IX

Mass mixing and *CP* violation

Aside from a concluding section on the strong *CP* problem, this chapter is about the *CP* violation of kaons. We set up the general framework for meson–antimeson mixing, which is also used in the weak interactions of heavy quarks, treated later in Chap. XIV. In this chapter we apply the formalism to K^0–$\bar{K}^0$ mixing and *CP*-violating processes involving kaons.

IX–1 K^0–$\bar{K}^0$ mixing

It is clear that K^0 and $\bar{K}^0$ *should* mix with each other. In addition to less obvious mechanisms discussed later, the most easily seen source of mixing occurs through their common $\pi\pi$ decays, i.e., $K^0 \leftrightarrow \pi\pi \leftrightarrow \bar{K}^0$. We can use second-order perturbation theory to study the phenomenon of mixing. Writing the wavefunctions in two-component form

$$|\psi(t)\rangle = \begin{pmatrix} a(t) \\ b(t) \end{pmatrix} \equiv a(t)|K^0\rangle + b(t)|\bar{K}^0\rangle, \tag{1.1}$$

we have the time development

$$i\frac{d}{dt}|\psi(t)\rangle = \left(M - \frac{i}{2}\Gamma\right)|\psi(t)\rangle, \tag{1.2}$$

where, to second order in perturbation theory, the quantity in parentheses is called the *mass matrix* and is given by[1]

[1] The factors $1/2m_K$ are required by the normalization convention of Eq. (C–3.7) for state vectors.

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$$\begin{aligned}\left[M - \frac{i}{2}\Gamma\right]_{ij} &\equiv \frac{\langle K_i^0|\mathcal{H}_{\rm eff}|K_j^0\rangle}{2m_K}\\ &= m_K^{(0)}\delta_{ij} + \frac{\langle K_i^0\,|\mathcal{H}_{\rm w}|\,K_j^0\rangle}{2m_K} + \frac{1}{2m_K}\sum_n \frac{\langle K_i^0|\mathcal{H}_{\rm w}|n\rangle\langle n|\mathcal{H}_{\rm w}|K_j^0\rangle}{m_K^{(0)} - E_n + i\epsilon}.\end{aligned} \tag{1.3}$$

Here, the absorptive piece Γ arises from use of the identity

$$\frac{1}{\omega - E_n + i\epsilon} = P\left(\frac{1}{\omega - E_n}\right) - i\pi\,\delta(E_n - \omega), \tag{1.4}$$

and hence involves only physical intermediate states

$$\Gamma_{ij} = \frac{1}{2m_K}\sum_n \langle K_i^0|\mathcal{H}_{\rm w}|n\rangle\langle n|\mathcal{H}_{\rm w}|K_j^0\rangle 2\pi\,\delta\,(E_n - m_K). \tag{1.5}$$

Because M and Γ are hermitian, we have $M_{21} = M_{12}^*$ and $\Gamma_{21} = \Gamma_{12}^*$. The diagonal elements of the mass matrix are required to be equal by *CPT* invariance, leading to a general form

$$M - \frac{i}{2}\Gamma = \begin{pmatrix} A & p^2 \\ q^2 & A \end{pmatrix}, \tag{1.6}$$

where A, p^2, and q^2 can be complex. The states $\bar{K}^0$ and K^0 are related by the unitary *CP* operation,

$$CP|K^0\rangle = \xi_K|\bar{K}^0\rangle \tag{1.7}$$

with $|\xi_K|^2 = 1$. Our convention will be to choose $\xi_K = -1$. The assumption of *CP* invariance would relate the off-diagonal elements in the mass matrix so as to imply $p = q$,

$$\langle K^0|\mathcal{H}_{\rm eff}|\bar{K}^0\rangle = \langle K^0\left|(CP)^{-1}CP\,\mathcal{H}_{\rm eff}\,(CP)^{-1}CP\right|\bar{K}^0\rangle = \langle \bar{K}^0\,|\mathcal{H}_{\rm eff}|\,K^0\rangle, \tag{1.8}$$

where $\langle \bar{K}^0\,|\mathcal{H}_{\rm eff}|\,K^0\rangle$ is defined in Eq. (1.3). Combined with the hermiticity of M and Γ, this would imply that M_{12} and Γ_{12} are real. In the actual *CP*-noninvariant world, this is not the case and we have instead for the eigenstates of the mass matrix,

$$|K_{\substack{L\\S}}\rangle = \frac{1}{\sqrt{|p|^2 + |q|^2}}\left[p|K^0\rangle \pm q|\bar{K}^0\rangle\right], \tag{1.9}$$

where, from the above discussion,

$$\frac{p}{q} = \sqrt{\frac{M_{12} - \frac{i}{2}\Gamma_{12}}{M_{12}^* - \frac{i}{2}\Gamma_{12}^*}}, \quad M_{12} - \frac{i}{2}\Gamma_{12} = \left\langle K^0\,|\mathcal{H}|\,\bar{K}^0\right\rangle. \tag{1.10}$$

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The difference in eigenvalues is given by

$$2qp = (m_L - m_S) - \frac{i}{2}(\Gamma_L - \Gamma_S)$$
$$= 2\left(M_{12} - \frac{i}{2}\Gamma_{12}\right)^{1/2}\left(M^*_{12} - \frac{i}{2}\Gamma^*_{12}\right)^{1/2} \simeq 2\mathrm{Re}\, M_{12} - i\mathrm{Re}\,\Gamma_{12}, \quad (1.11)$$

where the final relation is an approximation valid if *CP* violation is small ($1 \gg \mathrm{Im}\, M_{12}/\mathrm{Re}\, M_{12}$). The subscripts in K_L and K_S, standing for 'long' and 'short', refer to their respective lifetimes, whose ratio is substantial, $\tau_L/\tau_S \simeq 571$. To understand this large difference, we note that if *CP* were conserved ($p = q$), these states would become *CP* eigenstates $K^0_\pm$ (not to be confused with the charged kaons $K^\pm$!),

$$\begin{aligned} &|K_S\rangle \underset{p=q}{\longrightarrow} |K^0_+\rangle, && |K_L\rangle \underset{p=q}{\longrightarrow} |K^0_-\rangle, \\ &|K^0_\pm\rangle \equiv \tfrac{1}{\sqrt{2}}\left[|K^0\rangle \mp |\bar{K}^0\rangle\right], && CP|K^0_\pm\rangle = \pm|K^0_\pm\rangle. \end{aligned} \quad (1.12)$$

In this limit, which well approximates reality, K_S would decay only to *CP*-even final states like $\pi\pi$, whereas K_L would decay only to *CP*-odd final states, e.g., 3π. Since the phase space for the former considerably exceeds that of the latter at the rather low energy of the kaon mass, K_S has much the shorter lifetime. The states $K_{S,L}$, expanded in terms of *CP* eigenstates, are

$$\left|K_{\substack{L\\S}}\right\rangle = \frac{1}{\sqrt{1+|\bar{\epsilon}|^2}}\left[|K^0_\mp\rangle + \bar{\epsilon}|K^0_\pm\rangle\right], \qquad \frac{p}{q} = \frac{1+\bar{\epsilon}}{1-\bar{\epsilon}},$$
$$\bar{\epsilon} = \frac{p-q}{p+q} \simeq \frac{i}{2}\,\frac{\mathrm{Im}\, M_{12} - i\mathrm{Im}\,\Gamma_{12}/2}{\mathrm{Re}\, M_{12} - i\mathrm{Re}\,\Gamma_{12}/2} \simeq \frac{1}{2}\,\frac{M_{12} - M_{21} - \frac{i}{2}(\Gamma_{12} - \Gamma_{21})}{m_L - m_S - \frac{i}{2}(\Gamma_L - \Gamma_S)}. \quad (1.13)$$

K^0–$\bar{K}^0$ mixing can be observed experimentally from the time development of a state which is produced via a strong interaction process, and therefore starts out at $t = 0$ as either a pure K^0 or $\bar{K}^0$,

$$|K^0(t)\rangle = g_+(t)|K^0\rangle + \frac{q}{p}g_-(t)|\bar{K}^0\rangle,$$
$$|\bar{K}^0(t)\rangle = \frac{p}{q}g_-(t)|K^0\rangle + g_+(t)|\bar{K}^0\rangle,$$
$$g_\pm(t) = \frac{1}{2}e^{-\Gamma_L t/2}e^{-im_L t}\left[1 \pm e^{-\Delta\Gamma t/2}e^{i\Delta m t}\right], \quad (1.14)$$

where $\Delta\Gamma \equiv \Gamma_S - \Gamma_L$ and $\Delta m \equiv m_L - m_S$, each defined to be a *positive* quantity. From such experiments, the very precise value

$$\Delta m_{\mathrm{expt}} = (3.484 \pm 0.006) \times 10^{-12}\ \mathrm{MeV} \quad (1.15)$$

has been obtained.

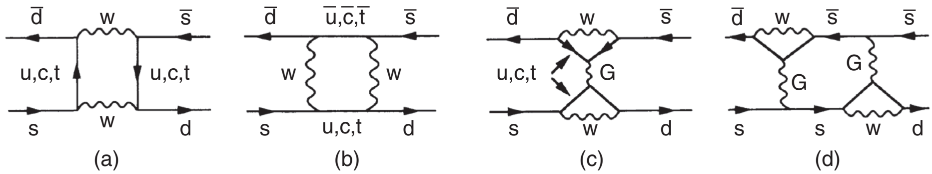

Fig. IX–1 Box (a),(b) and other contributions to *CP* violation.

CP-conserving mixing

There are two main classes of contributions, associated respectively with the short-distance box diagrams of Fig. IX–1(a),(b) and the long-distance contributions like those in Fig. IX–2,

$$\Delta m_{\text{theory}} = (\Delta m)^{\text{SD}}_{\text{theory}} + (\Delta m)^{\text{LD}}_{\text{theory}} \,. \tag{1.16a}$$

We shall consider the first of these here, the short distance component

$$(\Delta m)^{\text{SD}}_{\text{theory}} = 2\text{Re}\left\langle K^0 \left|\mathcal{H}^{\text{box}}_{\text{w}}\right| \bar{K}^0\right\rangle . \tag{1.16b}$$

Determining $(\Delta m)^{\text{SD}}_{\text{theory}}$ has long been, and continues to be, a significant topic in kaon physics. It involves almost all the field theory tools we describe in this book. Our discussion will of necessity include some advanced features in order to present a realistic picture of the current state of the art.

The construction of $\mathcal{H}^{\text{box}}_{\text{w}}$ follows a standard procedure: to a given order of *QCD* perturbation theory, first specify the Wilson coefficient at the scale $\mu = M_W$, then use the renormalization group (RG) to evolve down to a hadronic scale $\mu < m_c$ and finally match onto the effective three-quark (i.e. u, d, s) theory. The result of this is

$$\mathcal{H}^{\text{box}}_{\text{w}} = \mathcal{C}(\mu)\, O^{\Delta S=2}, \tag{1.17}$$

where $O^{\Delta S=2}$ is the local four-quark operator

$$O^{\Delta S=2} = \bar{d}\gamma_\mu(1+\gamma_5)s\ \bar{d}\gamma^\mu(1+\gamma_5)s, \tag{1.18}$$

and $\mathcal{C}(\mu)$ is the corresponding Wilson coefficient,

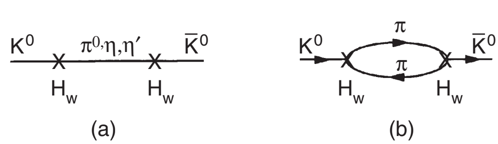

Fig. IX–2 Long-distance contributions to $K^0 - \bar{K}^0$ mixing.

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$$\mathcal{C}(\mu) = \frac{G_F^2}{16\pi^2}\left[\xi_c^2 H(x_c) m_c^2 \eta_{cc} + \xi_t^2 H(x_t) m_t^2 \eta_{tt} + 2\xi_c\xi_t \bar{G}(x_c, x_t) m_c^2 \eta_{ct}\right] b(\mu) \,, \tag{1.19}$$

with $\xi_i \equiv V_{id}^* V_{is}$ $(i = c, t)$ and $x_i \equiv m_i^2/M_W^2$. The above expression for $\mathcal{C}(\mu)$ is more complicated than the $\mathcal{C}_\pm(\mu)$ encountered in our earlier $\Delta S = 1$ discussion (cf. Eq. (VIII–3.11)) because the box amplitude for $\Delta S = 2$ has loop contributions from all the u, c, t quarks. Actually, Eq. (1.19) has already been simplified in that CKM unitarity has allowed removal of ξ_u and the tiny mass of the u quark has been neglected with respect to the heavy-quark masses m_c, m_t. The quantities $H(x_t)$, $H(x_c)$ and $\bar{G}(x_c, x_t)$ in Eq. (1.19) are so-called Inami–Lim functions [InL 81] that describe the quark-level loop amplitudes of Fig. IX–1(a),(b) in the no-QCD limit,

$$\begin{aligned} H(x) &= \left[\frac{1}{4} + \frac{9}{4}\frac{1}{1-x} - \frac{3}{2}\frac{1}{(1-x)^2}\right] - \frac{3}{2}\frac{x^2}{(1-x)^3}\ln x, \\ \bar{G}(x,y) &= y\left[-\frac{1}{y-x}\left(\frac{1}{4} + \frac{3}{2}\frac{1}{1-x} - \frac{3}{4}\frac{1}{(1-x)^2}\right)\ln x \right. \\ &\qquad \left. + (y \leftrightarrow x) - \frac{3}{4}\frac{1}{(1-x)(1-y)}\right]. \end{aligned} \tag{1.20}$$

This leaves in Eq. (1.19) the factors η_{cc}, η_{tt}, η_{ct}, and $b(\mu)$. These arise from calculating perturbative corrections to $\mathcal{H}_{\rm w}^{\rm box}$.[2] Such corrections will contain dependence on both the scale (μ) and renormalization scheme (say, the NDR approach, described in App. C–5). These cannot be present in the full amplitude and must be cancelled by analogous dependence in the matrix element $\langle K^0 \left| O^{\Delta S=2} \right| \bar{K}^0\rangle$. For convenience, the scale and scheme dependence present in the Wilson coefficient $\mathcal{C}(\mu)$ is placed into the factor $b(\mu)$, which for K^0–$\bar{K}^0$ mixing has the perturbative form

$$b(\mu) = \alpha_s(\mu)^{-2/9}\sum_{n=0}^{\infty} J^{(n)}\frac{\alpha_s(\mu)^n}{4\pi} = \alpha_s(\mu)^{-2/9}\left[1 + \frac{\alpha_s(\mu)}{4\pi}J^{(1)} + \cdots\right]. \tag{1.21}$$

Scheme dependence first appears in $J^{(1)}$ via the anomalous dimension $\gamma^{(1)}$ of operator $O^{\Delta S=2}$,

[2] It is customary to classify corrections according to the order of QCD perturbation theory used to determine them, e.g. 'leading' (LO), 'next-to-leading' (NLO), 'next-to-next-to leading' (NNLO) and so on. It is disturbing that $\eta_{cc}^{\rm NNLO} \simeq 1.87$ is about 36% larger than $\eta_{cc}^{\rm NLO} \simeq 1.38$. This is an unexpectedly large result, one which warrants further study.

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$$J^{(1)} = \frac{\gamma^{(0)}\beta^{(1)}}{2\beta_0^2} - \frac{\gamma^{(1)}}{2\beta^{(0)}} = 12 \cdot \frac{153 - 19n_f}{(33 - 2n_f)^2} - \frac{1}{6} \cdot \frac{4n_f - 63}{33 - 2n_f}, \tag{1.22}$$

shown here for n_f flavors and in NDR renormalization. The above expression for $b(\mu)$ serves at the same time to define the perturbative factors η_{cc}, η_{tt}, and η_{ct}. For completeness, we display the most recent determinations [BrG 12] of the $\{\eta_i\}$ (with perturbative order shown as well),

$$\eta_{cc}^{\rm NNLO} = 1.87(76) \qquad \eta_{tt}^{\rm NLO} = 0.5765(65) \qquad \eta_{ct}^{\rm NNLO} = 0.496(47). \tag{1.23}$$

The determination of $\langle K^0 \left| O^{\Delta S=2}(\mu) \right| \bar{K}^0\rangle$ at a hadronic scale $\mu < m_c$ involves nonperturbative physics, so its evaluation by analytical means is problematic. It has become standard to express this quantity relative to its vacuum saturation value and introduce a parameter B_K as

$$\left\langle K^0 \left| O^{\Delta S=2}(\mu) \right| \bar{K}^0 \right\rangle = \frac{16}{3} F_K^2 m_K^2\, B_K(\mu), \tag{1.24}$$

with $F_K = 110.4 \pm 0.6$ MeV.[3] There has been substantial progress in the calculation of nonperturbative quantities such as B_K using lattice QCD methods. The scale- and scheme-independent version is defined as

$$\hat{B}_K = b(\mu) B_K(\mu) \tag{1.25}$$

and the value used in [BrG 12] is

$$\hat{B}_K = 0.737 \pm 0.020. \tag{1.26}$$

Finally, given present values for the CKM elements and the t-quark mass, the most important contribution to the *real* part of $\mathcal{H}_{\rm w}^{\rm box}$ is found to be from the c quark. In view of this, and noting that $H(x_c) \simeq 1$ (cf. Eq. (1.20)), we then have

$$\text{Re}\, \mathcal{H}_{\rm w}^{\rm box} \simeq \frac{G_F^2}{16\pi^2} m_c^2\, \text{Re}\left(V_{\rm cd}^* V_{\rm cs}\right)^2\ \eta_{cc}^{\rm NNLO} b(\mu)\ O^{\Delta S=2}. \tag{1.27}$$

At this point, we have the ingredients for determining $(\Delta m)_{\rm theory}^{\rm SD}$ and one obtains [BrG 12]

$$(\Delta m)_{\rm theory}^{\rm SD} = (3.1 \pm 1.2) \times 10^{-15}\ \text{GeV}, \tag{1.28}$$

which is consistent with the value cited for $\Delta m_{\rm expt}$ in Eq. (1.15) within the quoted uncertainty.

[3] The reader should be wary of occasional notational confusion between F_K and $f_K = \sqrt{2} F_K$.

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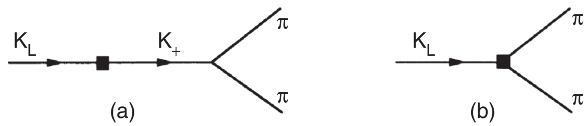

Fig. IX–3 Mechanisms for *CP* violation.

IX–2 The phenomenology of kaon *CP* violation

The $\pi\pi$ final state of kaon decay is even under *CP* provided the strong interactions are invariant under this symmetry. For the $\pi^0\pi^0$ system, this is clear since π^0 is itself a *CP* eigenstate, $CP|\pi^0\rangle = -|\pi^0\rangle$, and the two pions must be in an S-wave ($\ell = 0$) state,

$$CP|\pi^0\pi^0\rangle = (-1)^2(-1)^\ell|\pi^0\pi^0\rangle = +|\pi^0\pi^0\rangle. \tag{2.1}$$

The corresponding result for charged pions reflects the fact that π^+ and π^- are *CP*-conjugate partners, $CP|\pi^\pm\rangle = -|\pi^\mp\rangle$. We have seen that if *CP* were conserved, the two neutral kaons would organize themselves into *CP* eigenstates, with only K_S decaying into $\pi\pi$. Alternatively, K_L decays primarily into the $\pi\pi\pi$ final state, which is *CP*-odd if the pions are in relative S waves. The observation of *both* neutral kaons decaying into $\pi\pi$ is then a signal of *CP* violation.

There can be two sources of *CP* violation in $K_L \to \pi\pi$ decay. We have already seen that K^0–$\bar{K}^0$ mixing can generate a mixture of the *CP* eigenstates in physical kaons due to *CP* violation in the mass matrix. There also exists the possibility of *direct CP* violation in the weak decay amplitude, such that the *CP*-odd kaon eigenstate $|K^0_-\rangle$ makes a transition to $\pi\pi$. These two mechanisms are pictured in Fig. IX–3. The $K\pi\pi$ decay amplitudes have already been written down in Eq. (VIII–4.1) in terms of real-valued moduli A_0, A_2, and pion–pion scattering phases δ_0, δ_2. This decomposition is a consequence of Watson's theorem, and relies in part upon the assumption of time-reversal invariance. However, if direct *CP* violation occurs, A_0 and A_2 can themselves become complex-valued,

$$A_0 \equiv |A_0|e^{i\xi_0}, \qquad A_2 \equiv |A_2|e^{i\xi_2}, \tag{2.2}$$

with *CP* violation in the decay amplitude being characterized by the phases ξ_0 and ξ_2. Consequently, the $K_0 \to \pi\pi$ and $\bar{K}_0 \to \pi\pi$ decay amplitudes assume the modified form

$$\begin{aligned} A_{K^0\to\pi^+\pi^-} &= |A_0|e^{i\xi_0}e^{i\delta_0} + \frac{|A_2|}{\sqrt{2}}e^{i\xi_2}e^{i\delta_2}, \\ A_{\bar{K}^0\to\pi^+\pi^-} &= -|A_0|e^{-i\xi_0}e^{i\delta_0} - \frac{|A_2|}{\sqrt{2}}e^{-i\xi_2}e^{i\delta_2}. \end{aligned} \tag{2.3}$$

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Using the definitions of K_L and K_S in Eq. (1.13), a straightforward calculation leads to the following measures of CP violation:

$$\frac{\langle\pi^+\pi^-|\mathcal{H}_{\rm w}|K_L\rangle}{\langle\pi^+\pi^-|\mathcal{H}_{\rm w}|K_S\rangle} \equiv \eta_{+-} \equiv \epsilon + \epsilon', \qquad \frac{\langle\pi^0\pi^0|\mathcal{H}_{\rm w}|K_L\rangle}{\langle\pi^0\pi^0|\mathcal{H}_{\rm w}|K_S\rangle} \equiv \eta_{00} \equiv \epsilon - 2\epsilon', \quad (2.4)$$

where

$$\begin{aligned} \epsilon &= \bar{\epsilon} + i\xi_0, \\ \epsilon' &= \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left|\frac{A_2}{A_0}\right|(\xi_2-\xi_0) = \frac{ie^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left|\frac{A_2}{A_0}\right|\left(\frac{\text{Im } A_2}{\text{Re } A_2} - \frac{\text{Im } A_0}{\text{Re } A_0}\right). \end{aligned} \quad (2.5)$$

The expression for ϵ can be simplified by approximating the numerical value $\Delta m/\Delta\Gamma = 0.475 \pm 0.001$ by $\Delta m/\Delta\Gamma \simeq 1/2$. This yields the approximate relation,

$$\frac{i}{\Delta m + \dfrac{i}{2}\Delta\Gamma} \simeq \frac{e^{i\pi/4}}{\sqrt{2}}\frac{1}{\Delta m}, \quad (2.6)$$

which we shall use repeatedly in the analysis to follow. In addition, since the rate for $K \to \pi\pi$ is much larger than that for $K \to \pi\pi\pi$, and $K^0 \to \pi\pi$ is in turn dominated by the $I = 0$ final state because of the $\Delta I = 1/2$ rule, we have

$$\text{Im } \Gamma_{12} \simeq \xi_0\Gamma_S \simeq 2\xi_0\Delta m. \quad (2.7)$$

The above relations allow us to write

$$\begin{aligned} \epsilon &= \bar{\epsilon} + i\xi_0 \simeq \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(\frac{\text{Im } M_{12}}{\Delta m} - i\xi_0\right) + i\xi_0 \\ &= \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(\frac{\text{Im } M_{12}}{\Delta m} + \xi_0\right) = \frac{e^{i\frac{\pi}{4}}}{\sqrt{2}}\left(\frac{\text{Im } M_{12}}{2\text{Re } M_{12}} + \frac{\text{Im } A_0}{\text{Re } A_0}\right), \\ \epsilon' &= \frac{i\omega}{\sqrt{2}}e^{i(\delta_2-\delta_0)}(\xi_2-\xi_0) = \frac{i\omega e^{i(\delta_2-\delta_0)}}{\sqrt{2}}\left(\frac{\text{Im } A_2}{\text{Re } A_2} - \frac{\text{Im } A_0}{\text{Re } A_0}\right), \end{aligned} \quad (2.8)$$

where $\omega \equiv \text{Re } A_2/\text{Re } A_0 \simeq 1/22$. All CP-violating observables must involve an interference of two amplitudes. In Eq. (2.8), the quantity ϵ expresses the interference of $K^0 \to \pi\pi$ with $K^0 \to \bar{K}^0 \to \pi\pi$, while ϵ' involves interference of the $I = 0$ and $I = 2$ final states.

The formulae for ϵ and ϵ' exhibit an important theoretical property. Since the choice of phase convention for any meson M is arbitrary, its state vector may be modified by the global strangeness transformation $|M\rangle \to e^{i\lambda S}|M\rangle$. For the $\bar{K}^0$ and K^0 states, this becomes

$$|K^0\rangle \to e^{i\lambda}|K^0\rangle, \qquad |\bar{K}^0\rangle \to e^{-i\lambda}|\bar{K}^0\rangle, \quad (2.9)$$

which has the effect,

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$$\frac{\text{Im } A_I}{\text{Re } A_I} \to \frac{\text{Im } A_I}{\text{Re } A_I} + \lambda, \qquad \frac{\text{Im } M_{12}}{\text{Re } M_{12}} \to \frac{\text{Im } M_{12}}{\text{Re } M_{12}} - 2\lambda. \tag{2.10}$$

We see that the values of ϵ and ϵ' are left unchanged. Various phase conventions appear in the literature. In the *Wu–Yang* convention, λ is chosen so that the A_0 amplitude is real-valued. This is always possible to achieve by properly choosing the phase of the kaon state. However, it is inconvenient for the Standard Model, where the A_0 amplitude naturally picks up a CP-violating phase. We shall therefore employ the convention in which no such additional phases occur in the definitions of the kaon states.

It was in the $K \to \pi\pi$ system that CP violation was first observed. The current status of measurements is

$$\begin{aligned} |\epsilon| &= (2.228 \pm 0.011) \times 10^{-3}, \\ \text{Re}\left(\frac{\epsilon'}{\epsilon}\right) &= (1.66 \pm 0.26) \times 10^{-3}, \\ \varphi_{+-} \equiv \text{phase}(\eta_{+-}) &= (43.51 \pm 0.05)^\circ \\ \varphi_{00} \equiv \text{phase}(\eta_{00}) &= (43.52 \pm 0.05)^\circ. \end{aligned} \tag{2.11}$$

A violation of CP symmetry has also been observed in the semileptonic decays of K_L and K_S. These are related to matrix elements of the weak hadronic currents. Since K^0 must always decay into $e^+\nu_e\pi^-$ while $\bar{K}^0$ goes to $e^-\bar{\nu}_e\pi^+$, we have

$$\begin{aligned} A_{K_L\to\pi^-e^+\nu_e} &= \frac{1+\bar{\epsilon}}{\sqrt{2}}\, A_{K^0\to\pi^-e^+\nu_e}, \\ A_{K_L\to\pi^+e^-\bar{\nu}_e} &= \frac{1-\bar{\epsilon}}{\sqrt{2}}\, A_{\bar{K}^0\to\pi^+e^-\bar{\nu}_e}. \end{aligned} \tag{2.12}$$

If the semileptonic decays proceed as in the Standard Model, there is no direct CP violation in the transition amplitude, so that

$$\frac{\Gamma_{K_L\to\pi^-e^+\nu_e}}{\Gamma_{K_L\to\pi^+e^-\bar{\nu}_e}} = \frac{1+2\text{Re }\bar{\epsilon}}{1-2\text{Re }\bar{\epsilon}} \simeq 1 + 4\text{Re }\bar{\epsilon}. \tag{2.13}$$

Since $\text{Re }\bar{\epsilon} = \text{Re }\epsilon$, the above asymmetry is sensitive to the same parameter as appears in the $K_L \to \pi\pi$ studies. Here, measurement gives

$$\text{Re }\epsilon = (1.596 \pm 0.013) \times 10^{-3} = |\epsilon| \cos(44.3 \pm 0.8)^\circ, \tag{2.14}$$

which is consistent with the experimental values from $K \to \pi\pi$.

Finally, precision experiments also probe the CPT transformation. For example, two such predictions, involving kaon masses ($m_{K^0} = m_{\bar{K}^0}$) and phases ($\varphi_{+-} = \varphi_{00}$ up to very small corrections from ϵ'), are seen to be consistent with existing data,

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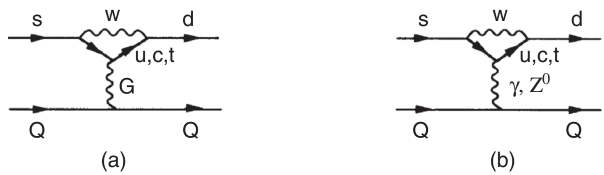

Fig. IX–4 (a) Penguin and (b) electroweak-penguin contributions to *CP* violation in $\Delta S = 1$ transitions.

$$\frac{|m_{K^0} - m_{\bar{K}^0}|}{m_{K^0}} \leq 8 \times 10^{-19},$$
$$\varphi_{00} - \varphi_{+-} = (0.2 \pm 0.4)^\circ. \qquad (2.15)$$

Further study of *CPT* invariance is left to Prob. IX–2.

IX–3 Kaon *CP* violation in the Standard Model

After diagonalization, there can remain a single phase in the CKM matrix. This phase generates the imaginary parts of amplitudes, which are required for *CP* violation. It is a physical requirement that results be invariant under rephasing of the quark fields. As a consequence, all observables must be proportional to

$$\mathrm{Im}\ \Delta^{(4)} = A^2\lambda^6\eta = c_1c_2c_3s_1^2s_2s_3s_\delta, \qquad (3.1)$$

written in the notation of Sect. II–4. This shows that all *CP*-violating signals must vanish if any of the CKM angles vanish. We shall now study the path whereby this phase is transferred from the lagrangian to experimental observables. For kaons, we have seen that the relevant amplitudes are those for K^0–$\bar{K}^0$ mixing ($\Delta S = 2$) and for $K \to \pi\pi$ decays ($\Delta S = 1$). Tree-level amplitudes in kaon decay can never be sensitive to the full rephasing invariant, so that one must consider loops. Typical diagrams are displayed in Fig. IX–4.

Experiment can help in simplifying the theoretical analysis. Note that ϵ' is sensitive to $\Delta S = 1$ physics through the penguin diagram [GiW 79], while ϵ is sensitive to $\Delta S = 2$ mass-matrix physics as well as to $\Delta S = 1$ effects. However, since experiment tells us that $|\epsilon| \gg |\epsilon'|$, it follows that the $\Delta S = 1$ contributions to ϵ must be small. Likewise, the long-distance contributions of Fig. IX–2 and the contribution of Fig. IX–1(d) must both be small because each also involves the $\Delta S = 1$ interaction. This leaves the box diagrams of Fig. IX–1(a),(b) as the dominant component of ϵ. Moreover, since the CKM phase δ is associated with the heavy-quark couplings, only the heavy-quark parts of the box diagrams are needed. Hence ϵ is very clearly short-distance dominated.

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Analysis of $|\epsilon|$

The evaluation of ϵ follows directly from Eq. (2.8). To begin, we shall ignore the tiny $\mathrm{Im}\, A_0/\mathrm{Re}\, A_0 \sim \mathcal{O}(10^{-5})$ dependence therein.[4] This leaves us with the issue of calculating $\mathrm{Im}\, M_{12}$. From the discussion of the 'box' hamiltonian $\mathcal{H}_{\mathrm{w}}^{\mathrm{box}}$ given in Sect. IX–1, we have

$$\begin{aligned}\mathrm{Im}\, M_{12} = &\frac{G_F^2}{3\pi^2} F_K^2 m_K \hat{B}_K A^2 \lambda^6 \bar{\eta} \\ &\times \left[\eta_{cc} m_c^2 H(x_c) - \eta_{tt} m_t^2 H(x_t) A^2 \lambda^4 (1-\bar{\rho}) - \eta_{ct} m_c^2 \bar{G}(x_c, x_t)\right]. \quad (3.2)\end{aligned}$$

Some CKM-related relations and definitions useful in obtaining the above form are

$$\mathrm{Re}\, \xi_c = -\lambda \left(1 - \frac{\lambda^2}{2}\right), \qquad \mathrm{Re}\, \xi_t = -\lambda \left(1 - \frac{\lambda^2}{2}\right) A^2 \lambda^4 (1-\bar{\rho}),$$
$$\mathrm{Im}\, \xi_c = -\mathrm{Im}\, \xi_t = -\eta A^2 \lambda^5,$$
$$\bar{\rho} \equiv \rho \left(1 - \frac{\lambda^2}{2}\right), \qquad \bar{\eta} \equiv \eta \left(1 - \frac{\lambda^2}{2}\right).$$

From Eq. (2.8) and Eq. (3.2), we obtain the Standard Model prediction,

$$\begin{aligned}|\epsilon|_{\mathrm{SM}} = &\frac{G_F^2}{3\sqrt{2}\pi^2} \frac{F_K^2 m_K \hat{B}_K A^2 \lambda^6 \bar{\eta}}{\Delta m_K} \\ &\times \left[\eta_{cc} m_c^2 - \eta_{tt} m_t^2 H(x_t) A^2 \lambda^4 (1-\bar{\rho}) - \eta_{ct} m_c^2 \bar{G}(x_c, x_t)\right], \quad (3.3)\end{aligned}$$

roughly in accord with the experimental value, given the uncertainties in several of the above factors, when lattice determinations of the $\hat{B}_K$ parameter are used.

Analysis of $|\epsilon'|$

The importance of ϵ' lies in the fact that it proves that *CP* violation also occurs in the direct $\Delta S = 1$ weak transition, which is a hallmark of the Standard Model's pattern of *CP* breaking. For this process, the *CP*-violating phases from the CKM elements can occur only in loop diagrams, and these appear in the penguin diagram and in the electroweak penguin process in which the gluon is replaced by a photon or a Z^0 boson, as shown in Fig. IX–4. At first sight, it appears surprising that the electroweak penguin plays any significant role, as it is suppressed by a power of α compared to the gluonic penguin. However, recall from Eq. (2.8), that ϵ' measures the relative phase difference of the $K \to \pi\pi$ amplitudes A_0 and A_2

$$|\epsilon'| = \frac{\omega}{\sqrt{2}} \left| \frac{\mathrm{Im}\, A_2}{\mathrm{Re}\, A_2} - \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0} \right| \simeq 0.032 \left| \frac{\mathrm{Im}\, A_2}{\mathrm{Re}\, A_2} - \frac{\mathrm{Im}\, A_0}{\mathrm{Re}\, A_0} \right|. \quad (3.4)$$

[4] Besides, the combination $\mathrm{Im}\, A_0/\mathrm{Re}\, A_0$ also contributes to ϵ' (as seen in Eq. (2.8)) and since $|\epsilon'| \ll |\epsilon|$, the contribution of this ratio to ϵ is presumably ignorable.

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The gluonic penguin only contributes an imaginary part to A_0 because its effect is purely $\Delta I = 1/2$. The electroweak penguin involves an extra factor of the electric charge $Q = \frac{1}{2}\lambda_3 + \frac{1}{2\sqrt{3}}\lambda_8$, which means that the corresponding operator has both $\Delta I = 1/2,\ 3/2$ components and can contribute an imaginary part to A_2. Because the real part of A_2 is much smaller than that of A_0, by a factor of $\omega \equiv \mathrm{Re}A_2/\mathrm{Re}A_0 \sim 1/22$, the effect of the electroweak penguin is enhanced by the small denominator. However, while both diagrams make important contributions, it does appear that the gluonic penguin is the larger effect.

The ingredients to ϵ' can be expressed numerically [CiFMRS 95] at the scale $\mu = 2$ GeV in the $\overline{\mathrm{MS}}$–NDR scheme as

$$\frac{\epsilon'}{\epsilon} = 2 \cdot 10^{-3} \left(\frac{\mathrm{Im}\,(V_{\mathrm{td}}^* V_{\mathrm{ts}})}{1.3 \times 10^{-3}}\right) \Big[2.0\ \mathrm{GeV}^{-3} \langle(\pi\pi)_{I=0}\,|O_6|K^0\rangle_{2\ \mathrm{GeV}}(1 - \Omega_{\mathrm{IB}}) \\ - 0.5\ \mathrm{GeV}^{-3}\langle(\pi\pi)_{I=2}\,|O_8|K^0\rangle_{2\ \mathrm{GeV}} - 0.06\Big]. \tag{3.5}$$

Here, we see the primary dependence of the gluonic penquin effect in the matrix elements of the penguin operator O_6, while the electroweak-penguin (EWP hereafter) operator is O_8. These operators refer back to the decomposition of Eq. (VIII–3.31). The factor Ω_{IB} describes isospin breaking.

As we mentioned in Sect. VIII–4, present lattice methods are able to calculate the A_2 amplitude with reasonable precision, while the isospin-zero final-state amplitude A_0 remains uncalculable. This means that the EWP contribution can be obtained, with the result [Bl *et al.* 12]

$$\mathrm{Re}\left(\frac{\epsilon'}{\epsilon}\right)_{\mathrm{EWP}} = -(6.25 \pm 0.44_{\mathrm{stat}} \pm 1.19_{\mathrm{syst}}) \times 10^{-4}, \tag{3.6}$$

which has the *opposite* sign from the experimental result and is about one third the magnitude. A chiral analysis that we will describe shortly agrees with this. This implies that the phase due to the gluonic penguin $\mathrm{Im}\,A_0/\mathrm{Re}\,A_0$ must be the larger effect and must have the same sign as the experimental determination. This seems reasonable in estimates which have been made, as discussed in [CiENPP 12]. However, it means that we do not yet have a precise prediction for ϵ' within the Standard Model.

Chiral analysis of $(\epsilon'/\epsilon)_{EWP}$

The chiral symmetry approach to low-energy hadron dynamics emphasized earlier in this book can be used to analyze the electroweak-penguin contribution to ϵ'/ϵ in the chiral limit.

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$$\lim_{p=0} \langle (\pi\pi)_{I=2} | O_8 | K^0 \rangle_\mu = -\frac{2}{F_\pi^{(0)}} \left[\frac{1}{3} \langle 0 | \mathcal{Q}_1 | 0 \rangle_\mu + \frac{1}{2} \langle 0 | \mathcal{Q}_8 | 0 \rangle_\mu \right]$$
$$\simeq -\frac{1}{F_\pi^{(0)}} \langle 0 | \mathcal{Q}_8 | 0 \rangle_\mu \qquad (3.7a)$$

where $\mathcal{Q}_8$ is the four-quark operator

$$\mathcal{Q}_8 \equiv \bar{q}\gamma^\mu \lambda^a \frac{\tau_3}{2} q \bar{q} \gamma_\mu \lambda^a \frac{\tau_3}{2} q - \bar{q}\gamma^\mu \gamma_5 \lambda^a \frac{\tau_3}{2} q \bar{q} \gamma_\mu \gamma_5 \lambda^a \frac{\tau_3}{2} q, \qquad (3.7b)$$

and, for notational simplicity, we have suppressed dependence on a second four-quark operator $\langle 0 | \mathcal{Q}_1 | 0 \rangle_\mu$ since $\langle 0 | \mathcal{Q}_8 | 0 \rangle_\mu \gg \langle 0 | \mathcal{Q}_1 | 0 \rangle_\mu$ [CiDGM 01].[5] This relation can be found either by constructing effective lagrangians or by use of the soft-pion theorem of App. B–3.

Thus, a chiral estimate of the EWP part of ϵ'/ϵ amounts to determining the vacuum matrix element of $\mathcal{Q}_8$. It turns out that such information is obtainable from the large Q^2 behavior of $V - A$ correlators measured in τ decay (cf. Sect. V–3),

$$\Delta\Pi(Q^2) \equiv \left(\Pi_{V,3} - \Pi_{A,3}\right)(Q^2). \qquad (3.8)$$

The operator-product expansion (OPE) reveals that $\Delta\Pi(Q^2)$ obeys the asymptotic behavior

$$\Delta\Pi(Q^2) \sim \frac{1}{Q^6} \left[a_6(\mu) + b_6(\mu) \ln \frac{Q^2}{\mu^2} \right] + \mathcal{O}(Q^{-8}), \qquad (3.9a)$$

where, from a two-loop study [CiDGM 01], we have

$$a_6(\mu) = 2\pi \langle 0 | \alpha_s \mathcal{Q}_8 | 0 \rangle_\mu + \frac{25}{4} \langle 0 | \alpha_s^2 \mathcal{Q}_8 | 0 \rangle_\mu + \cdots,$$
$$b_6(\mu) = -\langle 0 | \alpha_s^2 \mathcal{Q}_8 | 0 \rangle_\mu + \cdots, \qquad (3.9b)$$

where the ellipses represent higher-order terms in the OPE. Thus, the needed information (i.e. $\langle 0 | \mathcal{Q}_8 | 0 \rangle_\mu$) is contained in the large energy component of $\Delta\Pi(Q^2)$, but how can we access it? This problem has been solved in two different papers, which use two alternative approaches. In the first of these [CiDGM 01], one employs sum rules like

$$\langle 0 | \mathcal{Q}_8 | 0 \rangle_\mu = \int_0^\infty ds\, s^2 \frac{\mu^2}{s + \mu^2} \Delta\rho(s) + \cdots, \qquad (3.10)$$

where the ellipses denote contributions from $d > 6$ terms in the OPE. This approach yields a determination $\left[\epsilon'/\epsilon\right]_{\rm EWP}^{(0)} = (-22 \pm 7) \times 10^{-4}$, having a 32% uncertainty. The superscript indicates working in the chiral limit of massless u, d, s quarks. A second method [CiGM 03], which analyzes tau decay spectral functions by using a finite-energy sum rule (FESR), leads to $\left[\epsilon'/\epsilon\right]_{\rm EWP}^{(0)} = (-15.0 \pm 2.7) \times 10^{-4}$, having

[5] The effect of $\langle 0 | \mathcal{Q}_1 | 0 \rangle_\mu$ is, of course, included in the full analysis of [CiDGM 01].

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an 18% percent uncertainty. Upon including chiral corrections, the physical result $\left[\epsilon'/\epsilon\right]_{\rm EWP} = (-11.0 \pm 3.6) \times 10^{-4}$ is obtained. Together with the lattice evaluation quoted in Eq. (3.6), these evaluations firmly imply that $\left[\epsilon'/\epsilon\right]_{\rm EWP} < 0$ and that the QCD penguin effect must be large and positive in order to reproduce the experimental value for ϵ'/ϵ of Eq. (2.11).

IX–4 The strong *CP* problem

The possibility of a θ term in the QCD lagrangian raises potential problems (see Sect. III–5). For $\theta \neq 0$, QCD will in general violate parity and, even worse, time-reversal invariance. The strength of T violation (and hence, by the CPT theorem, CP violation) is known to be small, even by the standards of the weak interaction. This knowledge comes from both the observed $K_L \to 2\pi$ decay and bounds on electric dipole moments. From these it becomes clear that QCD must be T invariant to a very high degree. However, there is nothing within the Standard Model which would force the θ parameter to be small; indeed, it is a free parameter lying in the range $0 \leq \theta \leq 2\pi$. The puzzle of why $\theta \simeq 0$ in Nature is called the *strong CP problem*.

One is tempted to resolve the issue with an easy remedy first. If QCD were the only ingredient in our theory, we could remove the strong CP problem by imposing an additional discrete symmetry on the QCD lagrangian, the discrete symmetry being CP itself. This wouldn't really explain anything but would at least reduce a continuous problem to a discrete choice. In reality, this will not work for the full Standard Model since, as we have seen, the electroweak sector inherently violates CP. It would thus be improper to impose CP invariance upon the full lagrangian. Moreover, even if one could set $\theta_{\rm bare} = 0$ in QCD, electroweak radiative corrections would generate a nonzero value. These turn out to occur only at high orders of perturbation theory, and are expected to be divergent by power-counting arguments, although they have not been explicitly calculated. This divergence is not a fundamental problem because one could simply absorb $\theta_{\rm bare}$ plus the divergence into a definition of a renormalized parameter $\theta_{\rm ren}$, which could be inferred from experiment. However, we are then back to an arbitrary value of $\theta_{\rm ren}$ and to the problem of why $\theta_{\rm ren}$ is small.

The parameter $\bar{\theta}$

The situation is actually worse than this in the full Standard Model, as the quark mass matrix can itself shift the value of θ by an unknown amount. Recall that CP violation in the Standard Model arises from the Yukawa couplings between the Higgs doublet and the fermions. When the Higgs field picks up a vacuum

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expectation value, these couplings produce mass matrices for the quarks, which are neither diagonal nor *CP*-invariant. The mass matrices are diagonalized by separate left-handed and right-handed transformations, and *CP* violation is shifted to the weak mixing matrix. However, because different left-handed and right-handed rotations are generally required, one encounters an axial $U(1)$ rotation in this transformation to the quark mass eigenstates and, as discussed in Sect. III–5, this produces a shift in the value of θ. Let us determine the magnitude of this shift. Denoting by primes the original quark basis, one has the transformation to mass eigenstates given by (cf. Eqs. II–4.5,4.6)

$$\mathbf{m} = S_L^\dagger \mathbf{m}' S_R, \qquad \psi_L = S_L^\dagger \psi'_L, \qquad \psi_R = S_R^\dagger \psi'_R. \tag{4.1}$$

Here, we have combined the u and d mass matrices into a single mass matrix. Expressing $S_{L,R}$ as products of $U(1)$ and $SU(N)$ factors,

$$S_L = e^{i\varphi_L}\overline{S}_L, \qquad S_R = e^{i\varphi_R}\overline{S}_R, \tag{4.2}$$

with $\overline{S}_L$, $\overline{S}_R$ in $SU(N)$, one obtains an axial $U(1)$ transformation angle of $\varphi_R - \varphi_L$. From the discussion of Sect. III–5, this is seen to lead to a change in the θ parameter,

$$\theta \to \overline{\theta} = \theta + 2N_f\left(\varphi_L - \varphi_R\right), \tag{4.3}$$

where $N_f = 6$ for the three-generation Standard Model. However, noting that the final mass matrix $\mathbf{m}$ is purely real, we have

$$\begin{aligned}\arg(\det \mathbf{m}) = 0 &= \arg\left(\det S_L^\dagger \det \mathbf{m}' \det S_R\right) \\ &= \arg\left(\det S_L^\dagger\right) + \arg\left(\det \mathbf{m}'\right) + \arg\left(\det S_R\right) \\ &= 2N\left(\varphi_R - \varphi_L\right) + \arg\left(\det \mathbf{m}'\right),\end{aligned} \tag{4.4}$$

where we have used the $SU(N)$ property, $\det \overline{S}_R = \det \overline{S}_L = 1$. The resultant θ parameter is then

$$\overline{\theta} = \theta + \arg\left(\det \mathbf{m}'\right), \tag{4.5}$$

with $\mathbf{m}'$ being the original nondiagonal mass matrix. The real strong *CP* problem is to understand why $\overline{\theta}$ is small.

One possible solution to the strong *CP* problem occurs if one of the quark masses vanishes. In this case, the ability to shift θ by an axial transformation would allow one to remove the effect of θ by performing an axial phase transformation on the massless quark. Equivalently stated, any effect of θ must vanish if any quark mass vanishes. Unfortunately, phenomenology does not favor this solution. The u quark is the lightest, but a value $m_u \neq 0$ is favored.

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Connections with the neutron electric dipole moment

The $\overline{\theta}$ term is not the source of the observed CP violation in K decays. This can be seen because it occurs in a $\Delta S=0$ operator, and while this may ultimately generate effects in $\Delta S=1$ processes, its influence is stronger in the $\Delta S=0$ sector. In particular, it generates an electric dipole moment d_e for the neutron. Since no such dipole moment has been detected, one can obtain a bound on the magnitude of $\overline{\theta}$.

To determine the effect of $\bar{\theta}$, it is most convenient to use a chiral rotation to shift the $\overline{\theta}$ dependence back into the quark mass matrix. A small axial transformation produces the modified mass matrix

$$\mathcal{L}_{\text{mass}} = \bar{\psi}\begin{pmatrix} m_u & & \\ & m_d & \\ & & m_s \end{pmatrix}\psi + i\eta\bar{\psi}T\gamma_5\psi \equiv \bar{\psi}_L\tilde{M}\psi_R + \bar{\psi}_R\tilde{M}^\dagger\psi_L, \tag{4.6}$$

where η is a small parameter proportional to $\overline{\theta}$ having units of mass, and T is a 3×3 hermitian matrix. Consistency requires T to be proportional to the unit matrix. If this were not the case, and instead we wrote $T \equiv 1+\lambda_i T_i/2$, the effective lagrangian would start out with a term linear in the meson fields,

$$\mathcal{L}_{\text{eff}} \sim i\eta\,\mathrm{Tr}\left(TU^\dagger - UT^\dagger\right) = 2\frac{\eta}{F_\pi}\left(T_3\pi_0 + T_8\eta_8 + \cdots\right), \tag{4.7}$$

rather than the usual quadratic dependence. The vacuum would then be unstable because it could lower its energy by producing nonzero values of, say, the π_0 field. Thus, to incorporate θ-dependence without disturbing vacuum stability, one chooses $T=1$. The act of rotating away any dependence on $\overline{\theta}$ produces a nonzero value of $\arg(\det\tilde{M})$, and also determines η,

$$\begin{aligned} \overline{\theta} &= \arg(\det\tilde{M}) = \arg\left[(m_u+i\eta)\,(m_d+i\eta)\,(m_s+i\eta)\right], \\ \eta &\simeq \overline{\theta}\frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s} \qquad (\text{for small } \eta), \end{aligned} \tag{4.8}$$

such that the mass terms become

$$\begin{aligned} \mathcal{L}_{\text{mass}} &= m_u\bar{u}u + m_d\bar{d}d + m_s\bar{s}s \\ &\quad + i\overline{\theta}\frac{m_d m_d m_s}{m_u m_d + m_u m_s + m_d m_s}\left(\bar{u}\gamma_5 u + \bar{d}\gamma_5 d + \bar{s}\gamma_5 s\right). \end{aligned} \tag{4.9}$$

The last term is the CP-violating operator of the QCD sector. Note that, as expected, $\overline{\theta}$ vanishes if any quark is massless.

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A nonzero neutron electric dipole moment d_e requires both the action of the above CP-odd operator and that of the electromagnetic current,

$$d_e \bar{u}(\mathbf{p}')\sigma_{\mu\nu}q^\nu\gamma_5 u(\mathbf{p}) = \sum_I \langle n(\mathbf{p}') \left| \mathcal{L}^{CP\text{-}odd}_{\text{mass}} \right| I \rangle \frac{1}{E_n - E_I} \langle I \left| J^{\text{em}}_\mu \right| n(\mathbf{p}) \rangle, \quad (4.10)$$

where $q = p' - p$ and we have inserted a complete set of intermediate states $\{I\}$ in the neutron-to-neutron matrix element. For intermediate baryon states, the matrix elements of $\bar{\psi}\gamma_5\psi$ are dimensionless numbers of order unity and magnetic moment effects are of order the nucleon magneton, μ_n. Thus, we find for d_e,

$$d_e \simeq \bar{\theta} \frac{m_u m_d m_s}{m_u m_d + m_u m_s + m_d m_s} \frac{e\mu_n}{\Delta M}, \quad (4.11)$$

where ΔM is some typical energy denominator. Using $\Delta M = 300$ MeV, we obtain

$$d_e \sim \bar{\theta} \times 10^{-15} \text{ e-cm}. \quad (4.12)$$

Far more sophisticated methods have been used to calculate this, with results that have a spread of values [EnRV 13]. Our simple estimate is near the average. In explicit calculations, some subtlety is required because one must be sure that the evaluation correctly represents the $U(1)_A$ behavior of the theory. However, the precise value is not too important; the significant fact is that bounds on $d_e \lesssim 3 \times 10^{-26}$ e-cm require $\bar{\theta}$ to be tiny, $\bar{\theta} \lesssim 10^{-11}$.

The strong CP problem does not have a good resolution within the Standard Model. It would appear that the abnormally small value of $\bar{\theta}$, and of the cosmological constant as well, are indications that more physics is required beyond that contained in the Standard Model.

Problems

(1) **Strangeness gauge invariance**

(a) Physics must be invariant under a global strangeness transformation $|M\rangle \to \exp(i\lambda S)|M\rangle$, where λ is arbitrary. Explain why this is the case.

(b) Demonstrate that such a transformation has the effect

$$\frac{\text{Im } A_I}{\text{Re } A_I} \to \frac{\text{Im } A_I}{\text{Re } A_I} + \lambda, \quad \frac{\text{Im } M_{12}}{\text{Re } M_{12}} \to \frac{\text{Im } M_{12}}{\text{Re } M_{12}} - 2\lambda,$$

as claimed in Eq. (2.10), and that, while unphysical quantities such as $\bar{\epsilon}$, ξ_0 are affected by such a change, physical parameters such as ϵ, ϵ' are not.

(2) **Neutral kaon mass matrices and *CPT* invariance**

Some of the ideas discussed in this chapter can be addressed in terms of simple models of the neutral kaon mass matrix M which appears in Eq. (1.2).

(a) Consider the following *CP*-conserving parameterization as defined in the $(K^0, \bar{K}^0)$ basis:

$$M_0 = \begin{pmatrix} m_0 & \Delta \\ \Delta & m_0 \end{pmatrix},$$

where Δ is real-valued. Determine the basis states (K_-, K_+) in which $M_0 \to M_\pm$ becomes diagonal and obtain numerical values for m_0, Δ.

(b) Working in the (K_-, K_+) basis, extend the model of (a) to allow for *CP* violation by introducing a real-valued parameter δ,

$$M_\pm = \begin{pmatrix} m_- & 0 \\ 0 & m_+ \end{pmatrix} \quad \to \quad M_\pm{}' = \begin{pmatrix} m_- & -i\delta \\ i\delta & m_+ \end{pmatrix},$$

and assume there is no direct *CP* violation. This mass matrix corresponds to the *superweak* (SW) model. By expressing $M_\pm{}'$ in the $(K^0, \bar{K}^0)$ basis, use the analysis of Sects. IX–1,2 to predict $\varphi_\epsilon^{(\mathrm{SW})} \equiv$ phase ϵ and determine δ from the measured value of $|\epsilon|$.

(c) Finally, extend the model in (b) to

$$M_\pm{}'' = \begin{pmatrix} m_- & \chi \\ \chi^* & m_+ \end{pmatrix},$$

where Re χ is a T-conserving, *CP*-violating, and *CPT*-violating parameter. Show that the states which diagonalize $M_\pm{}''$ are

$$|K_S\rangle \;\simeq\; |K_+\rangle \;-\; \frac{\chi}{\mathcal{D}}|K_-\rangle,$$
$$|K_L\rangle \;\simeq\; |K_-\rangle \;+\; \frac{\chi^*}{\mathcal{D}}|K_+\rangle,$$

where $\mathcal{D} \equiv (m_L - m_S)/2 + i\Gamma_S/4$. Then evaluate η_{+-} and η_{00}, allowing for the presence of direct *CP* violation (i.e. $\epsilon' \neq 0$), and derive the following relation between phases,

$$|\chi|\left(\frac{2}{3}\varphi_{+-} + \frac{1}{3}\varphi_{00} - \varphi_\epsilon^{(\mathrm{SW})}\right) = \frac{1}{2m_{K^0}} \cdot \frac{|m_{\overline{K}^0} - m_{K^0}|}{m_L - m_S} \sin\varphi_\epsilon^{(\mathrm{SW})}.$$

The result $|m_{\overline{K}^0} - m_{K^0}|/m_{K^0} < 5 \times 10^{-18}$, which follows from this relation, is the best existing limit on *CPT* invariance.

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