{O}, m, A) := \mathrm{CAR}^{C^*}(\mathcal{Y}(\mathcal{O})_{\mathbb{R}}).$$

As explained in Subsect. 18.2.2, for the observable algebra we take $\mathrm{CAR}_{\mathrm{gi}}^{C^*}(\mathcal{Y})$.

Theorem 19.34 (1) $\mathfrak{A} = \mathfrak{A}(\mathbb{R}^{1,d})$.
(2) $\mathcal{O}_1 \subset \mathcal{O}_2$ *implies* $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.
(3) *If* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are causally separated, and* $B_i \in \mathfrak{A}(\mathcal{O}_i)$ *are elements of pure parity, then*

$$B_1 B_2 = (-1)^{|B_1||B_2|} B_2 B_1.$$

19.3.8 Fock quantization

Assume that A and m do not depend on t. For $\zeta \in \mathcal{Y}(m, A)$, set

$$r_t \zeta(s, \mathrm{x}) := \zeta(s - t, \mathrm{x}).$$

Clearly,

$$
\begin{aligned}
r_t &: \mathcal{Y}(m, A) \to \mathcal{Y}(m, A), \\
\hat{r}_t &: \mathfrak{A}(m, A) \to \mathfrak{A}(m, A)
\end{aligned}
$$

are one-parameter groups.

Write the Dirac equation in the Hamiltonian form (19.28), where now b does not depend on time. If $\operatorname{Ker} b = \{0\}$, we can apply the formalism of the positive energy quantization to the space $\mathcal{Y}(m, A)$, described in Subsect. 18.1.2 in the neutral case and in Subsect. 18.2.2 in the charged case. This construction leads to the Kähler anti-involution j, the one-particle space $\mathcal{Z}(m, A)$ and the positive one-particle Hamiltonian h on $\mathcal{Z}(m, A)$. We obtain a representation of $\mathfrak{A}(m, A)$ on $\Gamma_{\mathrm{a}}\big(\mathcal{Z}(m, A)\big)$ such that the dynamics $\hat{r}_t$ is implemented by the Hamiltonian $H := \mathrm{d}\Gamma(h)$:

$$
\hat{r}_t(B) = \mathrm{e}^{\mathrm{i}tH} B \mathrm{e}^{-\mathrm{i}tH}, \quad B \in \mathfrak{A}(m, A).
$$

19.3.9 Charge symmetry

We have the action of the $U(1)$ group

$$
\begin{aligned}
\mathrm{e}^{\mathrm{i}\theta} &: \mathcal{Y}(\mathcal{O}, m, A) \to \mathcal{Y}(\mathcal{O}, m, A), \\
\widehat{\mathrm{e}^{\mathrm{i}\theta}} &: \mathfrak{A}(\mathcal{O}, m, A) \to \mathfrak{A}(\mathcal{O}, m, A).
\end{aligned}
$$

If m, A do not depend on time, then we can define the charge operator $Q := \mathrm{d}\Gamma(q_{\mathcal{Z}})$, and we have

$$
\widehat{\mathrm{e}^{\mathrm{i}\theta}}(B) = \mathrm{e}^{\mathrm{i}\theta Q} B \mathrm{e}^{-\mathrm{i}\theta Q}, \qquad B \in \mathfrak{A}(m, A).
$$

19.3.10 Charge reversal

Recall that in Subsect. 15.3.2 we studied the existence of charge reversal in the context of Clifford relations. They are anti-linear operators on $\mathcal{V}$, denoted χ_+ or χ_-, defined by the following conditions:

(1) χ_+ is called a *real charge reversal* if

$$
\chi_+ \gamma(y) \chi_+^{-1} = \gamma(y), \quad \chi_+^2 = \mathbb{1};
$$

(2) χ_+ is called a *quaternionic charge reversal* if

$$
\chi_+ \gamma(y) \chi_+^{-1} = \gamma(y), \qquad \chi_+^2 = -\mathbb{1};
$$

(3) χ_- is called a *pseudo-real charge reversal* if

$$
\chi_- \gamma(y) \chi_-^{-1} = -\gamma(y), \qquad \chi_-^2 = \mathbb{1};
$$

(4) χ_- is called a *pseudo-quaternionic charge reversal* if

$$\chi_-\gamma(y)\chi_-^{-1} = -\gamma(y), \quad \chi_-^2 = -1\!\!1.$$

$\chi_\pm$ can be lifted in an obvious way to the dual phase space $\mathcal{Y}$, and then to the algebra $\mathfrak{A}$. We obtain the operators

$$\chi_\pm : \mathcal{Y}(\mathcal{O}, m, A) \to \mathcal{Y}(\mathcal{O}, \pm m, -A), \tag{19.29}$$

$$\hat{\chi}_\pm : \mathfrak{A}(\mathcal{O}, m, A) \to \mathfrak{A}(\mathcal{O}, \pm m, -A). \tag{19.30}$$

Note that (19.29) is anti-unitary and (19.30) is a (linear) $*$-homomorphism.

Recall from Subsect. 15.3.2 that in the case of an irreducible representation of Clifford relations, the existence and properties of $\chi_\pm$ can be summarized with the following table:

Table 19.1

$d(\mathrm{mod}\,8)$	χ_+^2	χ_-^2
0	—	$1\!\!1$
1	$1\!\!1$	$1\!\!1$
2	$1\!\!1$	—
3	$1\!\!1$	$-1\!\!1$
4	—	$-1\!\!1$
5	$-1\!\!1$	$-1\!\!1$
6	$-1\!\!1$	—
7	$-1\!\!1$	$1\!\!1$

If m, A do not depend on time, then we can introduce the unitary operator

$$\chi_{\pm,\mathcal{Z}} : \mathcal{Z}(m, A) \to \mathcal{Z}(\pm m, -A)$$

such that

$$\hat{\chi}_\pm(B) = \Gamma(\chi_{\pm,\mathcal{Z}})B\Gamma(\chi_{\pm,\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}(m, A).$$

19.3.11 Inversion of the sign in front of the mass

Suppose that η is a unitary operator on $\mathcal{V}$ such that $\eta\gamma^\mu = -\gamma^\mu\eta$. In the case of an irreducible representation, such an operator exists only if d is odd. It is then proportional to ω, whose definition (15.10) we recall:

$$\omega := \gamma^0\gamma^1\cdots\gamma^d.$$

If d is even, such operators exist only in reducible representations. Note that

$$\eta : \mathcal{Y}(\mathcal{O}, m, A) \to \mathcal{Y}(\mathcal{O}, -m, A),$$

$$\hat{\eta} : \mathfrak{A}(\mathcal{O}, m, A) \to \mathfrak{A}(\mathcal{O}, -m, A).$$

If m, A do not depend on time, we can introduce the unitary operator

$$\eta_{\mathcal{Z}} : \mathcal{Z}(m, A) \to \mathcal{Z}(-m, A)$$

such that

$$\hat{\eta}(B) = \Gamma(\eta_{\mathcal{Z}}) B \Gamma(\eta_{\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}(m, A).$$

19.3.12 Covariance under the Poincaré group

Recall that in the case of spinors the role of the Poincaré group is played by its double cover, $APin(\mathbb{R}^{1,d})$. We first describe its representations on spinor-valued functions that may seem the most natural from the mathematical point of view. It is not, however, the standard choice in quantum field theory. Then we describe another representation, which is preferred in standard textbooks.

Recall that in Subsect. 19.2.10 for $\Lambda = (a, L) \in AO(\mathbb{R}^{1,d})$ we defined an operator u_Λ acting on $C^\infty(\mathbb{R}^{1,d})$. Let us define the analog of u_Λ for spinor-valued functions.

The group $Pin(\mathbb{R}^{1,d})$ will be treated as a subgroup of $L(\mathcal{V})$ by the Pin representation. Let $\tilde{\Lambda} = (a, \tilde{L}) \in APin(\mathbb{R}^{1,d})$. Let $\Lambda = (a, L)$ be the corresponding element of the Poincaré group $AO(\mathbb{R}^{1,d})$. $\tilde{\Lambda}$ acts on $\zeta \in C^\infty(\mathbb{R}^{1,d}, \mathcal{V})$ by

$$u_{\tilde{\Lambda}} \zeta(x) := \tilde{L} \zeta(\Lambda^{-1} x),$$

or $u_{\tilde{\Lambda}} \zeta = \tilde{L} \zeta \circ \Lambda^{-1}$. We obtain the unitary map

$$u_{\tilde{\Lambda}} : \mathcal{Y}(\mathcal{O}, m, A) \to \mathcal{Y}(\Lambda \mathcal{O}, \det L \, u_\Lambda m, u_\Lambda A). \tag{19.31}$$

Actually, the standard choice for the action of the Poincaré group is different. It involves additionally the charge reversal operator χ_+ and the operator η that inverts the sign of the mass. We set

$$w_{\tilde{\Lambda}} := \begin{cases} u_{\tilde{\Lambda}}, & \text{for even orthochronous } \Lambda, \\ \eta u_{\tilde{\Lambda}}, & \text{for odd orthochronous } \Lambda, \\ \chi_+ u_{\tilde{\Lambda}}, & \text{for even anti-orthochronous } \Lambda, \\ \eta \chi_+ u_{\tilde{\Lambda}}, & \text{for odd anti-orthochronous } \Lambda. \end{cases}$$

We obtain the transformations

$$w_{\tilde{\Lambda}} : \mathcal{Y}(\mathcal{O}, m, A) \to \mathcal{Y}(\Lambda \mathcal{O}, u_\Lambda m, \rho_\Lambda u_\Lambda A), \tag{19.32}$$

$$\hat{w}_{\tilde{\Lambda}} : \mathfrak{A}(\mathcal{O}, m, A) \to \mathfrak{A}(\Lambda \mathcal{O}, u_\Lambda m, \rho_\Lambda u_\Lambda A). \tag{19.33}$$

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(Recall that ρ_Λ denotes the temporal parity of Λ.) (19.32) is unitary and (19.33) is a linear $*$-homomorphism for orthochronous Λ. (19.32) is anti-unitary and (19.33) is an anti-linear $*$-homomorphism for anti-orthochronous Λ. In irreducible representations the operator χ_+ exists only if $d \neq 0$ (mod 4), and the operator η exists only if d is odd (one can then choose $\eta = \omega$). Thus in irreducible representations the definition of $w_{\tilde{\Lambda}}$ is possible if $d \equiv 1, 3$ (mod 4), which includes the physical case $d = 3$.

Assume now that $m(x)$ and $A(x)$ do not depend on time, so that one can consider the Fock quantization. Consider an element $\Lambda = (a, L)$ such that $u_\Lambda m$ and $u_\Lambda A$ do not depend on time as well. As before, this is the case if Λ or $\mathrm{T}\Lambda$ belong to $\mathbb{R}^{1,d} \rtimes O(\mathbb{R}^d)$. Note that $O_\pm(\mathbb{R}^d) \subset O_\pm^\uparrow(\mathbb{R}^{1,d})$ and $\mathrm{T}O_\pm(\mathbb{R}^d) \subset O_\mp^\downarrow(\mathbb{R}^{1,d})$, where T is the time reversal. From this we deduce the existence of operators

$$w_{\Lambda,\mathcal{Z}} : \mathcal{Z}(m, V, \mathrm{A}) \to \mathcal{Z}(m \circ \Lambda^{-1}, V \circ \Lambda^{-1}, \rho_\Lambda L\mathrm{A} \circ \Lambda^{-1})$$

satisfying

$$\hat{w}_\Lambda(A) = \Gamma(w_{\Lambda,\mathcal{Z}})A\Gamma(w_{\Lambda,\mathcal{Z}})^{-1}.$$

The operator $w_{\Lambda,\mathcal{Z}}$ is unitary for orthochronous Λ; otherwise it is anti-unitary.

19.3.13 Parity reversal

Consider the parity reversal operator $\mathrm{P} := \mathrm{diag}(1, -1, \dots, -1)$. This operator is even if d is even and odd if d is odd. Let $\tilde{\mathrm{P}}$ be an element of $Pin(\mathbb{R}^{1,d})$ or $Pin(\mathbb{R}^{d,1})$ covering P (which equals $\pm\gamma_1 \cdots \gamma_d$).

The standard operator implementing the parity reversal (which works in any dimension) is $\pi := w_{\tilde{\mathrm{P}}} = \eta u_{\tilde{\mathrm{P}}}$. We have

$$\begin{aligned} \pi &: \mathcal{Y}(\mathcal{O}, m, V, \mathrm{A}) \to \mathcal{Y}(\mathrm{P}\mathcal{O}, m \circ \mathrm{P}, V \circ \mathrm{P}, -\mathrm{A} \circ \mathrm{P}), \\ \hat{\pi} &: \mathfrak{A}(\mathcal{O}, m, V, \mathrm{A}) \to \mathfrak{A}(\mathrm{P}\mathcal{O}, m \circ \mathrm{P}, V \circ \mathrm{P}, -\mathrm{A} \circ \mathrm{P}). \end{aligned}$$

Therefore, if $V \circ \mathrm{P} = V$, $\mathrm{A} \circ \mathrm{P} = -\mathrm{A}$ and $m \circ \mathrm{P} = m$, then the system is invariant w.r.t. π. Note that $\pi^2 = -\mathbb{1}$.

If m, A do not depend on time, then we can introduce the unitary operator

$$\pi_{\mathcal{Z}} : \mathcal{Z}(m, V, \mathrm{A}) \to \mathcal{Z}(m \circ \mathrm{P}, V \circ \mathrm{P}, -\mathrm{A} \circ \mathrm{P})$$

such that

$$\hat{\pi}(B) = \Gamma(\pi_{\mathcal{Z}})B\Gamma(\pi_{\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}(m, A).$$

19.3.14 Time reversal

Recall that $\mathrm{T} := \mathrm{diag}(-1, 1, \dots, 1)$ denotes the time reversal operator. It is an odd anti-orthochronous operator. Let $\tilde{\mathrm{T}}$ be an element of $\tilde{L}$ covering T (which equals $\pm\gamma_0$).

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The operator u_{T} is the so-called *Racah time reversal* (see Subsect. 18.2.4). The *Wigner time reversal*, which is the standard choice, is $\tau := w_T = \eta\chi_+ u_T$. We have

$$\tau : \mathcal{Y}(\mathcal{O}, m, V, \mathrm{A}) \to \mathcal{Y}(\mathrm{T}\mathcal{O}, m \circ \mathrm{T}, V \circ \mathrm{T}, -\mathrm{A} \circ \mathrm{T}), \tag{19.34}$$

$$\hat{\tau} : \mathfrak{A}(\mathcal{O}, m, V, \mathrm{A}) \to \mathfrak{A}(\mathrm{T}\mathcal{O}, m \circ \mathrm{T}, V \circ \mathrm{T}, -\mathrm{A} \circ \mathrm{T}). \tag{19.35}$$

(19.34) is anti-unitary and (19.35) is an anti-linear $*$-homomorphism.

If m, A do not depend on time, then we can introduce the anti-unitary operator

$$\tau_{\mathcal{Z}} : \mathcal{Z}(m, V, \mathrm{A}) \to \mathcal{Z}(m, V, -\mathrm{A})$$

such that

$$\hat{\tau}(B) = \Gamma(\tau_{\mathcal{Z}}) B \Gamma(\tau_{\mathcal{Z}})^{-1}, \quad B \in \mathfrak{A}(m, V, \mathrm{A}).$$

Note the identity

$$\pi\chi_+\tau = \pm u_{\mathrm{PT}}.$$

We thus obtain the unitary operator

$$\pm u_{\mathrm{PT}} : \mathcal{Y}(m, A) \to \mathcal{Y}(m \circ \mathrm{PT}, -A \circ \mathrm{PT}),$$

which is an important ingredient of the famous PCT theorem.

19.3.15 Dirac equation in the momentum representation

Let us assume that m is constant and $A = 0$. The Dirac equation can be written as

$$(\gamma^\mu \partial_\mu - m)\zeta = 0. \tag{19.36}$$

If ζ is a solution, then

$$(-\Box + m^2)\zeta = 0.$$

Therefore,

$$\widehat{\zeta}(\tau, \mathrm{k}) = (2\pi)^{\frac{1}{2}} \left(f^+(\mathrm{k}) \frac{\delta\big(\tau - \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} + f^-(\mathrm{k}) \frac{\delta\big(\tau + \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} \right)$$

defines invariantly the functions $f^\mp$ on the lower and upper hyperboloid of mass m.

If we introduce $\vartheta_t(\mathrm{x}) = \zeta(t, \mathrm{x})$, then (19.36) can be rewritten in the Hamiltonian form

$$\partial_t \vartheta_t = \mathrm{i} b \vartheta_t, \quad \text{for} \ \ b := -\alpha \cdot D + \beta m. \tag{19.37}$$

b is self-adjoint on $L^2(\mathbb{R}^d, \mathcal{V})$.

Proposition 19.35 *Let ζ_i, $i = 1, 2$, be two solutions of (19.36). Then we have*

$$\begin{aligned}&\overline{\zeta}_1 \cdot \zeta_2 \\ &\quad = \frac{1}{2m} \int \Big(\overline{f_1^+(\mathrm{k})} \cdot \beta f_2^+(\mathrm{k}) \frac{\delta\big(\tau - \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} - \overline{f_1^-(\mathrm{k})} \cdot \beta f_2^-(\mathrm{k}) \frac{\delta\big(\tau + \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} \Big) \mathrm{d}\tau \mathrm{d}\mathrm{k},\end{aligned}$$

$$\widehat{\mathrm{sgn}(b)}\zeta(\tau, \mathrm{k}) = (2\pi)^{\frac{1}{2}} \Big(f^+(\mathrm{k}) \frac{\delta\big(\tau - \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} - f^-(\mathrm{k}) \frac{\delta\big(\tau + \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} \Big).$$

Proof Set $b(\mathrm{k}) = \gamma^0 \gamma \cdot \mathrm{k} - \mathrm{i} m \gamma^0$, so that $\widehat{b\vartheta}(\mathrm{k}) = b(\mathrm{k})\hat{\vartheta}(\mathrm{k})$. Note that $b(\mathrm{k})$ is self-adjoint on $\big(\mathcal{V}, (\cdot|\cdot)\big)$ and $b(\mathrm{k})^2 = \epsilon^2(\mathrm{k})$. Hence,

$$\mathbb{1} = P_+(\mathrm{k}) + P_-(\mathrm{k})$$

for $P_\pm(\mathrm{k}) = \mathbb{1}_{\{\mp\epsilon(\mathrm{k})\}}(b(\mathrm{k}))$.

If ϑ_t is a solution of (19.36) with the initial condition ϑ, then

$$\hat{\vartheta}_t(\mathrm{k}) = \mathrm{e}^{\mathrm{i}tb(\mathrm{k})} \hat{\vartheta}(\mathrm{k}).$$

Taking the Fourier transform we obtain

$$\hat{\zeta}(\tau, \mathrm{k}) = (2\pi)^{\frac{1}{2}} \Big(\delta\big(\tau - \epsilon(\mathrm{k})\big) P_+(\mathrm{k}) + \delta\big(\tau + \epsilon(\mathrm{k})\big) P_-(\mathrm{k}) \Big) \hat{\vartheta}(\mathrm{k}).$$

Therefore,

$$f^\pm(\mathrm{k}) = 2\epsilon(\mathrm{k}) P_\pm(\mathrm{k}) \hat{\vartheta}(\mathrm{k}). \tag{19.38}$$

We clearly have $\gamma^0 b(\mathrm{k}) + b(\mathrm{k})\gamma^0 = 2\mathrm{i}m$, which implies that

$$P_\pm(\mathrm{k}) = \pm \mathrm{i} m^{-1} \epsilon(\mathrm{k}) P_\pm(\mathrm{k}) \gamma^0 P_\pm(\mathrm{k}). \tag{19.39}$$

Recall that $\beta = \mathrm{i}\gamma^0$. Hence, setting $\vartheta^\pm(\mathrm{k}) := P^\pm(\mathrm{k})\vartheta(\mathrm{k})$,

$$\begin{aligned}&\frac{\mathrm{i}}{2m} \int \overline{f_1^+(\mathrm{k})} \cdot \gamma^0 f_2^+(\mathrm{k}) \frac{\delta\big(\tau - \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} \mathrm{d}\tau \mathrm{d}\mathrm{k} - \frac{\mathrm{i}}{2m} \overline{f_1^-(\mathrm{k})} \cdot \gamma^0 f_2^-(\mathrm{k}) \frac{\delta\big(\tau + \epsilon(\mathrm{k})\big)}{2\epsilon(\mathrm{k})} \mathrm{d}\tau \mathrm{d}\mathrm{k} \\ &\quad = \frac{\mathrm{i}}{2m} \int_{\mathbb{R}^d} 2\epsilon(\mathrm{k}) \overline{\hat{\vartheta}_1^+(\mathrm{k})} \cdot \gamma^0 \hat{\vartheta}_2^+(\mathrm{k}) \mathrm{d}\mathrm{k} - \frac{\mathrm{i}}{2m} \int_{\mathbb{R}^d} 2\epsilon(\mathrm{k}) \overline{\hat{\vartheta}_1^-(\mathrm{k})} \cdot \gamma^0 \hat{\vartheta}_2^-(\mathrm{k}) \mathrm{d}\mathrm{k} \\ &\quad = \int_{\mathbb{R}^d} \overline{\hat{\vartheta}_1^+(\mathrm{k})} \cdot \hat{\vartheta}_2^+(\mathrm{k}) \mathrm{d}\mathrm{k} + \int_{\mathbb{R}^d} \overline{\hat{\vartheta}_1^-(\mathrm{k})} \cdot \hat{\vartheta}_2^-(\mathrm{k}) \mathrm{d}\mathrm{k} \\ &\quad = \int_{\mathbb{R}^d} \overline{\hat{\vartheta}_1(\mathrm{k})} \cdot \hat{\vartheta}_2(\mathrm{k}) \mathrm{d}\mathrm{k} \ = \ \int_{\mathbb{R}^d} \overline{\vartheta_1(\mathrm{x})} \cdot \vartheta_2(\mathrm{x}) \mathrm{d}\mathrm{x} \ = \ \overline{\zeta}_1 \cdot \zeta_2.\end{aligned}$$

□

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19.4 Partial differential equations on manifolds

In this section we introduce basic notation and terminology for the analysis of partial differential equations on manifolds. We will be mostly interested in Lorentzian manifolds, especially the so-called globally hyperbolic manifolds, which serve as models for curved space-times. The material introduced in this section will be needed in Sects. 19.5 and 19.6, where we describe quantization of the algebraic Klein–Gordon and Dirac equations on curved space-times.

19.4.1 Manifolds

Let $\mathcal{X}$ be a manifold of dimension d. We will denote by $C_{\mathrm{c}}^{\infty}(\mathcal{X})$ the space of compactly supported smooth functions on $\mathcal{X}$ and by $\mathcal{D}'(\mathcal{X})$ its dual space – the space of distributions on $\mathcal{X}$.

$\mathrm{T}\mathcal{X}$, resp. $\mathrm{T}^{\#}\mathcal{X}$ denote the *tangent*, resp. *cotangent bundle over* $\mathcal{X}$ with fibers $\mathrm{T}_x\mathcal{X}$, resp. $\mathrm{T}_x^{\#}\mathcal{X}$ equal to the *tangent*, resp. *cotangent space to* $\mathcal{X}$ *at* $x \in \mathcal{X}$. Smooth sections of $\mathrm{T}\mathcal{X}$, resp. $\mathrm{T}^{\#}\mathcal{X}$ are called *vector fields*, resp. *differential* 1-*forms on* $\mathcal{X}$.

Suppose that (an open subset of) $\mathcal{X}$ is parametrized by local coordinates $x = (x^1, \dots, x^d)$ from (an open subset of) $\mathbb{R}^d$. Then we have a natural local frame in $\mathrm{T}\mathcal{X}$, traditionally denoted $(\partial_{x^1}, \dots, \partial_{x^d})$. Its dual frame is denoted $(\mathrm{d}x^1, \dots, \mathrm{d}x^d)$. We will use the coordinate-dependent notation, tacitly identifying $\mathcal{X}$ (or its open subset) with $\mathbb{R}^d$ (or its open subset). We will use the Einstein summation convention.

By a *(parametrized) curve* in $\mathcal{X}$ we will mean a continuous piecewise C^1 map from an interval in $\mathbb{R}$ into $\mathcal{X}$. A curve is called *inextensible* if none of its piecewise C^1 reparametrizations can be continuously extended beyond its endpoints.

Let $\mathcal{V}$ be a finite-dimensional vector space. In the rest of the section we will discuss differential operators acting on $C^{\infty}(\mathcal{X}, \mathcal{V})$ – the space of smooth functions $\mathcal{X} \to \mathcal{V}$. Of course, our discussion can be easily generalized to differential operators on smooth sections of a *vector bundle* $(E, \mathcal{X})$ with base $\mathcal{X}$ and fibers isomorphic to $\mathcal{V}$. Using local trivializations of E, one can locally reduce the analysis to the trivial bundle $\mathcal{X} \times \mathcal{V}$ considered here. All the objects introduced below have natural definitions covariant under change of coordinates and of local frames, which a reader with a little familiarity with vector bundles can easily guess.

19.4.2 Integration on pseudo-Riemannian manifolds

A manifold $\mathcal{X}$ is called *pseudo-Riemannian* if it is equipped with a smooth pseudo-Euclidean form, called the *metric tensor* $\mathcal{X} \ni x \mapsto g(x) = [g_{\mu\nu}(x)] \in \otimes_{\mathrm{s}}^2 \mathrm{T}_x^{\#}\mathcal{X}$. It equips $\mathrm{T}_x\mathcal{X}$ with a scalar product, so that $g_{\mu\nu} = (\partial_{x^\mu} | \partial_{x^\nu})$. We will set $|g|(x) := |\det[g_{\mu\nu}(x)]|$.

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The inverse of $[g_{\mu\nu}(x)]$ will be denoted by $[g^{\mu\nu}(x)]$. Clearly, it induces the dual scalar product in $\mathrm{T}_x^\# \mathcal{X}$: $(\mathrm{d}x^\mu|\mathrm{d}x^\nu) = g^{\mu\nu}$.

Let $\mathrm{d}x$ denote the Lebesgue measure on $\mathbb{R}^d$ transported by a local chart to the manifold $\mathcal{X}$. $\mathrm{d}v$ will denote the measure $|g|^{\frac{1}{2}}\mathrm{d}x$ on $\mathcal{X}$. It does not depend on the coordinates. Thus if f is a function on $\mathcal{X}$, its integral over $\mathcal{X}$ is denoted $\int f \mathrm{d}v$. We equip $C_\mathrm{c}(\mathcal{X})$ with the scalar product

$$(f|g) := \int \overline{f} g \mathrm{d}v. \tag{19.40}$$

Let $\mathcal{S}$ be a smooth *hypersurface of* $\mathcal{X}$ (that is, a sub-manifold of co-dimension 1). We can find local coordinates such that

$$\mathcal{S} = \{(x^1, \dots, x^d) \ : \ x^d = 0\}. \tag{19.41}$$

We define a measure on $\mathcal{S}$ by $\mathrm{d}s := |h|^{\frac{1}{2}}\mathrm{d}x^1 \dots \mathrm{d}x^{d-1}$, where $|h| = \det[h_{ij}]$ and $[h_{ij}]$ is obtained from $[g_{ij}]$ by discarding the last column and the last line (so that $i, j = 1, \dots, d-1$). The measure $\mathrm{d}s$ does not depend on the coordinates. If f is a smooth function on $\mathcal{S}$, then we write $\int_\mathcal{S} f \mathrm{d}s$ for its integral over $\mathcal{S}$.

We say that $\mathcal{S}$ has an *external orientation* if for any $x \in \mathcal{S}$ a unit normal vector has been chosen, which depends continuously on x. A hypersurface given by (19.41) has a natural external orientation: in the direction of the coordinate x^d. The co-vector $|g^{dd}|^{-\frac{1}{2}}\mathrm{d}x^d$ restricted to $\mathcal{S}$ is called the *normal co-vector*. It will be denoted n_μ. In the coordinates that we use we have

$$n_\mu = \begin{cases} 0, & \mu = 1, \dots, d-1, \\ |g^{dd}|^{-\frac{1}{2}}, & \mu = d. \end{cases}$$

Note that $|g| = |h||g^{dd}|^{-1}$.

If $x \mapsto [f^\mu(x)]$ is a vector field, then we define its *flux across* $\mathcal{S}$ as

$$\int f^\mu n_\mu \mathrm{d}s. \tag{19.42}$$

Again, (19.42) does not depend on coordinates. To shorten the notation, we will write $\mathrm{d}s_\mu$ instead of $n_\mu \mathrm{d}s$.

The *Stokes theorem* says that if Ω is an open subset of $\mathcal{X}$ with a sufficiently regular boundary $\partial\Omega$, then

$$\int_\Omega |g|^{-\frac{1}{2}} \nabla_\mu |g|^{\frac{1}{2}} f^\mu \mathrm{d}v = \int_{\partial\Omega} f^\mu \mathrm{d}s_\mu, \tag{19.43}$$

where $\nabla_\mu = \partial_{x^\mu}$ is the μth partial derivative.

19.4.3 Lorentzian manifolds

We will use the terminology for vectors in the Minkowski space introduced in Def. 19.1.

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Definition 19.36 *A pseudo-Riemannian manifold* $\mathcal{X}$ *is called* Lorentzian *if the signature of its metric tensor is* $(-1,1,\dots,1)$.

We say that $\mathcal{X}$ *is* time-orientable *if there exists a global continuous time-like vector field on* $\mathcal{X}$. *If* v *is such a vector field and* $x \in \mathcal{X}$, *a time-like vector* $v' \in T_x\mathcal{X}$ *is future, resp. past oriented if* $\pm v(x) \cdot g(x)v' > 0$. *The manifold* $\mathcal{X}$ *equipped with such a continuous choice of future/past directions is called* time-oriented.

In the remaining part of this subsection, $\mathcal{X}$ is a time-oriented Lorentzian manifold.

Definition 19.37 *A curve in* $\mathcal{X}$ *is called* time-like, causal, *resp.* light-like *if all its tangent vectors are such and all pairs of tangent vectors at break points are in the same causal cone. A curve in* $\mathcal{X}$ *is called* space-like *if all its tangent vectors are such.*

Definition 19.38 *Let* $x \in \mathcal{X}$. *The* causal, *resp.* time-like future, *resp.* past *of* x *is the set of all* $y \in \mathcal{X}$ *that can be reached from* x *by a causal, resp. time-like future-, resp. past-directed curve, and is denoted* $J^{\pm}(x)$, *resp.* $I^{\pm}(x)$. *For* $\mathcal{U} \subset \mathcal{X}$, *its* causal, *resp.* time-like future, *resp.* past *is defined as*

$$J^{\pm}(\mathcal{U}) = \bigcup_{x\in\mathcal{U}} J^{\pm}(x), \quad I^{\pm}(\mathcal{U}) = \bigcup_{x\in\mathcal{U}} I^{\pm}(x).$$

We define also the causal, *resp.* time-like shadow:

$$J(\mathcal{U}) = J^{+}(\mathcal{U}) \cup J^{-}(\mathcal{U}), \quad I(\mathcal{U}) = I^{+}(\mathcal{U}) \cup I^{-}(\mathcal{U}).$$

Definition 19.39 *A* Cauchy hypersurface *is a hypersurface* $\mathcal{S} \subset \mathcal{X}$ *such that each inextensible time-like curve intersects* $\mathcal{S}$ *at exactly one point.*

If $\mathcal{S}$ is a smooth space-like Cauchy hypersurface, it will always be equipped with the external orientation given by the future directed normal vector at each point of $\mathcal{S}$.

Let us quote the following result from the theory of Lorentzian manifolds:

Theorem 19.40 *Let* $\mathcal{X}$ *be a connected Lorentzian manifold. The following are equivalent:*

(1) *The following two conditions hold:*
  (1a) for any $x, y \in \mathcal{X}$, $J^{+}(x) \cap J^{-}(y)$ *is compact,*
  (1b) (causality condition) *there are no closed causal curves.*
(2) *There exists a Cauchy hypersurface.*
(3) $\mathcal{X}$ *is isometric to* $\mathbb{R} \times \mathcal{S}$ *with metric* $-\beta \mathrm{d}t^2 + g_t$, *where* β *is a smooth positive function,* g_t *is a Riemannian metric on* $\mathcal{S}$ *depending smoothly on* $t \in \mathbb{R}$, *and each* $\{t\} \times \mathcal{S}$ *is a smooth space-like Cauchy hypersurface in* $\mathcal{X}$.

The above theorem is quoted from Bär–Ginoux–Pfäffle (2007) except for (1b), where in this reference the so-called strong causality condition is given. The fact that the strong causality condition can be replaced by the causality condition is a recent result of Bernal–Sanchez (2007).

Definition 19.41 *A connected Lorentzian manifold satisfying the equivalent conditions of the above theorem is called* globally hyperbolic.

We recall that a Riemannian manifold $(\mathcal{S}, h)$ is *geodesically complete* if all its geodesics can be infinitely extended. By the Hopf–Rinow theorem, this condition is equivalent to the condition that $\mathcal{S}$, equipped with the Riemannian distance, is complete as a metric space; see Sakai (1996).

Example 19.42 *Let $(\mathcal{S}, h)$ be a Riemannian manifold and $I \subset \mathbb{R}$ an open interval. Let $f : I \to]0, \infty[$ be a smooth function. Then $\mathcal{X} = I \times \mathcal{S}$ with the metric $-\mathrm{d}t^2 + f(t)^2 h$ is globally hyperbolic iff $(\mathcal{S}, h)$ is geodesically complete.*

Example 19.43 *Let $(\mathcal{S}, h)$ be a Riemannian manifold and $\tau : \mathcal{S} \to]0, +\infty[$ a smooth function, $v \in T^{\#}\mathcal{S}$ a smooth 1-form. Then $\mathcal{X} = \mathbb{R} \times \mathcal{S}$, equipped with the Lorentzian metric $-\tau(\mathrm{x})\big(\mathrm{d}t - v_j(\mathrm{x})\mathrm{d}\mathrm{x}^j\big)^2 + h_{jk}(\mathrm{x})\mathrm{d}\mathrm{x}^j\mathrm{d}\mathrm{x}^k$, is called a* stationary *space-time. If $v = 0$ it is called* static. *It is called* uniformly static *if there exists $c > 0$ such that $c \le \tau(\mathrm{x}) \le c^{-1}$. A uniformly static space-time is globally hyperbolic iff $(\mathcal{S}, h)$ is geodesically complete; see Fulling (1989) .*

It is straightforward to generalize the notion of space-compact functions from the Minkowski space (see Def. 19.2) to a Lorentzian manifold.

Definition 19.44 *A function $f \in C(\mathcal{X})$ is called* space-compact *iff there exists a compact $K \subset \mathcal{X}$ such that* $\operatorname{supp} f \subset J(K)$. *It is called* future, *resp.* past space-compact *iff there exists a compact $K \subset \mathcal{X}$ such that* $\operatorname{supp} f \subset J^{\pm}(K)$.

The set of smooth space-compact functions will be denoted $C^\infty_{\mathrm{sc}}(\mathcal{X})$. The set of smooth future, resp. past space-compact functions will be denoted $C^\infty_{\pm\mathrm{sc}}(\mathcal{X})$.

Finally, let us give the definition of the causal dependence.

Definition 19.45 *Let $\mathcal{X}$ be globally hyperbolic and $\mathcal{O} \subset \mathcal{X}$. We say that $x \in \mathcal{X}$ is* causally dependent on $\mathcal{O}$ *if there exists a neighborhood $\mathcal{U}$ of x and a smooth Cauchy surface $\mathcal{S}$ such that every causal curve starting from $\mathcal{U}$ intersects $\mathcal{S}$ in $\mathcal{O}$.*

If $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{X}$ we say that $\mathcal{O}_1$ is causally dependent on $\mathcal{O}_2$, *if every $x \in \mathcal{O}_1$ is causally dependent on $\mathcal{O}_2$.*

19.4.4 First-order partial differential equations

We assume that the manifold $\mathcal{X}$ is equipped with a measure, which in local coordinates equals $|g|^{\frac{1}{2}}(x)\mathrm{d}x$. (In this subsection, $|g|^{\frac{1}{2}}$ does not have to come from a metric tensor.)

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Let $\mathcal{V}$ be a finite-dimensional vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. To simplify notation we will assume that $\mathbb{K} = \mathbb{C}$. The formulas in the real case are obtained from the complex case by dropping the bars and replacing the Hermitian conjugation * with the transposition $^\#$.

In this subsection we study first-order differential equations for functions with values in $\mathcal{V}$. With a large class of such equations we will associate a locally defined vector-valued sesquilinear form J^μ whose divergence vanishes. In physics J^μ is often called a *conserved current*.

This construction has a special importance for hyperbolic equations on hyperbolic manifolds, where it leads to an invariantly defined bilinear or sesquilinear form on the space of solutions. In the case of the Klein–Gordon equation (which, of course, can be reduced to a first-order equation) this form is symplectic for $\mathbb{K} = \mathbb{R}$ and charged symplectic for $\mathbb{K} = \mathbb{C}$. The Klein–Gordon equation on curved space-times will be considered in Sect. 19.5. In the case of the generalized Dirac equation, which on a curved space-time we will consider in Sect. 19.6, this form is a positive definite scalar product. In both cases these forms play a fundamental role in the quantization of the classical equation.

In many textbooks on quantum field theory the conserved current J^μ is derived from the Lagrangian by the Noether theorem using the invariance w.r.t. the charge symmetry $\zeta \mapsto \mathrm{e}^{\mathrm{i}\theta}\zeta$, $\theta \in U(1)$. In the derivation that we give, complex numbers do not enter at all.

Let us consider a first-order linear equation for $\zeta \in C^\infty(\mathcal{X}, \mathcal{V})$. Every such equation can be written in the form

$$\alpha^\mu(x)\nabla_\mu\zeta(x) + \frac{1}{2}\big(|g|^{-\frac{1}{2}}(x)\nabla_\mu|g|^{\frac{1}{2}}(x)\alpha^\mu(x)\big)\zeta(x) + \theta(x)\zeta(x) = 0, \tag{19.44}$$

where $\mathcal{X} \ni x \mapsto \alpha^\mu(x), \theta(x) \in L(\mathcal{V}, \mathcal{V}^*)$, and μ enumerates the coordinates of $\mathcal{X}$. The following theorem describes conditions that guarantee the existence of a conserved current for (19.44).

Theorem 19.46 *Suppose that either Condition (19.45) or Condition (19.46) is satisfied:*

$$\alpha^\mu(x)^* = \alpha^\mu(x), \ \ \theta(x)^* = -\theta(x), \tag{19.45}$$

$$\text{or} \quad \alpha^\mu(x)^* = -\alpha^\mu(x), \ \ \theta(x)^* = \theta(x). \tag{19.46}$$

Define

$$J^\mu(\zeta_1, \zeta_2, x) := \overline{\zeta_1(x)}\cdot\alpha^\mu(x)\zeta_2(x).$$

Let ζ_1, ζ_2 be solutions of (19.44). Then

$$\nabla_\mu|g|^{\frac{1}{2}}(x)J^\mu(\zeta_1, \zeta_2, x) = 0. \tag{19.47}$$

Proof

$$\begin{aligned}|g|^{-\frac{1}{2}}\nabla_\mu|g|^{\frac{1}{2}}J^\mu = &\overline{\Big(\alpha^{\mu*}\nabla_\mu + \frac{1}{2}\big(|g|^{-\frac{1}{2}}\nabla_\mu|g|^{\frac{1}{2}}\alpha^\mu\big)^* - \theta^*\Big)\zeta_1} \cdot \zeta_2 \\ &+\overline{\zeta}_1 \cdot \Big(\alpha^\mu\nabla_\mu + \frac{1}{2}\big(|g|^{-\frac{1}{2}}\nabla_\mu|g|^{\frac{1}{2}}\alpha^\mu\big) + \theta\Big)\zeta_2 = 0.\end{aligned}$$

□

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Note that in the complex case one can pass from Condition (19.45) to (19.46) by multiplying α^μ and θ by i.

19.5 Generalized Klein–Gordon equation on curved space-time

Throughout this section we assume that $\mathcal{X} \ni x \mapsto g(x) = [g_{\mu\nu}(x)]$ is the metric tensor on a pseudo-Riemannian manifold $\mathcal{X}$. In the first two subsections only we allow $\mathcal{X}$ to be an arbitrary pseudo-Riemannian manifold. Starting with Subsect. 19.5.3, we will assume that $\mathcal{X}$ is globally hyperbolic.

$\mathcal{V}$ will be a real or complex finite-dimensional vector space. For simplicity, most formulas will be given for the complex case. We will consider a vector bundle with a base $\mathcal{X}$ and fiber $\mathcal{V}$. For simplicity, we will always assume that the bundle is trivial and trivialized to $\mathcal{X} \times \mathcal{V}$.

In this section we describe algebraic quantization of a large class of second-order equations on $\mathcal{X}$ with values in $\mathcal{V}$. We will always assume that the principal term of these equations is given by the metric tensor. In the case of Lorentzian manifolds, such equations will be called *generalized Klein–Gordon equations*. Solutions of these equations propagate causally.

Generalized Klein–Gordon equations possess a conserved current, which is symplectic in the real case and charged symplectic in the complex case. Therefore, it is natural to quantize these equations using the CCR. The bosonic algebraic quantization leads to a net of algebras satisfying the Einstein causality.

This section is to a large extent a generalization of Sect. 19.2 to a curved space-time. Unlike in Sect. 19.2, we limit our discussion to the algebraic quantization. We do not discuss the positive energy quantization on a bosonic Fock space, which is possible for the Klein–Gordon equation on a stationary space-time; see Kay (1978).

19.5.1 Klein–Gordon operators

Let $\mathcal{X} \ni x \mapsto \Gamma_\mu(x) \in L(\mathcal{V})$, $\mu = 0, 1, \dots, d$, be smooth functions.

Definition 19.47

$$\nabla_\mu^\Gamma \zeta := (\nabla_\mu + \Gamma_\mu)\zeta, \quad \zeta \in C^\infty(\mathcal{X}, \mathcal{V}),$$

is called the covariant derivative of ζ with the connection Γ.

Let $\mathcal{X} \ni x \mapsto \rho(x) \in L(\mathcal{V})$ be a smooth function.

Definition 19.48 *The operator*

$$\Box = \Box(\Gamma, \rho) := -|g|^{-\frac{1}{2}} \nabla_\mu^\Gamma |g|^{\frac{1}{2}} g^{\mu\nu} \nabla_\nu^\Gamma + \rho, \tag{19.48}$$

acting on $C^\infty(\mathcal{X}, \mathcal{V})$, will be called the generalized Klein–Gordon operator with the connection Γ and the mass-squared term ρ.

We will study the equation

$$\Box\zeta = 0. \tag{19.49}$$

Remark 19.49 *The name* Klein–Gordon equation *is usually associated with the Lorentz signature. Its analog for the Euclidean signature has the traditional name of the* Helmholtz equation.

Let us fix a smooth map $\mathcal{X} \ni x \mapsto \lambda(x) \in L(\mathcal{V}, \mathcal{V}^*)$ with values in positive definite forms on $\mathcal{V}$. One can then introduce a scalar product on $C^\infty(\mathcal{X}, \mathcal{V})$ by

$$\overline{\zeta}_1 \cdot \zeta_2 := \int_{\mathcal{X}} \overline{\zeta_1(x)} \cdot \lambda(x) \zeta_2(x) \mathrm{d}v(x). \tag{19.50}$$

Remark 19.50 *In the literature, a complex vector bundle $\mathcal{X} \times \mathcal{V} \to \mathcal{X}$ equipped with a scalar product $\lambda(x)$ is often called a* Hermitian bundle*, which is not quite correct, since it is equipped with a* positive definite *Hermitian form. A connection satisfying (19.51) is also called* Hermitian.

We will assume the following conditions:

$$\nabla_\mu \lambda = \Gamma_\mu^* \lambda + \lambda \Gamma_\mu, \qquad \lambda\rho = \rho^* \lambda. \tag{19.51}$$

If (19.51) holds, for any smooth open set $\Omega \subset \mathcal{X}$, *Green's formula* is valid:

$$\begin{aligned} &\int_\Omega \left(\overline{\zeta_1(x)} \cdot \lambda(x) \Box \zeta_2(x) - \overline{\Box \zeta_1(x)} \cdot \lambda(x) \zeta_2(x) \right) \mathrm{d}v(x) \qquad (19.52) \\ &= - \int_{\partial\Omega} g^{\mu,\nu}(x) \left(\overline{\zeta_1(x)} \cdot \lambda(x) \nabla_\mu^\Gamma \zeta_2(x) - \overline{\nabla_\mu^\Gamma \zeta_1(x)} \cdot \lambda(x) \zeta_2(x) \right) \mathrm{d}s_\nu(x). \end{aligned}$$

This formula follows easily from (19.43).

In the following theorem we describe a conserved current associated with equation (19.49):

Theorem 19.51 *Assume (19.51). Let ζ_1, ζ_2 be solutions of (19.49). Then*

$$\begin{aligned} &J^\mu(\zeta_1, \zeta_2, x) \\ &\quad := \overline{\zeta_1(x)} \cdot \lambda(x) g^{\mu\nu}(x) \nabla_\nu^\Gamma \zeta_2(x) - \overline{\nabla_\nu^\Gamma \zeta_1(x)} \cdot \lambda(x) g^{\mu\nu}(x) \zeta_2(x) \end{aligned}$$

satisfies

$$\nabla_\mu |g|^{\frac{1}{2}}(x) J^\mu(\zeta_1, \zeta_2, x) = 0. \tag{19.53}$$

Before we prove the above theorem let us remark that we can always assume that $\lambda(x)$ does not depend on x. Then we can drop $\lambda(x)$ from our notation altogether and replace the condition (19.51) with

$$\Gamma_\mu \text{ are anti-self-adjoint and } \rho \text{ is self-adjoint.} \tag{19.54}$$

In fact let us first fix a scalar product $(\cdot|\cdot)$ on $\mathcal{V}$ so that $\lambda(x) > 0$ becomes a self-adjoint operator on $\mathcal{V}$. We have

$$\nabla_\mu^\Gamma \zeta = \lambda^{-\frac{1}{2}} \nabla_\mu^{\tilde{\Gamma}} \lambda^{\frac{1}{2}} \zeta, \quad \text{for } \tilde{\Gamma}_\mu = \lambda^{\frac{1}{2}} \Gamma_\mu \lambda^{-\frac{1}{2}} - \left(\nabla_\mu \lambda^{\frac{1}{2}}\right) \lambda^{-\frac{1}{2}}.$$

From (19.51), we obtain that $\tilde{\Gamma}_\mu$ is anti-self-adjoint for $(\cdot|\cdot)$. In fact, we can rewrite (19.51) as

$$(\nabla_\mu \lambda^{\frac{1}{2}})\lambda^{\frac{1}{2}} + \lambda^{\frac{1}{2}} \nabla_\mu \lambda^{\frac{1}{2}} = \Gamma_\mu^* \lambda + \lambda \Gamma_\mu,$$

which gives $\tilde{\Gamma}_\mu^* = -\tilde{\Gamma}_\mu$. Moreover, $\tilde{\rho} = \lambda^{\frac{1}{2}} \rho \lambda^{-\frac{1}{2}}$ is self-adjoint for $(\cdot|\cdot)$. Considering the function $\tilde{\zeta} = \lambda^{\frac{1}{2}} \zeta$, we obtain (19.54).

Proof of Thm. 19.51. The theorem follows by direct computation. However, it is perhaps instructive to give a proof that reduces it to a special case of Thm. 19.46.

As remarked above, we can assume (19.54). Let us now introduce $\zeta'_\mu := \nabla_\mu^\Gamma \zeta$. The equation (19.49) yields

$$\left(g^{\mu\nu}\nabla_\mu + \frac{1}{2}|g|^{-\frac{1}{2}}\left(\nabla_\mu |g|^{\frac{1}{2}} g^{\mu\nu}\right)\right)\zeta'_\nu = \rho\zeta + \left(-|g|^{-\frac{1}{2}}\frac{1}{2}(\nabla_\mu |g|^{\frac{1}{2}} g^{\mu\nu}) - g^{\mu\nu}\Gamma_\mu\right)\zeta'_\nu,$$

$$-\left(g^{\mu\nu}\nabla_\nu + |g|^{-\frac{1}{2}}\frac{1}{2}\left(\nabla_\nu |g|^{\frac{1}{2}} g^{\mu\nu}\right)\right)\zeta = \left(-|g|^{-\frac{1}{2}}\frac{1}{2}(\nabla_\nu |g|^{\frac{1}{2}} g^{\mu\nu}) + g^{\mu\nu}\Gamma_\nu\right)\zeta - g^{\nu,\mu}\zeta'_\nu.$$

This can be rewritten as

$$\alpha^\mu \nabla_\mu \begin{bmatrix} \zeta \\ \zeta' \end{bmatrix} + \frac{1}{2}|g|^{-\frac{1}{2}}(\nabla_\mu |g|^{\frac{1}{2}} \alpha^\mu) \begin{bmatrix} \zeta \\ \zeta' \end{bmatrix} + \theta \begin{bmatrix} \zeta \\ \zeta' \end{bmatrix} = 0, \tag{19.55}$$

for

$$\theta = \begin{bmatrix} -\rho & |g|^{-\frac{1}{2}}\frac{1}{2}(\nabla_\mu |g|^{\frac{1}{2}} g^{\mu\nu}) + g^{\mu\nu}\Gamma_\mu \\ |g|^{-\frac{1}{2}}\frac{1}{2}(\nabla_\nu |g|^{\frac{1}{2}} g^{\mu\nu}) - g^{\mu\nu}\Gamma_\nu & g^{\mu\nu} \end{bmatrix},$$

$$\alpha^\mu = \begin{bmatrix} 0 & g^{\mu\nu} \\ -g^{\mu\nu} & 0 \end{bmatrix}.$$

Identifying $\mathcal{V}$ with $\mathcal{V}^*$ using the scalar product $(\cdot|\cdot)$, we obtain an equation of the form (19.44) with $\mathcal{V}$ replaced by $\mathcal{V} \oplus \mathcal{V}^d$. Clearly, $\theta^* = \theta$ and $\alpha^{\mu *} = -\alpha^\mu$. Thus (19.55) satisfies the condition (19.46). Hence,

$$J^\mu = \begin{bmatrix} \zeta_1 \\ \zeta'_1 \end{bmatrix} \alpha^\mu \begin{bmatrix} \zeta_2 \\ \zeta'_2 \end{bmatrix} = \overline{\zeta_1} g^{\mu\nu} \zeta'_{2\nu} - \overline{\zeta'_{1\nu}} g^{\mu\nu} \zeta_2$$

is a conserved current. □

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19.5.2 Lagrangian of the Klein–Gordon equation

The equation (19.49) can be obtained as an Euler–Lagrange equation, where the Lagrangian is

$$L(\zeta,\partial\zeta) := -\frac{1}{2}(\partial_\mu+\Gamma_\mu)\zeta\cdot g^{\mu\nu}|g|^{\frac{1}{2}}(\partial_\mu+\Gamma_\mu)\zeta - \frac{1}{2}\zeta\cdot|g|^{\frac{1}{2}}\rho\zeta, \text{in the real case,}$$

$$L(\zeta,\overline{\zeta},\partial\zeta,\partial\overline{\zeta}) := -\overline{(\partial_\mu+\Gamma_\mu)\zeta}\cdot g^{\mu\nu}|g|^{\frac{1}{2}}(\partial_\mu+\Gamma_\mu)\zeta - \overline{\zeta}\cdot|g|^{\frac{1}{2}}\rho\zeta, \text{in the complex case.}$$

(For simplicity, we have assumed that λ does not depend on x and that (19.54) holds.)

19.5.3 Green's functions of hyperbolic Klein–Gordon equations

From here until the end of the section we assume that $\mathcal{X}$ is globally hyperbolic.

The following theorem is a classic result from the theory of hyperbolic equations. Its proof can be found e.g. in Bär–Ginoux–Pfäffle (2007).

Theorem 19.52 *For any $f \in C_{\mathrm{c}}^\infty(\mathcal{X},\mathcal{V})$, there exist unique functions $\zeta^\pm \in C_{\pm\mathrm{sc}}^\infty(\mathcal{X},\mathcal{V})$ that solve*

$$\Box\zeta^\pm = f.$$

Moreover,

$$\zeta^\pm(x) = (G^\pm f)(x) := \int G^\pm(x,y)f(y)\mathrm{d}v(y),$$

where $G^\pm \in \mathcal{D}'(\mathcal{X}\times\mathcal{X}, L(\mathcal{V}))$ satisfy

$$\Box G^\pm = G^\pm\Box = \mathbb{1}, \quad \operatorname{supp} G^\pm \subset \big\{(x,y)\ :\ x \in J^\pm(y)\big\}.$$

If in addition (19.51) holds, then $G^{\pm *} = G^\mp$.

Note that by duality $G^\pm$ can be applied to distributions of compact support.

Definition 19.53 *G^+, resp. G^- is called the* retarded*, resp.* advanced Green's function*.*

$$G := G^+ - G^-$$

is called the Pauli–Jordan function*.*

In what follows, until the end of the section, we assume (19.51). Note that $G^* = -G$ or, in other words,

$$\overline{v_1}\cdot\lambda(x)G(x,y)v_2 = -\overline{G(y,x)v_1}\cdot\lambda(y)v_2,\ \ v_1,v_2\in\mathcal{V}.$$

19.5.4 Cauchy problem

Theorem 19.54 *Let $\mathcal{S}$ be a smooth Cauchy hypersurface. Let $\varsigma, \vartheta \in C_{\mathrm{c}}^\infty(\mathcal{S}, \mathcal{V})$. Then there exists a unique $\zeta \in C_{\mathrm{sc}}^\infty(\mathcal{X}, \mathcal{V})$ that solves*

$$\square\zeta = 0 \tag{19.56}$$

with initial conditions $\zeta\big|_{\mathcal{S}} = \varsigma$, $n^\mu \nabla_\mu^\Gamma \zeta\big|_{\mathcal{S}} = \vartheta$.

It satisfies $\operatorname{supp}\zeta \subset J(\operatorname{supp}\varsigma \cup \operatorname{supp}\vartheta)$ *and is given by*

$$\begin{aligned}\zeta(x) = &-\int_{\mathcal{S}} \big(\nabla_{y^\mu} G(x,y) - G(x,y)\Gamma_\mu(y)\big)\varsigma(y) g^{\mu\nu}(y)\mathrm{d}s_\nu(y)\\ &+\int_{\mathcal{S}} G(x,y)\vartheta(y)\mathrm{d}s(y).\end{aligned} \tag{19.57}$$

Proof The existence and uniqueness of solutions are well known. Let us prove (19.57). We apply Green's formula (19.52) to $\zeta_2 = \zeta$ and to $\zeta_1 = G^\mp f$, $f \in C_{\mathrm{c}}^\infty(\mathcal{X}, \mathcal{V})$, $\Omega = J^\pm(\mathcal{S})$. We obtain

$$\int_{J^+(\mathcal{S})} \overline{f} \cdot \lambda\zeta \mathrm{d}v = \int_{\mathcal{S}} g^{\mu\nu} \left(\overline{G^- f} \cdot \lambda \nabla_\mu^\Gamma \zeta - \overline{\nabla_\mu^\Gamma G^- f} \cdot \lambda\zeta\right) \mathrm{d}s_\nu, \tag{19.58}$$

$$\int_{J^-(\mathcal{S})} \overline{f} \cdot \lambda\zeta \mathrm{d}v = \int_{\mathcal{S}} g^{\mu\nu} \left(-\overline{G^+ f} \cdot \lambda \nabla_\mu^\Gamma \zeta + \overline{\nabla_\mu^\Gamma G^+ f} \cdot \lambda\zeta\right) \mathrm{d}s_\nu. \tag{19.59}$$

Adding (19.58) and (19.59) we get

$$\int_{\mathcal{X}} \overline{f} \cdot \lambda\zeta \mathrm{d}v = \int_{\mathcal{S}} g^{\mu\nu} \left(-\overline{Gf} \cdot \lambda \nabla_\mu^\Gamma \zeta + \overline{\nabla_\mu^\Gamma Gf} \cdot \lambda\zeta\right) \mathrm{d}s_\nu.$$

This can be rewritten as

$$\begin{aligned}&\int_{\mathcal{X}} \overline{f(x)} \cdot \lambda(x)\zeta(x)\mathrm{d}v(x)\\ &= -\int_{\mathcal{X}} \mathrm{d}v(x) \int_{\mathcal{S}} \overline{G(y,x)f(x)} \cdot \lambda(y)\nabla_\mu^\Gamma \zeta(y) g^{\mu\nu}(y)\mathrm{d}s_\nu(y)\\ &\quad+ \int_{\mathcal{X}} \mathrm{d}v(x) \int_{\mathcal{S}} \overline{\nabla_{y_\mu}^\Gamma G(y,x)f(x)} \cdot \lambda(y)\zeta(y) g^{\mu\nu}(y)\mathrm{d}s_\nu(y)\\ &= \int_{\mathcal{X}} \overline{f(x)} \cdot \lambda(x)\mathrm{d}v(x) \int_{\mathcal{S}} G(x,y)\nabla_\mu^\Gamma \zeta(y) g^{\mu\nu}(y)\mathrm{d}s_\nu(y)\\ &\quad- \int_{\mathcal{X}} \overline{f(x)} \cdot \lambda(x)\mathrm{d}v(x) \int_{\mathcal{S}} \left(\nabla_{y^\mu} G(x,y) - G(x,y)\Gamma_\mu(y)\right)\zeta(y) g^{\mu\nu}(y)\mathrm{d}s_\nu(y),\end{aligned}$$

where in the last line we use $G^* = -G$. Thus (19.57) is true. $\square$

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19.5.5 Symplectic space of solutions of the Klein–Gordon equation

Let $\mathcal{Y} = \mathcal{Y}(\Gamma, \rho)$ denote the set of solutions of (19.56) in $C^\infty_{\rm sc}(\mathcal{X}, \mathcal{V})$.

Theorem 19.55 *Let $\zeta_1, \zeta_2 \in \mathcal{Y}$. Define $J^\mu(\zeta_1, \zeta_2, x)$ as in Thm. 19.51. Then*

$$\overline{\zeta_1}\cdot\omega\zeta_2 := \int_{\mathcal{S}} J^\mu(\zeta_1, \zeta_2, x)\mathrm{d}s_\mu(x) \tag{19.60}$$

does not depend on the choice of a Cauchy hypersurface $\mathcal{S}$ and defines a charged symplectic form on $\mathcal{Y}$.

Proof Let $\mathcal{S}_1$, $\mathcal{S}_2$ be two Cauchy hypersurfaces. We can find a third Cauchy hypersurface $\mathcal{S}_0$ that lies in the future of $\operatorname{supp}\zeta_i \cap \mathcal{S}_j$, $i = 1, 2$, $j = 1, 2$. Applying the Stokes theorem and (19.53) to the domain between $\mathcal{S}_0$ and $\mathcal{S}_1$, we show that the integrals (19.60) on $\mathcal{S}_0$ and $\mathcal{S}_1$ coincide. By the same argument, the integrals (19.60) on $\mathcal{S}_0$ and $\mathcal{S}_2$ coincide. $\square$

Note that in terms of the Cauchy data we have

$$\overline{\zeta_1}\cdot\omega\zeta_2 = \int_{\mathcal{S}} \left(\overline{\vartheta_1}\cdot\lambda\varsigma_2 - \overline{\varsigma_1}\cdot\lambda\vartheta_2\right) \mathrm{d}s.$$

19.5.6 Solutions parametrized by test functions

Theorem 19.56 (1) *For any $f \in C^\infty_{\rm c}(\mathcal{X}, \mathcal{V})$, $Gf \in \mathcal{Y}$.*
(2) *Every element of $\mathcal{Y}$ is of this form.*
(3) $\overline{Gf_1}\cdot\omega Gf_2 = \int \overline{f_1(x)}\cdot\lambda(x)G(x, y)f_2(y)\mathrm{d}v(x)\mathrm{d}v(y)$.

Proof Gf is a solution of (19.56), since G is a solution of (19.56) in its first variable. The fact that Gf is space-compact follows from the support properties of $G^\pm$. Hence, (1) is true.

Now let $\zeta \in \mathcal{Y}$. Since ζ is space-compact, we can find cutoff functions $\chi^\pm \in C^\infty_{\pm\rm sc}(\mathcal{X})$ such that $\chi^+ + \chi^- = 1$ on $\operatorname{supp}\zeta$. Moreover, it follows from Condition (1a) of Thm. 19.40 that $\operatorname{supp}\nabla\chi^\pm \cap \operatorname{supp}\zeta$ is compact. Setting

$$\zeta^\pm := \chi^\pm\zeta, \quad f := \Box\zeta^+ = -\Box\zeta^-,$$

we see that $f \in C^\infty_{\rm c}(\mathcal{X})$. Hence, by Thm. 19.52, $\zeta^\pm = \pm G^\pm f$, and $\zeta = Gf$. This proves (2).

Let $f_1, f_2 \in C^\infty_{\rm c}(\mathcal{X}, \mathcal{V})$. In a sufficiently far future we have $Gf_i = G^+f_i$, $i = 1, 2$. Hence, for a Cauchy surface S in a far future we have

$$\begin{aligned}
\overline{Gf_1}\omega Gf_2 &= \int_{\mathcal{S}} g^{\mu\nu}\left(\overline{\nabla^\Gamma_\mu G^+f_1}\cdot\lambda G^+f_2 - \overline{G^+f_1}\cdot\lambda\nabla^\Gamma_\mu G^+f_2\right)\mathrm{d}s_\nu\\
&= \int_{J^-(\mathcal{S})}\left(\overline{\Box G^+f_1}\cdot\lambda G^+f_2 - \overline{G^+f_1}\cdot\lambda\Box G^+f_2\right)\mathrm{d}v\\
&= \int_{\mathcal{X}}\left(\overline{f_1}\cdot\lambda G^+f_2 - \overline{G^+f_1}\cdot\lambda f_2\right)\mathrm{d}v\\
&= \int_{\mathcal{X}}\left(\overline{f_1}\cdot\lambda G^+f_2 - \overline{f_1}\cdot\lambda G^-f_2\right)\mathrm{d}v.
\end{aligned}$$

In the first line we use the definition of ω, in the second Green's formula, in the third the fact that f_i are compactly supported, and in the last the fact that $G^{+*} = G^-$. $\Box$

Let $\mathcal{O}$ be an open subset of $\mathcal{X}$. We define

$$\mathcal{Y}(\mathcal{O}) := \big\{ Gf \ : \ f \in C_c^\infty(\mathcal{O}, \mathcal{V}) \big\}.$$

Theorem 19.57 (1) *$\mathcal{Y}(\mathcal{X}) = \mathcal{Y}$.*
(2) *$\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{Y}(\mathcal{O}_1) \subset \mathcal{Y}(\mathcal{O}_2)$.*
(3) *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are causally separated and $\zeta_i \in \mathcal{Y}(\mathcal{O}_i)$, $i = 1, 2$, then*

$$\overline{\zeta}_1 \cdot \omega \zeta_2 = 0.$$

(4) *If $\mathcal{O}_1$ is causally dependent on $\mathcal{O}_2$, then $\mathcal{Y}(\mathcal{O}_1) \subset \mathcal{Y}(\mathcal{O}_2)$.*
(5) *ω is non-degenerate on $\mathcal{Y}$.*

Proof (1) follows from Thm. 19.56 (2). (2) is obvious. (3) follows from the definition of ω and the support properties of $G^\pm$ in Thm. 19.52. To prove (4), it suffices to show that if $\zeta = Gf$ for $f \in C_c^\infty(\mathcal{O}_1)$, then $\zeta = Gg$ for $g \in C_c^\infty(\mathcal{O}_2)$. Using a partition of unity of $\mathcal{O}_1$, we can assume that there exists a Cauchy hypersurface $\mathcal{S}$ such that any causal curve starting from $\mathcal{O}_1$ intersects S in $\mathcal{O}_2$. If $\zeta = Gf$ for $f \in C_c^\infty(\mathcal{O}_1)$, we set $\varsigma = \zeta\big|_{\mathcal{S}}$, $\vartheta = n^\mu \nabla_\mu^\Gamma \zeta\big|_{\mathcal{S}}$. By Thm. 19.52, the Cauchy data (ϑ, ς) are supported in a compact set $N \subset \mathcal{S}$, and by Thm. 19.54 we obtain that $\operatorname{supp} \zeta \subset J^+(N) \cup J^-(N) \subset \mathcal{O}_2^+ \cup \mathcal{O}_2^-$, where $\mathcal{O}_2^\pm = \mathcal{O}_2 \cup \mathcal{S}^\pm$ and $\mathcal{S}^\pm$ are the future, resp. past of $\mathcal{S}$. Hence, we can find cutoff functions $\chi^\pm$ supported in $\mathcal{O}_2^\pm$ such that $\zeta = \chi^+\zeta + \chi^-\zeta =: \zeta^+ + \zeta^-$. Setting

$$g := \Box \zeta^+ = -\Box \zeta^-,$$

we obtain that $\operatorname{supp} g \subset \mathcal{O}_2^+ \cap \mathcal{O}_2^- = \mathcal{O}_2$. Moreover, since $\zeta^\pm$ is past, resp. future space-compact, we see by Thm. 19.52 that $\zeta^\pm = \pm G^\pm g$. Hence, $\zeta = Gg$. This completes the proof of the theorem. $\Box$

19.5.7 Algebraic quantization

Let $\mathfrak{A} := \mathrm{CCR}^{\mathrm{Weyl}}(\mathcal{Y})$. More generally, if $\mathcal{O}$ is an open bounded subset in $\mathcal{X}$, let $\mathfrak{A}(\mathcal{O})$ be the sub-algebra of $\mathfrak{A}$ generated by $W(Gf + \overline{Gf})$, where $f \in C_c^\infty(\mathcal{O})$. In other words, $\mathfrak{A}(\mathcal{O}) = \mathrm{CCR}^{\mathrm{Weyl}}\big(\mathcal{Y}(\mathcal{O})\big)$.

The family of algebras $\mathfrak{A}(\mathcal{O})$ satisfies the following properties, which express the Einstein causality:

Theorem 19.58 (1) *$\mathfrak{A}(\mathcal{X}) = \mathfrak{A}$.*
(2) *$\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.*

(3) *If* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are space-like separated and* $B_i \in \mathfrak{A}(\mathcal{O}_i)$, $i = 1, 2$, *then*

$$B_1 B_2 = B_2 B_1.$$

(4) *If* $\mathcal{O}_1$ *is causally dependent on* $\mathcal{O}_2$, *then* $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.

19.6 Generalized Dirac equation on curved space-time

The setting of this section is similar to that of Sect. 19.5. Again, we assume that $\mathcal{X}$ is a pseudo-Riemannian manifold. Starting with Subsect. 19.6.3, we assume in addition that $\mathcal{X}$ is globally hyperbolic.

Similarly, as in the previous section we consider also a finite-dimensional space $\mathcal{V}$ and the vector bundle $\mathcal{X} \times \mathcal{V} \to \mathcal{X}$.

The goal of this section is to describe the algebraic quantization of a large class of first-order equations on a curved space-time. We will always assume that the principal term of this equation is of the form $\gamma^\mu(x)\partial_\mu$, where $\gamma^\mu(x)$ satisfy the Clifford relations given by the metric tensor $[g_{\mu\nu}(x)]$. Such an equation will be called a *generalized Dirac equation*. In the case of Lorentzian manifolds, solutions of this equation have causal propagation.

We will see that generalized Dirac equations possess a conserved current. In the case of globally hyperbolic manifolds, this current defines a scalar product on the space of solutions. Therefore, it is natural to quantize this equation using the CAR. Its algebraic quantization leads to a net of algebras satisfying the fermionic version of the Einstein causality.

This section is to a large extent a generalization of Sect. 19.3 to a curved space-time. Unlike in Sect. 19.3, we limit our discussion to the algebraic quantization. We do not discuss the positive energy quantization on a fermionic Fock space, which is possible for the Dirac equation on a stationary space-time.

19.6.1 Dirac operators

Recall that $\mathcal{X}$ is a pseudo-Riemannian manifold. $\mathcal{V}$ can be a real or complex space. For simplicity, we will consider only the complex case.

Definition 19.59 *A map* $\mathcal{X} \ni x \mapsto \gamma^\mu(x) \in L(\mathcal{V})$ *satisfying*

$$[\gamma^\mu(x), \gamma^\nu(x)]_+ = 2g^{\mu\nu}(x) \tag{19.61}$$

is called a spinor structure *on* $\mathcal{X}$.

Definition 19.60 *Let* $\mathcal{X} \ni x \mapsto \theta(x) \in L(\mathcal{V})$. *The operator on* $C^\infty(\mathcal{X}, \mathcal{V})$

$$\mathbb{D} = \mathbb{D}(\theta) := \gamma^\mu(x)\nabla_\mu + \theta(x) \tag{19.62}$$

is called a generalized Dirac operator.

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We will study the equation

$$\mathbb{D}\zeta = 0. \tag{19.63}$$

Let us fix a smooth map $\mathcal{X} \ni x \mapsto \lambda(x) \in L(\mathcal{V}, \mathcal{V}^*)$ such that $\lambda(x)$ is non-degenerate for all $x \in \mathcal{X}$. We will often assume that

$$\lambda(x)\gamma^\mu(x) \ \text{ is self-adjoint,} \tag{19.64}$$

$$\lambda(x)\theta(x) - \frac{1}{2}|g|^{-\frac{1}{2}}(x)\nabla_\mu |g|^{\frac{1}{2}}(x)\lambda(x)\gamma^\mu(x) \ \text{ is anti-self-adjoint.} \tag{19.65}$$

Using Thm. 19.46 and the Stokes theorem, we obtain the following version of Green's formula:

$$\begin{aligned}&\int_\Omega \big(\overline{\zeta_1(x)}\cdot\lambda(x)\mathbb{D}\zeta_2(x) - \overline{\mathbb{D}\zeta_1(x)}\cdot\lambda(x)^*\zeta_2(x)\big)\mathrm{d}v(x)\\ &= \int_{\partial\Omega} \overline{\zeta_1(x)}\cdot\lambda(x)\gamma^\mu(x)\zeta_2(x)\mathrm{d}s_\mu(x).\end{aligned} \tag{19.66}$$

For $\zeta_1, \zeta_2 \in C^\infty(\mathcal{X}, \mathcal{V})$, we define

$$J^\mu(\zeta_1, \zeta_2, x) = \overline{\zeta_1(x)}\cdot\lambda(x)\gamma^\mu(x)\zeta_2(x). \tag{19.67}$$

If ζ_1, ζ_2 are solutions of the Dirac equation, by Thm. 19.46, we have

$$\nabla_\mu |g|^{\frac{1}{2}}(x) J^\mu(\zeta_1, \zeta_2, x) = 0. \tag{19.68}$$

Note also that if we have another Dirac operator $\mathbb{D}_1 := \gamma^\mu(x)\nabla_\mu + \theta_1(x)$, then

$$\Box := \mathbb{D}\mathbb{D}_1$$

is a second-order operator of the form considered in Subsect. 19.5.1.

19.6.2 Lagrangian of the Dirac equation

(19.63) can be obtained as the Euler–Lagrange equation for the following Lagrangians:

$$\begin{aligned}L_1(\zeta, \overline{\zeta}, \partial\zeta, \partial\overline{\zeta}) &:= -|g|^{\frac{1}{2}}\overline{\zeta}\cdot\lambda(\gamma^\mu\partial_\mu + \theta)\zeta, \qquad \text{or}\\ L(\zeta, \overline{\zeta}, \partial\zeta, \partial\overline{\zeta}) &:= -\frac{1}{2}|g|^{\frac{1}{2}}\big(\overline{\zeta}\cdot\lambda\gamma^\mu\partial_\mu\zeta + \overline{\partial_\mu\zeta}\cdot\lambda\gamma^\mu\zeta\big)\\ &\quad -\overline{\zeta}\cdot\big(|g|^{\frac{1}{2}}\lambda\theta - \frac{1}{2}(\partial_\mu|g|^{\frac{1}{2}}\lambda\gamma^\mu)\big)\zeta.\end{aligned}$$

19.6.3 Green's functions of hyperbolic Dirac equations

Until the end of the section we assume that $\mathcal{X}$ is globally hyperbolic and $\mathbb{D}$ is a generalized Dirac operator on $\mathcal{X}$.

Theorem 19.61 *For any $f \in C_{\mathrm{c}}^{\infty}(\mathcal{X},\mathcal{V})$, there exist unique functions $\zeta^{\pm} \in C_{\pm\mathrm{sc}}^{\infty}(\mathcal{X},\mathcal{V})$ that solve*

$$\mathbb{D}\zeta^{\pm} = f. \tag{19.69}$$

Moreover

$$\zeta^{\pm}(x) = (S^{\pm}f)(x) := \int S^{\pm}(x,y)f(y)\mathrm{d}v(y), \tag{19.70}$$

where $S^{\pm} \in \mathbb{D}'\big(\mathcal{X}\times\mathcal{X}, L(\mathcal{V})\big)$ satisfy

$$\mathbb{D}S^{\pm} = S^{\pm}\mathbb{D} = \mathbb{1}, \qquad \operatorname{supp} S^{\pm} \subset \big\{(x,y)\ :\ x \in J^{\pm}(y)\big\}. \tag{19.71}$$

If in addition (19.64) and (19.65) hold, then

$$\lambda(x)^{*}S^{+}(x,y) = -S^{-}(x,y)^{*}\lambda(y). \tag{19.72}$$

Proof By the remark in Subsect. 19.6.1, $\mathbb{D}^2$ is a generalized Klein–Gordon operator. Let $G^{\pm}$ be the retarded, resp. advanced Green's functions of $\mathbb{D}^2$. Clearly, $\zeta^{\pm} = \mathbb{D}G^{\pm}f$ are solutions of (19.69). To prove the uniqueness, we note that if $\mathbb{D}\zeta^{\pm} = 0$ and $\operatorname{supp}\zeta^{\pm} \subset J^{\pm}(K)$ for some compact K, then $\mathbb{D}^2\zeta^{\pm} = 0$. Hence, $\zeta^{\pm} = 0$ by Thm. 19.52. We set then $S^{\pm} = \mathbb{D}G^{\pm}$. This proves (19.70) and (19.71). Let us prove (19.72). We need to show that

$$\int \overline{f}_2 \cdot \lambda^{*}S^{+}f_1\mathrm{d}v = -\int \overline{f}_2 \cdot S^{+*}\lambda f_1\mathrm{d}v. \tag{19.73}$$

It is enough to set $f_i = \mathbb{D}\zeta_i$ for $\zeta_i = S^{+}f_i$. Now, $\lambda\mathbb{D}$ is anti-Hermitian for the scalar product (19.40), hence

$$\begin{aligned}
\int \overline{\mathbb{D}\zeta_2}\cdot\lambda^{*}S^{+}\mathbb{D}\zeta_1\mathrm{d}v &= \int \overline{\lambda\mathbb{D}\zeta_2}\cdot S^{+}\mathbb{D}\zeta_1\mathrm{d}v \\
&= -\int \overline{\zeta}_2\cdot\lambda\mathbb{D}S^{+}\mathbb{D}\zeta_1\mathrm{d}v = -\int \overline{\zeta}_2\cdot\lambda\mathbb{D}\zeta_1\mathrm{d}v \\
&= -\int \overline{S^{+}\mathbb{D}\zeta_2}\cdot\lambda\mathbb{D}\zeta_1\mathrm{d}v = -\int \overline{\mathbb{D}\zeta_2}\cdot S^{+*}\lambda\mathbb{D}\zeta_1\mathrm{d}v.
\end{aligned}$$

□

Note that by duality $S^{\pm}$ can be applied to distributions of compact support.

Definition 19.62 *S^{+}, resp. S^{-} is called the* retarded*, resp.* advanced Green's function*. We also set*

$$S := S^{+} - S^{-}.$$

Note that

$$\lambda(x)^{*}S(x,y) = S(x,y)^{*}\lambda(x). \tag{19.74}$$

19.6.4 Cauchy problem

Until the end of the section we assume that $\mathcal{X} \ni x \mapsto \lambda(x)$ has been chosen so that (19.64) and (19.65) hold. We also assume that $\lambda(x)$ is non-degenerate for any $x \in \mathcal{X}$.

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Theorem 19.63 *Let $\mathcal{S}$ be a smooth Cauchy surface. Let $\vartheta \in C_{\mathrm{c}}^\infty(\mathcal{S},\mathcal{V})$. Then there exists a unique $\zeta \in C_{\mathrm{sc}}^\infty(\mathcal{X},\mathcal{V})$ that solves*

$$\mathbb{D}\zeta = 0 \tag{19.75}$$

with initial conditions $\zeta\big|_{\mathcal{S}} = \vartheta$. It satisfies $\operatorname{supp}\zeta \subset J(\operatorname{supp}\vartheta)$ *and is given by*

$$\zeta(x) = -\int_{\mathcal{S}} S(x,y)\gamma^\mu(y)\vartheta(y)\mathrm{d}s_\mu(y). \tag{19.76}$$

Proof The existence and uniqueness is well known. Let us prove (19.76).

We apply Green's Formula (19.66) to $\zeta_2 = \zeta$ and $\zeta_1 = S^\mp f$, $f \in C_{\mathrm{c}}^\infty(\mathcal{X})$, $\Omega = J^\pm(\mathcal{S})$, obtaining

$$\int_{J^+(\mathcal{S})} \overline{f}\cdot\lambda^*\zeta\mathrm{d}v = \int_{\mathcal{S}} \overline{S^- f}\cdot\lambda\gamma^\mu\zeta\mathrm{d}s_\mu, \tag{19.77}$$

$$\int_{J^-(\mathcal{S})} \overline{f}\cdot\lambda^*\zeta\mathrm{d}v = -\int_{\mathcal{S}} \overline{S^+ f}\cdot\lambda\gamma^\mu\zeta\mathrm{d}s_\mu. \tag{19.78}$$

Adding (19.77) and (19.78), we get

$$\int_{\mathcal{X}} \overline{f}\cdot\lambda^*\zeta\mathrm{d}v = -\int_{\mathcal{S}} \overline{Sf}\cdot\lambda\gamma^\mu\zeta\mathrm{d}s_\mu.$$

This can be rewritten as

$$\begin{aligned}\int_{\mathcal{X}} \overline{f(x)}\cdot\lambda(x)^*\zeta(x)\mathrm{d}v(x) &= -\int_{\mathcal{X}}\mathrm{d}v(x)\int_{\mathcal{S}} \overline{S(y,x)f(x)}\cdot\lambda(y)\gamma^\mu\zeta(y)\mathrm{d}s_\mu(y)\\ &= -\int_{\mathcal{X}} \overline{f(x)}\cdot\lambda(x)^*\mathrm{d}v(x)\int_{\mathcal{S}} S(x,y)\gamma^\mu(y)\zeta(y)\mathrm{d}s_\mu(y),\end{aligned}$$

where in the last line we use (19.74). □

19.6.5 Unitary space of solutions of the Dirac equation

Let $\mathcal{Y}$ denote the set of solutions of the Dirac equation in $C_{\mathrm{sc}}^\infty(\mathcal{X},\mathcal{V})$. To equip $\mathcal{Y}$ with a scalar product an additional positivity condition is required. We assume that for all $x \in \mathcal{X}$

$$\lambda(x)\gamma^\mu(x)v_\mu > 0, \text{ if } v \in \mathrm{T}_x\mathcal{X} \text{ is time-like and future directed.} \tag{19.79}$$

By Lemma 19.9, it suffices to assume that there exists a time-like future directed vector field v such that

$$\lambda(x)\gamma^\mu(x)v_\mu(x) > 0, \quad x \in \mathcal{X}.$$

Theorem 19.64 *Let $\zeta_1,\zeta_2 \in \mathcal{Y}$. Define $J^\mu(\zeta_1,\zeta_2,x)$ as in (19.67). Then*

$$\overline{\zeta}_1\cdot\zeta_2 := \int_{\mathcal{S}} J^\mu(\zeta_1,\zeta_2,x)\mathrm{d}s_\mu(x)$$

does not depend on the choice of a Cauchy hypersurface $\mathcal{S}$ and defines a positive definite Hermitian form on $\mathcal{Y}$.

Proof To show that $\overline{\zeta}_1 \cdot \zeta_2$ is independent of $\mathcal{S}$ we apply the Stokes theorem as in Thm. 19.55, using (19.68). The fact that it is positive definite follows from the positivity condition (19.79). □

19.6.6 Solutions parametrized by test functions

Theorem 19.65 (1) *For any $f \in C_{\mathrm{c}}^{\infty}(\mathcal{X}, \mathcal{V})$, $Sf \in \mathcal{Y}$.*
(2) *Every element of $\mathcal{Y}$ is of this form.*
(3) $\overline{Sf_1} \cdot Sf_2 = \int \overline{f_1(x)} \cdot \lambda(x)^* S(x,y) f_2(y) \mathrm{d}v(x) \mathrm{d}v(y)$.

Proof (1) follows from the fact that S solves (19.75) in its first coordinate.

(2). Let $\zeta \in \mathcal{Y}$. We can write $\zeta = \zeta^+ + \zeta^-$, where $\zeta^+ \in C_{\pm\mathrm{sc}}^{\infty}$. Set

$$f := \mathbb{D}\zeta^+ = -\mathbb{D}\zeta^-.$$

Then $f \in C_{\mathrm{c}}^{\infty}(\mathcal{X}, \mathcal{V})$, $\zeta^{\pm} = \pm S^{\pm} f$, and hence $\zeta = Sf$.

(3). Let $f_1, f_2 \in C_{\mathrm{c}}^{\infty}(\mathcal{X}, \mathcal{V})$. In a sufficiently far future we have $Sf_i = S^+ f_i$, $i = 1, 2$. Hence, for a late Cauchy surface $\mathcal{S}$,

$$\begin{aligned}
\overline{Sf_1} \cdot Sf_2 &= \int_{\mathcal{S}} \overline{S^+ f_1} \cdot \lambda \gamma^{\mu} S^+ f_2 \mathrm{d}s_{\mu} \\
&= \int_{J^-(\mathcal{S})} \big(\overline{\mathbb{D} S^+ f_1} \cdot \lambda^* S^+ f_2 - \overline{S^+ f_1} \cdot \lambda \mathbb{D} S^+ f_2 \big) \mathrm{d}v \\
&= \int_{\mathcal{X}} \big(\overline{f_1} \cdot \lambda^* S^+ f_2 - \overline{S^+ f_1} \cdot \lambda f_2 \big) \mathrm{d}v \\
&= \int_{\mathcal{X}} \big(\overline{f_1} \cdot \lambda^* S^+ f_2 - \overline{f_1} \cdot \lambda^* S^- f_2 \big) \mathrm{d}v.
\end{aligned}$$
□

Let $\mathcal{O}$ be an open subset of $\mathcal{X}$. We define

$$\mathcal{Y}(\mathcal{O}) := \big\{ Sf \ : \ f \in C_{\mathrm{c}}^{\infty}(\mathcal{O}, \mathcal{V}) \big\}.$$

Theorem 19.66 (1) $\mathcal{Y}(\mathcal{X}) = \mathcal{Y}$.
(2) *$\mathcal{O}_1 \subset \mathcal{O}_2$ implies $\mathcal{Y}(\mathcal{O}_1) \subset \mathcal{Y}(\mathcal{O}_2)$.*
(3) *If $\mathcal{O}_1$ and $\mathcal{O}_2$ are space-like separated and $\zeta_i \in \mathcal{Y}(\mathcal{O}_i)$, $i = 1, 2$, then*

$$\overline{\zeta}_1 \cdot \zeta_2 = 0.$$

(4) *If $\mathcal{O}_1$ is causally dependent on $\mathcal{O}_2$, then $\mathcal{Y}(\mathcal{O}_1) \subset \mathcal{Y}(\mathcal{O}_2)$.*

19.6.7 Algebraic quantization

Set $\mathfrak{A} := \mathrm{CAR}^{C^*}(\mathcal{Y})$. Note that it is a graded algebra. More generally, if $\mathcal{O}$ is an open bounded subset in $\mathcal{X}$, let $\mathfrak{A}(\mathcal{O})$ be the C^*-sub-algebra of $\mathfrak{A}$ generated by $\psi(Sf)$, where $f \in C_{\mathrm{c}}^{\infty}(\mathcal{O})$. In other words, $\mathfrak{A}(\mathcal{O}) = \mathrm{CAR}^{C^*}(\mathcal{Y}(\mathcal{O}))$. The family of algebras $\mathfrak{A}(\mathcal{O})$ satisfies the following properties:

Theorem 19.67 (1) $\mathfrak{A}(\mathcal{X}) = \mathfrak{A}$.

(2) $\mathcal{O}_1 \subset \mathcal{O}_2$ *implies* $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.

(3) *If* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are causally separated, and* $B_i \in \mathfrak{A}(\mathcal{O}_i)$ *are elements of parity* $|B_i|$, *then*

$$B_1 B_2 = (-1)^{|B_1||B_2|} B_2 B_1.$$

(4) *If* $\mathcal{O}_1$ *is causally dependent on* $\mathcal{O}_2$, *then* $\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$.

19.7 Notes

The material of the first three sections is discussed in essentially all textbooks on quantum field theory, such as Jauch–Röhrlich (1976), Schweber (1962), Weinberg (1995) and Srednicki (2007).

Mathematical aspects of quantum field theory on curved space-time were studied by Dimock (1980, 1982). A review of this subject can be found in monographs by Wald (1994) and Fulling (1989).

A short and readable monograph on wave equations on Lorentzian manifolds is Bär–Ginoux–Pfäffle (2007).

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