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Strong Digraph Groups

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Abstract. **A digraph group is a group defined by non-empty presentation with the property that each relator is of the form $R(x, y)$, where x and y are distinct generators and $R(\cdot,\cdot)$ is determined by some fixed cyclically reduced word $R(a, b)$ that involves both a and b. Associated to each such presentation is a digraph whose vertices correspond to the generators and whose arcs correspond to the relators. In this paper we consider digraph groups for strong digraphs that are digon-free and triangle-free. We classify when the digraph group is finite and show that in these cases it is cyclic, giving its order. We apply this result to the Cayley digraph of the generalized quaternion group, to circulant digraphs, and to cartesian and direct products of strong digraphs.**

1 Introduction

Given a finite digraph Γ with vertex set $V(\Gamma)$ and arc set $A(\Gamma)$ (without loops or parallel arcs) and an element $R(a, b)$ in the free group of rank 2 generated by a and b, the *digraph group* $G_\Gamma(R)$ is the group defined by the presentation

$$P_\Gamma(R) = \langle x_v \ (v \in V(\Gamma)) \mid R(x_u, x_v) \ ([u, v] \in A(\Gamma))\rangle.$$

These groups were introduced in [7] and contain the *graph groups* or *right angled Artin groups* as special cases (by setting $R(a, b) = aba^{-1}b^{-1}$). Formal definitions of undefined terms of the introduction are given in Section 2.

Digraph groups with balanced presentations (i.e. digraph groups corresponding to digraphs with an equal number of vertices and arcs) were considered in [7] and digraph groups corresponding to digraphs with one more arc than vertices were considered in [5]. Two families of finite, non-cyclic, digraph groups were obtained in [6, Corollaries A2, B2]. In this article we consider digraph groups $G_\Gamma(R)$ where Γ is a strong digraph that is digon-free and triangle-free. We state our results in terms of the period of the digraph (that is, the greatest common divisor of the lengths of its cycles), the one-relator group $K = \langle a, b \mid R(a, b)\rangle$, and integers α, β, which denote the exponent sums of a and of b^{-1} in $R(a, b)$, respectively.

In this context, [17, Theorem 3] concerns digraph groups $G_\Gamma(R)$ where Γ is a cycle of length at least 4, and can be expressed as follows:

***Theorem* 1.1** *([17]) Let Γ be a cycle of length $n \geq 4$ and let $R(a, b)$ be a cyclically reduced word that involves both a and b with exponent sums α and β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b)\rangle$. Then $G_\Gamma(R)$ is finite if and only if $\alpha \neq 0$, $\beta \neq 0$, $\gcd\{\alpha, \beta\} = 1$, $\alpha^n \neq \beta^n$, $a^\alpha = b^\beta$ in K, in which case $G_\Gamma(R)$ is cyclic of order $|\alpha^n - \beta^n|$.*

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When Γ is a cycle the digraph group $G_\Gamma(R)$ is an example of a *cyclically presented group* [12, Chapter III, Section 9]. As observed in [17], Theorem 1.1 cannot be directly extended to the cases $n = 2$ and $n = 3$; that is, to the cases where Γ is neither digon-free nor triangle-free. Examples that demonstrate this include the Macdonald groups $Mac(a, a)$ which, for $a \in \mathbb{Z}$, $a \neq 0, 1, 2$, are finite of rank 2 [13],[16]; the Fox groups $\langle x, y \mid xy^n = y^l x, yx^n = x^l y\rangle$ which are finite of rank 2 if $(n, l) = 1$, $n \neq l$ [4]; the Mennicke groups $M(q, q, q)$, which are finite of rank 3 for each $q \geq 3$ [15]; and the Johnson groups $J(q, q, q)$ which are finite of rank 3 for each even $q \geq 2$ [11],[12, page 70]. A strong digraph is one in which there is a path joining every pair of vertices, and so the cycle of length n is a strong digraph of period n. In this article we generalize Theorem 1.1 by replacing the cycle of length at least 4 by a non-trivial, strong digraph that is digon-free and triangle-free. Our main result is the following:

***Theorem* 1.2** *Let Γ be a non-trivial, strong digraph of period p that is digon-free and triangle-free and let $R(a, b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b)\rangle$. Then $G_\Gamma(R)$ is finite if and only if $\alpha \neq 0$, $\beta \neq 0$, $\gcd\{\alpha, \beta\} = 1$, $\alpha^p \neq \beta^p$, $a^\alpha = b^\beta$ in K, in which case $G_\Gamma(R)$ is cyclic of order $|\alpha^p - \beta^p|$.*

In Section 5 we apply this to the Cayley digraph of the generalized quaternion group, to circulant digraphs, and to cartesian and direct products of strong digraphs.

2 Preliminaries

A *digraph* (or *directed graph*) Γ consists of a finite set $V(\Gamma)$ of *vertices* and a set $A(\Gamma)$ of *arcs*, which are ordered pairs $[u, v]$ of distinct vertices (as such, Γ does not contain parallel arcs or loops); it is *non-trivial* if it has at least two vertices. The *underlying graph* of Γ is the graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$ consisting of all unordered pairs $\{u, v\}$ (*edges*), where $[u, v] \in A(\Gamma)$. A *digon* is a subdigraph of Γ consisting of vertices $u, v \in V(\Gamma)$ and arcs $[u, v], [v, u] \in A(\Gamma)$. A *triangle* is a subdigraph of Γ consisting of vertices $u, v, w \in V(\Gamma)$ and either arcs $[u, v], [v, w], [w, u] \in A(\Gamma)$ or arcs $[u, v], [v, w], [u, w] \in A(\Gamma)$. A digraph is said to be *digon-free* (resp. *triangle-free*) if it contains no digons (resp. triangles). A *tournament* is a digraph Γ in which exactly one of $[u, v], [v, u] \in A(\Gamma)$ for each pair of vertices $u, v \in V(\Gamma)$. A *walk* (of *length* $n - 1$) in a digraph Γ is a collection of vertices $v_1, v_2, \ldots, v_n$ such that $[v_i, v_{i+1}] \in A(\Gamma)$ for each $1 \leq i < n$; it is *closed* if $v_n = v_1$. We denote such a walk $v_1 \to v_2 \to \cdots \to v_n$. A *path* is a walk in which the vertices are distinct. A *cycle* (of *length* n) is a collection of distinct vertices $v_1, v_2, \ldots, v_n$ such that $[v_i, v_{i+1}] \in A(\Gamma)$ for each $1 \leq i < n$ and $[v_n, v_1] \in A(\Gamma)$. The *period* $p(\Gamma)$ of a digraph Γ is the greatest common divisor of the lengths of its cycles, and so $p(\Gamma)$ is equal to the greatest common divisor of the lengths of its closed walks. A digraph Γ is *strong* (or *strongly connected*) if for each $u, v \in V(\Gamma)$ there is a path from u to v (and hence, also a path from v to u); it follows that every vertex of a non-trivial, strong digraph is contained in some cycle [9, page 64]. A *weak component* (or *weakly connected component*) of a digraph Γ is a maximal subdigraph of Γ whose underlying graph is connected. A digraph is *weak* (or *weakly connected*) if it has exactly one weak component. A digraph Γ is *bipartite* if there is a *vertex partition* $V(\Gamma) =$

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$V_1 \cup V_2$ such that if $[u, v] \in A(\Gamma)$ then either $u \in V_1$ and $v \in V_2$ or $u \in V_2$ and $v \in V_1$. A strong digraph is bipartite if and only if it has no odd length cycles [9, Theorem 6.14] that is, if and only if its period is even.

The *cartesian product* $\Gamma_1 \square \Gamma_2$ of digraphs Γ_1, Γ_2 is the digraph Γ with $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$ and $A(\Gamma) = \{[(u, u'), (v, v')] \mid [u, v] \in A(\Gamma_1), u' = v' \text{ or } [u', v'] \in A(\Gamma_2), u = v\}$. The *direct product* $\Gamma_1 \times \Gamma_2$ of digraphs Γ_1, Γ_2 is the digraph Γ with $V(\Gamma) = V(\Gamma_1) \times V(\Gamma_2)$ and $A(\Gamma) = \{[(u, u'), (v, v')] \mid [u, v] \in A(\Gamma_1) \text{ and } [u', v'] \in A(\Gamma_2)\}$. Both graph products $\square$ and $\times$ are associative [8].

Given a finite group G with generating set S that does not contain the identity of G, the *Cayley digraph* $\mathrm{Cay}(G, S)$ is the digraph Γ with $V(\Gamma) = G$ and arc set

$$A(\Gamma) = \{[g, h] \mid h = gs \text{ for some } s \in S\}.$$

Thus, every finite Cayley digraph is strong. The *circulant digraph* $\mathrm{circ}_n\{d_1, \dots, d_t\}$, where $n \geq 2$ and $1 \leq d_1, \dots, d_t < n$ are distinct integers, is the digraph Γ with $V(\Gamma) = \{0, 1, \dots, n-1\}$ and arcs $[i, i + d_j]$ for each $0 \leq i < n$, $1 \leq j \leq t$ (where the entries are taken mod n) [1]. It is weakly connected if and only if $\gcd\{n, d_1, \dots, d_t\} = 1$, in which case it is strongly connected and is the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_n, \{d_1, \dots, d_t\})$.

3 Digraph groups

First we observe that if Γ has weakly connected components $\Gamma_1, \dots, \Gamma_k$ with $k \geq 2$, then the presentation $P_\Gamma(R)$ decomposes as the disjoint union of the presentations $P_{\Gamma_1}(R), \dots, P_{\Gamma_k}(R)$ so $G_\Gamma(R)$ is isomorphic to the free product $G_{\Gamma_1}(R) * \cdots * G_{\Gamma_k}(R)$. Therefore, without loss of generality, in considering digraph groups $G_\Gamma(R)$ we may assume that Γ is weak.

***Lemma* 3.1** *([17]) Let Γ be a non-trivial, weak digraph that is digon-free and triangle-free and let $R(a, b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b)\rangle$. If $G_\Gamma(R)$ is finite then $\alpha \neq 0, \beta \neq 0$ and $a^\alpha = b^\beta$ in K.*

Proof By [17, Theorem 4], if K has Property W_1 (see [17] for the definition) then $G_\Gamma(R)$ is infinite, so we may assume that K does not satisfy Property W_1. Then by [17, Proposition, page 248], we have $\alpha \neq 0, \beta \neq 0$ and $a^\alpha = b^\beta$ in K (see [7, page 5] for further discussion on this point). ■

***Lemma* 3.2** *Let Γ be a digraph and let $R(a, b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b)\rangle$. If $a^\alpha = b^\beta$ in K then $G_\Gamma(R)$ is a quotient of $G_\Gamma(a^\alpha b^{-\beta})$ so, in particular, if $G_\Gamma(a^\alpha b^{-\beta})$ is abelian then $G_\Gamma(R) \cong G_\Gamma(a^\alpha b^{-\beta})$.*

Proof Since $a^\alpha = b^\beta$ in K, the presence of a relator $R(x_u, x_v)$ in $P_\Gamma(R)$ implies that the relation $x_u^\alpha = x_v^\beta$ holds in $G_\Gamma(R)$. Therefore the corresponding relators $x_u^\alpha x_v^{-\beta}$ can be added to the defining presentation for $G_\Gamma(R)$, so $G_\Gamma(R)$ is a quotient of $G_\Gamma(a^\alpha b^{-\beta})$. If $G_\Gamma(a^\alpha b^{-\beta})$ is abelian, then $G_\Gamma(R)$ is abelian and so the relators $R(x_u, x_v)$ are equivalent to the relators $x_u^\alpha x_v^{-\beta}$, so can be removed; that is $G_\Gamma(R) \cong G_\Gamma(a^\alpha b^{-\beta})$. ■

Lemma 3.3 *([17])* *Let Γ be a digraph that is not a tournament, and let $R(a,b)$ be a cyclically reduced word which has exponent sums α, β in a and b^{-1}, respectively. If $G_\Gamma(R)$ is finite then* $\gcd\{\alpha,\beta\} = 1$.

Proof Since Γ is not a tournament there exists a pair of vertices $u, v \in V(\Gamma)$ that are not connected by an arc. As in [17, page 248] (or [7, Proof of Lemma 3.3]), killing all generators of $G_\Gamma(R)$ except x_u, x_v and then adjoining relators x_u^d, x_v^d, where $d = \gcd\{\alpha,\beta\}$, gives that $G_\Gamma(R)$ maps onto the group $\langle x_u, x_v \mid x_u^d, x_v^d\rangle \cong \mathbb{Z}_d * \mathbb{Z}_d$, which is infinite if $d > 1$. ■

Lemma 3.4 *Let Γ be a digraph and let $R(a,b)$ be a cyclically reduced word which has exponent sums α, β in a and b^{-1}, respectively. Then $G_\Gamma(R)$ maps epimorphically to $\mathbb{Z}_{|\alpha-\beta|}$. In particular, if $\alpha = \beta$ then $G_\Gamma(R)$ is infinite.*

Proof Let $\phi : G_\Gamma(R) \to \mathbb{Z}_{|\alpha-\beta|}$ be given by $\phi(x_v) = 1 \in \mathbb{Z}_{|\alpha-\beta|}$ for each $v \in V(\Gamma)$. Then for each relator $R(x_u, x_v)$ of $G_\Gamma(R)$ we have $\phi(R(x_u,x_v)) = \alpha \cdot 1 - \beta \cdot 1 = 0 \in \mathbb{Z}_{|\alpha-\beta|}$ so ϕ is a homomorphism; and since $\phi(x_v) = 1$ for some $v \in V(\Gamma)$, it is an epimorphism. ■

Lemma 3.5 *Let Γ be a bipartite digraph and let $R(a,b)$ be a cyclically reduced word which has exponent sums α, β in a and b^{-1}, respectively, and suppose* $\gcd\{\alpha,\beta\} = 1$*. Then $G_\Gamma(R)$ maps epimorphically to $\mathbb{Z}_{|\alpha^2-\beta^2|}$. In particular, if $\alpha = \pm\beta$ then $G_\Gamma(R)$ is infinite.*

Proof Suppose Γ has vertex partition $V(\Gamma) = V_1 \uplus V_2$. Let $\phi : G_\Gamma(R) \to \mathbb{Z}_{|\alpha^2-\beta^2|}$ be given by $\phi(x_u) = \alpha$ if $u \in V_1$ and $\phi(x_u) = \beta$ if $u \in V_2$. Then given an arc $[u, v]$ of $G_\Gamma(R)$, we have $\phi(R(x_u,x_v)) = \alpha^2 - \beta^2 = 0$ if $u \in V_1$ and $\phi(R(x_u,x_v)) = \alpha\beta - \beta\alpha = 0$ if $u \in V_2$. Thus ϕ is a homomorphism. Since $\gcd\{\alpha,\beta\} = 1$, there exist $r, s \in \mathbb{Z}$ such that $r\alpha + s\beta = 1$, and so if $u \in V_1, v \in V_2$ then $\phi(x_u^r x_v^s) = r\alpha + s\beta = 1 \in \mathbb{Z}_{|\alpha^2-\beta^2|}$, so ϕ is an epimorphism. ■

Lemma 3.6 *Let Γ be a non-trivial, weak digraph that is digon-free and triangle-free and let $R(a,b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and suppose $\alpha = -\beta$. Then $G = G_\Gamma(R)$ is finite if and only if $\alpha = -\beta = \pm 1$, $a^\alpha = b^\beta$ in K, and Γ is not bipartite, in which case $G \cong \mathbb{Z}_2$.*

Proof Suppose G is finite. Then Lemma 3.1 implies $\alpha \neq 0, \beta \neq 0$ and $a^\alpha = b^\beta$ in K, and Lemma 3.3 implies $\gcd\{\alpha,\beta\} = 1$, so $\alpha = -\beta = \pm 1$. Moreover, Γ is not bipartite by Lemma 3.5. Under these conditions $a = b^{-1}$ in K and so, given an arc $[u, v] \in A(\Gamma)$, we have $x_u = x_v^{-1}$ in G. Fix a vertex $w \in V(\Gamma)$. Then, since the underlying graph of Γ is connected, for any $v \in V(\Gamma)$ we have either $x_v = x_w$ or $x_v = x_w^{-1}$. Therefore G is cyclic. Since Γ is not bipartite there is an odd length cycle $1 \to 2 \to 3 \to \cdots \to r \to 1$, say. Then $x_1 = x_2^{-1} = x_3 = \cdots = x_{r-1}^{-1} = x_r = x_1^{-1}$, so $x_1^2 = 1$. Moreover, each generator of G is equal to any other generator or its inverse, so G is generated by x_1, which satisfies the relation $x_1^2 = 1$; that is, G is cyclic of order at most 2. Then, by Lemma 3.4, G maps onto $\mathbb{Z}_2$, so $G \cong \mathbb{Z}_2$. ■

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By Lemma 3.6 we may assume $\alpha \neq -\beta$. In this situation we summarize Lemmas 3.1 – 3.4 in the following result.

***Theorem* 3.7** *Let Γ be a non-trivial, weak digraph that is digon-free and triangle-free and let $R(a,b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, where $\alpha \neq -\beta$, and let $K = \langle a, b \mid R(a,b)\rangle$. If $G_\Gamma(R)$ is finite then $\alpha \neq 0, \beta \neq 0$, $\alpha \neq \beta$, $\gcd\{\alpha,\beta\} = 1$ and $a^\alpha = b^\beta$ in K, in which case $G_\Gamma(R)$ is a quotient of $G_\Gamma(a^\alpha b^{-\beta})$ so, in particular, if $G_\Gamma(a^\alpha b^{-\beta})$ is abelian then $G_\Gamma(R) \cong G_\Gamma(a^\alpha b^{-\beta})$.*

4 Strong digraph groups

In this section we prove Theorem 1.2.

***Lemma* 4.1** *Let Γ be a digraph and let $G = G_\Gamma(a^\alpha b^{-\beta})$ where $\gcd\{\alpha,\beta\} = 1$. If v is a vertex of a cycle of length Δ in Γ then $x_v^{|\alpha^\Delta - \beta^\Delta|} = 1$ in G.*

Proof This is essentially stated implicitly in [17, page 248], and proofs are given in [7, Lemma 3.4], [3, Lemma 3.4]. ■

***Lemma* 4.2** *Let Γ be a digraph and let $G = G_\Gamma(a^\alpha b^{-\beta})$ where $\gcd\{\alpha,\beta\} = 1$. Let $u \in V(\Gamma)$ be a vertex of some cycle, and suppose there is a path from u to a vertex w. Then the generator x_u of G is equal to a power of the generator x_w.*

Proof Label the path from u to w as $u = 1 \to 2 \to 3 \to \cdots \to (n-1) \to n = w$. First consider the arc $[1,2]$. Since 1 is the vertex of some cycle, of length Δ, say, Lemma 4.1 implies $x_1^\gamma = 1$, where $\gamma = |\alpha^\Delta - \beta^\Delta|$. Now $\gcd\{\alpha,\gamma\} = 1$ so there exist $r, s \in \mathbb{Z}$ such that $r\alpha + s\gamma = 1$, so $r\alpha \equiv 1 \bmod \gamma$. Moreover, there is a relation $x_1^\alpha = x_2^\beta$ in Γ so

$$x_1 = x_1^{r\alpha + s\gamma} = (x_1^\alpha)^r (x_1^\gamma)^s = (x_2^\beta)^r = x_2^{r\beta}.$$

Therefore x_1 is equal to a power of x_2. Moreover $x_1^\gamma = 1$ so $(x_2^{r\beta})^\gamma = 1$, i.e. $x_2^{r\beta\gamma} = 1$. But also $x_1^\alpha = x_2^\beta$ so $(x_2^{r\beta})^\alpha = x_2^\beta$, so $x_2^{(1-r\alpha)\beta} = 1$, i.e. $x_2^{s\beta\gamma} = 1$. Thus $x_2^{(r\beta\gamma, s\beta\gamma)} = 1$, i.e. $x_2^{|\beta\gamma|} = 1$. (The argument in this paragraph is that of [7, proof of Lemma 3.1].)

Now consider the arc $[2,3]$. Let $\gamma' = \beta\gamma$. Noting that $\gcd\{\alpha,\gamma'\} = 1$, repeating the argument of the previous paragraph gives that x_2 is equal to a power of x_3 and $x_3^{|\beta\gamma'|} = 1$; that is, $x_3^{|\beta^2\gamma|} = 1$. Continuing in this way, we see that for each $1 \leq i < n$ the generator x_i is equal to a power of x_{i+1} and $x_i^{|\beta^{i-1}\gamma|} = 1$. Therefore x_1 is equal to a power of x_n; that is, x_u is equal to a power of x_w, as required. ■

***Lemma* 4.3** *Let Γ be a non-trivial, strong digraph of period p and let $\alpha, \beta \in \mathbb{Z}$ satisfy $\alpha \neq 0$, $\beta \neq 0$, $\alpha \neq \pm\beta$, $\gcd\{\alpha,\beta\} = 1$. Then $G_\Gamma(a^\alpha b^{-\beta})$ is finite and cyclic, of order dividing $|\alpha^p - \beta^p|$.*

Proof Fix a vertex $w \in V(\Gamma)$. Since Γ is non-trivial and strong, for every vertex $u \in V(\Gamma)$, u is a vertex of some cycle and there is a walk from u to w. Therefore, by Lemma 4.2, the generator x_u is equal to some power of x_w. Therefore every generator of G is equal to some power of x_w, so G is cyclic, generated by x_w. Since the choice of w was arbitrary, G is cyclic and generated by x_v for any $v \in V(\Gamma)$ and, by Lemma 4.1, if Δ is the length of any cycle of which v is a vertex, $x_v^{|\alpha^\Delta - \beta^\Delta|} = 1$ in G. That is, G is cyclic of order dividing $f(\Delta) = |\alpha^\Delta - \beta^\Delta|$. Applying this observation to every vertex $v \in V(\Gamma)$ and every cycle of Γ of which v is a vertex, we see that G is cyclic of order dividing $\gcd\{f(\Delta) \mid \Delta \text{ is the length of some cycle of } \Gamma\}$. That is, G is cyclic of order dividing $|\alpha^p - \beta^p|$. ∎

Lemma 4.4 *Let Γ be a strong digraph of period $p \geq 2$, let Λ be the cycle of length p and let $R(a, b)$ be a cyclically reduced word. Then $G_\Gamma(R)$ maps epimorphically to $G_\Lambda(R)$.*

Proof (In this proof, subscripts of v_*, V_* and y_* terms are to be taken mod$p$.) By [2, Theorem 10.5.1] there exists a vertex partition $V(\Gamma) = V_0 \cup V_1 \cup \cdots \cup V_{p-1}$ such that if $[u, v] \in A(\Gamma)$ then $u \in V_i$ and $v \in V_{i+1}$ for some $0 \leq i < p$. By adjoining all relators $R(x_{v_i}, x_{v_{i+1}})$, where $v_i \in V_i, v_{i+1} \in V_{i+1}$ to the defining presentation for $G = G_\Gamma(R)$ we see that G maps onto

$$H = \langle x_{v_i}\ (v_i \in V_i, 0 \leq i < p) \mid R(x_{v_i}, x_{v_{i+1}})\ (v_i \in V_i, v_{i+1} \in V_{i+1}, 0 \leq i < p)\rangle.$$

For each $0 \leq i < p$, equating all generators x_{v_i} $(v_i \in V_i)$, and introducing generators $y_i = x_{v_i}$, we see that H maps onto

$$\begin{aligned} K &= \left\langle \begin{array}{l} x_{v_i}\ (v_i \in V_i, 0 \leq i < p), \\ y_i\ (0 \leq i < p) \end{array} \middle| \begin{array}{l} R(x_{v_i}, x_{v_{i+1}})\ (v_i \in V_i, v_{i+1} \in V_{i+1}, 0 \leq i < p), \\ y_i = x_{v_i}\ (v_i \in V_i, 0 \leq i < p) \end{array} \right\rangle \\ &= \langle y_i\ (0 \leq i < p) \mid R(y_i, y_{i+1})\ (0 \leq i < p)\rangle \\ &= G_\Lambda(R) \end{aligned}$$

where $V(\Lambda) = \{0, 1, \ldots, p-1\}$ and $A(\Lambda) = \{[i, i+1] \mid 0 \leq i < p\}$. ∎

We can now prove Theorem 1.2.

Proof Suppose first $\alpha = -\beta$. Then by Lemma 3.6 $G = G_\Gamma(R)$ is finite if and only if $\alpha = \pm 1$, Γ is not bipartite, and $a^\alpha = b^\beta$ in K, in which case $G \cong \mathbb{Z}_2$. Equivalently, G is finite if and only if $\alpha \neq 0, \beta \neq 0, \gcd\{\alpha, \beta\} = 1, \alpha^p \neq \beta^p$, and $a^\alpha = b^\beta$ in K, in which case $G \cong \mathbb{Z}_{|\alpha^p - \beta^p|}$.

Suppose then $\alpha \neq -\beta$. If G is finite then Theorem 3.7 implies $\alpha \neq 0, \beta \neq 0, \alpha \neq \beta$ (equivalently, $\alpha^p \neq \beta^p$), $\gcd\{\alpha, \beta\} = 1$ and $a^\alpha = b^\beta$ in K. Under these conditions Lemma 4.3 implies that $G_\Gamma(a^\alpha b^{-\beta})$ is finite and cyclic of order dividing $\gamma = |\alpha^p - \beta^p|$. Then $G \cong G_\Gamma(a^\alpha b^{-\beta})$ by Theorem 3.7. We now show that G maps epimorphically to $\mathbb{Z}_\gamma$. If $p = 1$ this follows from Lemma 3.4, so assume $p \geq 2$. By Lemma 4.4 $G = G_\Gamma(a^\alpha b^{-\beta})$ maps epimorphically to $G_\Lambda(a^\alpha b^{-\beta})$, where Λ is the cycle of length p. But $G_\Lambda(a^\alpha b^{-\beta}) \cong \mathbb{Z}_\gamma$, by [17, page 248] (or [7, Lemma 3.4], [3, Lemma 3.4]). Thus $G \cong \mathbb{Z}_\gamma$, as required. ∎

5 Applications

In this section we obtain corollaries to Theorem 1.2 for particular types of strong digraphs.

5.1 A Cayley digraph

We consider the Cayley digraph of the *generalized quaternion group*

$$Q_{2n} = \langle c, d \mid c^{2n}, c^n d^{-2}, cdcd^{-1} \rangle$$

with respect to the generating set $\{c, d\}$.

***Lemma* 5.1** *Let $\Gamma = \mathrm{Cay}(Q_{2n}, \{c, d\})$ where $n \geq 2$. Then Γ is digon-free and triangle-free and the period $p(\Gamma) = \gcd\{n, 2\}$.*

Proof Since $c^2, d^2, c^3, d^3, c^2d^{\pm 1}, d^2c^{\pm 1} \neq e$ in Q_{2n} (where e denotes the identity) the digraph Γ is digon-free and triangle-free. It contains the following cycles:

- $e \to d \to d^2 \to d^3 \to d^4 = c^{2n} = e$,
- $e \to c \to c^2 \to \cdots \to c^n \to c^n d \to c^n d^2 = c^n c^n = c^{2n} = e$,
- $e \to c \to c^2 \to \cdots \to c^{2n-1} \to c^{2n-1}d \to c^{2n-1}dc = c^{2n-2}d \to c^{2n-2}dc = c^{2n-3}d \to \cdots \to c^n d \to c^n d^2 = c^n c^n = c^{2n} = e$,

of lengths $4, n+2, 3n$, respectively, and so $p = p(\Gamma)$ divides $\gcd\{4, n+2, 3n\}$, so $p = 1$ if n is odd and $p|2$ if n is even. If n is even then Γ is bipartite with vertex partition

$$V(\Gamma) = \{c^i, c^i d \mid \text{where } i \text{ is even}\} \cup \{c^i, c^i d \mid \text{where } i \text{ is odd}\}$$

and so Γ has no odd length cycles and hence $p = 2$. ∎

Noting that Cayley digraphs are strong, we obtain the following.

***Corollary* 5.2** *Let $\Gamma = \mathrm{Cay}(Q_{2n}, \{c, d\})$ where $n \geq 2$, let $p = \gcd\{n, 2\}$, and let $R(a, b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b) \rangle$. Then $G = G_\Gamma(R)$ is finite if and only if $\alpha \neq 0, \beta \neq 0$, $\gcd\{\alpha, \beta\} = 1$, $\alpha^p \neq \beta^p$, $a^\alpha = b^\beta$ in K, in which case G is cyclic of order $|\alpha^p - \beta^p|$.*

5.2 Circulant digraphs

***Lemma* 5.3** *Let $\Gamma = \mathrm{circ}_n\{d_1, \ldots, d_t\}$, where $n \geq 2$ and $1 \leq d_i < n$ for each $1 \leq i \leq t$, and where $\gcd\{n, d_1, \ldots, d_t\} = 1$. Then $p(\Gamma) = \gcd\{n, d_1 - d_2, d_1 - d_3, \ldots, d_1 - d_t\}$.*

Proof Let $p = p(\Gamma)$ and $r = \gcd\{n, d_1 - d_2, d_1 - d_3, \ldots, d_1 - d_t\}$. Observe that

$$(n - d_2)d_1 + d_1 d_2 + 0d_3 + \cdots + 0d_t \equiv 0 \bmod n$$

so there is a closed walk with $(n - d_2)$ arcs corresponding to the generator d_1 and d_1 arcs corresponding to the generator d_2 and 0 arcs corresponding to the remaining generators

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(and so has length $n + d_1 - d_2$). Similarly there are closed walks of lengths $n + d_1 - d_3, \dots, n+d_1-d_t$. Also, there is a closed walk of length n, consisting of d_1 arcs. Therefore p divides $\gcd\{n, n+d_1-d_2, n+d_1-d_3, \dots, n+d_1-d_t\} = r$.

Consider a cycle of length l which has l_i arcs corresponding to the generator d_i, for each $1 \le i \le t$. Then $l = l_1 + \dots + l_t$ and $\sum_{j=1}^t l_j d_j \equiv 0 \bmod n$, and hence $\sum_{j=1}^t l_j d_j \equiv 0 \bmod r$ so (since $d_j \equiv d_1 \bmod r$ for each $1 \le j \le t$) we have $\sum_{j=1}^t l_j d_1 \equiv 0 \bmod r$. That is, $l d_1 \equiv 0 \bmod r$. Now if $\delta | r$ and $\delta | d_1$ then $\delta | d_2, \dots, d_t$ so $\delta | (n, d_1, \dots, d_t) = 1$, so $\delta = 1$. Thus $\gcd\{r, d_1\} = 1$, and so $l \equiv 0 \bmod r$. Hence r divides the length of any cycle in Γ, and so r divides p. Hence $p = r$. ■

Noting that $\mathrm{circ}_n\{d_1, \dots, d_t\}$ is digon-free and triangle-free if and only if $d_i + d_j \not\equiv 0$ and $d_i + d_j + d_k \not\equiv 0 \bmod n$ for each $1 \le i, j, k \le t$, and that circulant digraphs are Cayley digraphs, and hence strong, we obtain the following.

***Corollary* 5.4** *Let $\mathrm{circ}_n\{d_1, \dots, d_t\}$, where $n \ge 4$ and $1 \le d_i < n$ for each $1 \le i \le t$, and where $\{d_1, \dots, d_t\}$ is a generating set for $\mathbb{Z}_n$ and suppose $d_i + d_j \not\equiv 0$ and $d_i + d_j + d_k \not\equiv 0 \bmod n$ for each $1 \le i, j, k \le t$, and let $p = \gcd\{n, d_1 - d_2, d_1 - d_3, \dots, d_1 - d_t\}$. Let $R(a,b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a,b) \rangle$. Then $G = G_\Gamma(R)$ is finite if and only if $\alpha \neq 0, \beta \neq 0$, $\gcd\{\alpha, \beta\} = 1$, $\alpha^p \neq \beta^p$, $a^\alpha = b^\beta$ in K, in which case G is cyclic of order $|\alpha^p - \beta^p|$.*

5.3 Cartesian products of strong digraphs

***Lemma* 5.5** *Let $\Gamma = \Gamma_1 \square \dots \square \Gamma_t$ be the cartesian product of digon-free and triangle-free strong digraphs $\Gamma_1, \dots, \Gamma_t$ with periods $p_1, \dots, p_t$ respectively. Then Γ is strong, digon-free, and triangle-free, with period $p(\Gamma) = \gcd\{p_1, \dots, p_t\}$.*

Proof Since $\Gamma_1, \dots, \Gamma_t$ are strong, so is Γ [8, Theorem 10.3.2]. Since $\Gamma_1, \dots, \Gamma_t$ are digon-free and triangle-free, so is Γ [18, Lemma 2.4].

Let $r = \gcd\{p_1, \dots, p_t\}$ and $p = p(\Gamma)$. Since every cycle of each Γ_i has a corresponding cycle in Γ, the lengths of the cycles in the Γ_i's are lengths of cycles in Γ, so p divides the lengths of all cycles in all Γ_i's, so p divides the greatest common divisor of these lengths. That is, p divides r.

We shall say that an arc $[(u_1, \dots, u_t), (v_1, \dots, v_t)] \in A(\Gamma)$ is of *type* r if $u_i = v_i$ for each $1 \le i \le t$, $i \neq r$. Consider a cycle of length l that involves l_r arcs of type r ($1 \le r \le t$). Then, for each $1 \le i \le t$, $l_i \equiv 0 \mod p_i$ and so $l_i \equiv 0 \mod r$. Therefore $l = l_1 + \dots + l_t \equiv 0 \bmod r$. Thus r divides the length of any cycle in Γ, so l divides the greatest common divisor of the lengths of the cycles in Γ. That is, r divides p, and hence $r = p$. ■

Example 5.6 (Cartesian product of cycles) Let $\Gamma = \Gamma_1 \square \dots \square \Gamma_t$ where Γ_i ($1 \le i \le t$) is the cycle of length $m_i \ge 4$ ($1 \le i \le t$). Then Γ is strong, digon-free and triangle-free, with period $\gcd\{m_1, \dots, m_t\}$. (Note that Γ is the Cayley digraph $\mathrm{Cay}(\mathbb{Z}_{m_1} \oplus \dots \oplus \mathbb{Z}_{m_t}, \{e_1, \dots, e_t\})$, where $e_1 = (1, 0, \dots, 0), e_2 = (0, 1, 0, \dots, 0), \dots, e_t = (0, 0, \dots, 1)$.)

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***Corollary* 5.7** *Let $\Gamma = \Gamma_1 \square \dots \square \Gamma_t$ be the cartesian product of digon-free and triangle-free, strong digraphs $\Gamma_1, \dots, \Gamma_t$ with periods $p_1, \dots, p_t$ respectively, and let $p = \gcd\{p_1, \dots, p_t\}$. Let $R(a, b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b)\rangle$. Then $G = G_\Gamma(R)$ is finite if and only if $\alpha \neq 0, \beta \neq 0$, $\gcd\{\alpha, \beta\} = 1$, $\alpha^p \neq \beta^p$, $a^\alpha = b^\beta$ in K, in which case G is cyclic of order $|\alpha^p - \beta^p|$.*

5.4 Direct products of strong digraphs

***Lemma* 5.8** *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_t$ be the direct product of strong digraphs $\Gamma_1, \dots, \Gamma_t$ with periods $p_1, \dots, p_t$ respectively, where $\gcd\{p_1, \dots, p_t\} = 1$, and where at least one of $\Gamma_1, \dots, \Gamma_t$ is digon-free and at least one is triangle-free. Then Γ is strong, digon-free, and triangle-free, with period $p(\Gamma) = p_1 \cdots p_t$.*

Proof The digraph Γ is strong with period $p(\Gamma) = p_1 \cdots p_t$ by [14] (see [14, Theorem 1(ii),(iii)] and page 251 and Proposition 4 of [10]; see also [8, Theorem 10.3.2]). If Γ contains a digon (resp. triangle) then each of $\Gamma_1, \dots, \Gamma_t$ contains a digon (resp. triangle), a contradiction. Hence Γ is digon-free and triangle-free. ∎

Example 5.9 (Direct product of an oriented diamond and a cycle) Let $\Gamma = \Gamma_1 \times \Gamma_2$ where Γ_1 is the oriented diamond digraph with $V(\Gamma_1) = \{u, v, w, t\}$ and $A(\Gamma_1) = \{[u, v], [u, w], [v, t], [w, t], [t, u]\}$, and Γ_2 is the cycle of length $n \geq 2$ where $n \not\equiv 0 \bmod 3$. Then Γ_1 is strong and digon-free with period 3 and Γ_2 is strong and triangle-free with period n, so Γ is strong, digon-free and triangle free, of period $p(\Gamma) = 3n$.

***Corollary* 5.10** *Let $\Gamma = \Gamma_1 \times \cdots \times \Gamma_t$ be the direct product of strong digraphs $\Gamma_1, \dots, \Gamma_t$ with periods $p_1, \dots, p_t$ respectively, where $\gcd\{p_1, \dots, p_t\} = 1$, and where at least one of $\Gamma_1, \dots, \Gamma_t$ is digon-free and at least one is triangle-free, and let $p = p_1 \cdots p_t$. Let $R(a, b)$ be a cyclically reduced word that involves both a and b and which has exponent sums α, β in a and b^{-1}, respectively, and let $K = \langle a, b \mid R(a, b)\rangle$. Then $G = G_\Gamma(R)$ is finite if and only if $\alpha \neq 0$, $\beta \neq 0$, $\gcd\{\alpha, \beta\} = 1$, $\alpha^p \neq \beta^p$, $a^\alpha = b^\beta$ in K, in which case G is cyclic of order $|\alpha^p - \beta^p|$.*

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