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ON THE STEINITZ MODULE AND CAPITULATION OF IDEALS

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Abstract. Let L be a finite extension of a number field K with ring of integers $\mathcal{O}_L$ and $\mathcal{O}_K$ respectively. One can consider $\mathcal{O}_L$ as a projective module over $\mathcal{O}_K$. The highest exterior power of $\mathcal{O}_L$ as an $\mathcal{O}_K$ module gives an element of the class group of $\mathcal{O}_K$, called the Steinitz module. These considerations work also for algebraic curves where we prove that for a finite unramified cover Y of an algebraic curve X, the Steinitz module as an element of the Picard group of X is the sum of the line bundles on X which become trivial when pulled back to Y. We give some examples to show that this kind of result is not true for number fields. We also make some remarks on the *capitulation* problem for both number field and function fields. (An ideal in $\mathcal{O}_K$ is said to capitulate in L if its extension to $\mathcal{O}_L$ is a principal ideal.)

§1. Introduction

Let L be a finite extension of a number field K, and let $\mathcal{O}_L$ and $\mathcal{O}_K$ denote the ring of integers in L and K respectively. It is easy to see that $\mathcal{O}_L$ is a projective module of rank equal to the degree of the field extension $d = [L : K]$. It is well known that any projective module, such as $\mathcal{O}_L$, over a Dedekind domain, such as $\mathcal{O}_K$, can be written as a sum of a free module and an ideal. This ideal gives a well defined element in the ideal class group of $\mathcal{O}_K$, called the Steinitz module. We will denote this ideal class by $\mathrm{St}_{L/K}$. It is the purpose of this paper to make some remarks on this module and its relation to *capitulation*: An ideal in $\mathcal{O}_K$ is said to capitulate in $\mathcal{O}_L$ if its extension to $\mathcal{O}_L$ is a principal ideal. We refer to the paper of Miyake [Mi], as well as the report by Kisilevsky [K] on the work of Olga Taussky-Todd in which he discusses capitulation in some detail, including a comment of E. Artin who seems to have once asked Olga Tausky-Todd if she was still working 'on those hopeless questions'! The situation seems to prevail even today. We refer to the book of Narkiewicz [Na, pp. 397–403], together with its exhaustive bibliography, for the known literature on Steinitz module.

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The concepts introduced in the previous paragraph work as well for algebraic curves over general fields too where $\mathcal{O}_L$ and $\mathcal{O}_K$ are to be thought of as structure sheaves on algebraic curves to be denoted by X_L and X_K, respectively. In this case the Steinitz module is an invertible sheaf on the curve X_K defined most simply as the determinant line bundle associated to the vector bundle $\mathcal{O}_L$ over X_K. When the map $X_L \to X_K$ is an abelian unramified covering, the Steinitz class $\mathrm{St}_{L/K}$ is the product of the elements in the (finite) kernel of the induced map $\mathrm{Pic}^0(X_K) \to \mathrm{Pic}^0(X_L)$ (see Proposition 1). In the number field case, the analogous statement for unramified abelian extensions L/K is false as we show by some examples.

After a few generalities in Section 2, we study the function field case in Section 3. In the number field case, we have only been able to provide counter-examples to certain results about capitulation and Steinitz modules available for the function field.

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§2. Generalities

For later calculations, it will be useful to have the structure of any ideal in $\mathcal{O}_L$ as an $\mathcal{O}_K$ module. This follows once the structure of $\mathcal{O}_L$ as an $\mathcal{O}_K$-module is determined. In the following lemma, and in the rest of the paper, we denote the norm of an ideal $\mathcal{A}$ in $\mathcal{O}_L$ by $\mathrm{Nm}(\mathcal{A})$ which is an ideal in $\mathcal{O}_K$.

Lemma 1. *Let $\mathcal{A}$ be an ideal in $\mathcal{O}_L$. Then $\mathcal{A}$ thought of as a module over $\mathcal{O}_K$ is isomorphic to*

$$\mathcal{A} = \mathcal{O}_K^{n-1} \oplus \mathrm{Nm}(\mathcal{A}) \cdot \mathrm{St}_{L/K}\,.$$

Proof. Assume first that $\mathcal{A}$ is a prime ideal. Let $\mathcal{A}_K = \mathcal{A} \cap \mathcal{O}_K$ be the corresponding prime ideal in $\mathcal{O}_K$. We have $\mathrm{Nm}(\mathcal{A}) = \mathcal{A}_K^d$ where d is the degree of the residue field extension $\mathcal{O}_L/\mathcal{A}$ over $\mathcal{O}_K/\mathcal{A}_K$. Clearly $[\mathcal{O}_L/\mathcal{A}]$ thought of as an element in the Grothendieck group of finitely generated $\mathcal{O}_K$-modules $K_0[\mathcal{O}_K]$ is

$$d[\mathcal{O}_K/\mathcal{A}_K] = [\mathcal{O}_K/\mathcal{A}_K^d] = [\mathcal{O}_K/\mathrm{Nm}(\mathcal{A})].$$

Therefore,

$$[\mathcal{O}_L/\mathcal{A}] = [\mathcal{O}_K/\mathrm{Nm}(\mathcal{A})]\,.$$

This relation clearly continues to hold good for general ideals $\mathcal{A} = \prod \wp_i^{n_i}$ as

$$\begin{aligned}[\mathcal{O}_L/\mathcal{A}] &= \sum n_i\,[\mathcal{O}_L/\wp_i]\\ &= \sum n_i\,[\mathcal{O}_K/\mathrm{Nm}(\wp_i)]\\ &= [\mathcal{O}_K/\mathrm{Nm}(\mathcal{A})]\,.\end{aligned}$$

From the exact sequence,

$$0 \to \mathcal{A} \to \mathcal{O}_L \to \mathcal{O}_L/\mathcal{A} \to 0,$$

we have,

$$\begin{aligned}[\mathcal{A}] &= [\mathcal{O}_L] - [\mathcal{O}_K/\mathrm{Nm}(\mathcal{A})]\\ &= (n-1)[\mathcal{O}_K] + \mathrm{St}_{L/K} - [\mathcal{O}_K/\mathrm{Nm}(\mathcal{A})]\\ &= (n-2)[\mathcal{O}_K] + \mathrm{St}_{L/K} + [\mathrm{Nm}\mathcal{A}]\\ &= (n-1)[\mathcal{O}_K] + \mathrm{St}_{L/K} \cdot [\mathrm{Nm}\mathcal{A}].\end{aligned}$$

In the above, we have used the identity $[I_1] + [I_2] = [\mathcal{O}_K] + [I_1 \cdot I_2]$. Now noting that two projective modules which are equal in the K-group are actually isomorphic, the proof of the lemma follows.

We use this lemma to prove the following well-known result.

LEMMA 2. *The square of the Steinitz module* $\mathrm{St}_{L/K}$ *is the discriminant ideal of* L *over* K *as an element of the class group of* K*.*

Proof. There is a nondegenerate K-bilinear form $\mathrm{tr} : L \times L \to K$ given by $(x, y) \to \mathrm{tr}(xy)$. Let δ^{-1} denote the fractional ideal in $\mathcal{O}_L$ consisting of those elements d in L such that $\mathrm{tr}(dy)$ belongs to $\mathcal{O}_K$ for all y in $\mathcal{O}_L$. The inverse of δ^{-1} is the different ideal of $\mathcal{O}_L$.

We note that there is an isomorphism of $\mathcal{O}_L$-modules:

$$\mathrm{Hom}_{\mathcal{O}_K}[\mathcal{O}_L, \mathcal{O}_K] \cong \delta^{-1}.$$

For this observe that for any $d \in \delta^{-1}$, we have $\phi_d(x) = \mathrm{tr}(dx) \in \mathrm{Hom}_{\mathcal{O}_K}[\mathcal{O}_L, \mathcal{O}_K]$ for $x \in \mathcal{O}_L$. The mapping $d \to \phi_d$ gives a surjection from δ^{-1} to $\mathrm{Hom}_{\mathcal{O}_K}[\mathcal{O}_L, \mathcal{O}_K]$

as any element of $\mathrm{Hom}_{\mathcal{O}_K}[\mathcal{O}_L, \mathcal{O}_K]$ is the restriction of an element in $\mathrm{Hom}_K[L, K]$ which because of the nondegeneracy of the trace form, is of the form $\phi_d(x)$ for some d in L. As ϕ_d restricted to $\mathcal{O}_L$ gives an element in $\mathrm{Hom}_{\mathcal{O}_K}[\mathcal{O}_L, \mathcal{O}_K]$, $d \in \delta^{-1}$.

Now write, $\mathcal{O}_L = \mathcal{O}_K^{n-1} \oplus \mathrm{St}_{L/K}$, and note that as proved in the previous lemma, for any ideal $\mathcal{A}$ in $\mathcal{O}_L$ thought of as a module over $\mathcal{O}_K$,

$$\mathcal{A} = \mathcal{O}_K^{n-1} \oplus \mathrm{Nm}(\mathcal{A}) \cdot \mathrm{St}_{L/K} .$$

Therefore from the isomorphism $\mathrm{Hom}_{\mathcal{O}_K}[\mathcal{O}_L, \mathcal{O}_K] \cong \delta^{-1}$, we get

$$\mathcal{O}_K^{n-1} + \mathrm{St}_{L/K}^{-1} \cong \mathcal{O}_K^{n-1} + \mathrm{Nm}(\delta^{-1}) \cdot \mathrm{St}_{L/K},$$

or,

$$\mathrm{St}_{L/K}^2 \cong \mathrm{Nm}(\delta).$$

We note an immediate consequence of the lemma above.

Corollary 1. *If L is an unramified extension of K its Steinitz class $\mathrm{St}_{L/K}$ is of order dividing 2.*

Remark 1. It is a well known theorem of Hecke (an existence theorem, proved by analytic methods!) that the different ideal itself is a square in the ideal class group. By Lemma 2, Steinitz module gives an explicit square root of the discriminant ideal. The example in [FST] of a different with an odd class, in the situation of curves over certain fields, shows that there may not be a similar, explicit, algebraic construction for a square root of the different ideal.

The following lemma will be useful in the calculation of the Steinitz module.

Lemma 3. *Let L be a degree n extension of a number field K. Suppose that we have an exact sequence of $\mathcal{O}_K$ modules*

$$0 \to \mathcal{O}_K^n \to \mathcal{O}_L \to M \to 0,$$

with M isomorphic to $\mathcal{O}_K/\wp$ for a ideal $\wp$ in $\mathcal{O}_K$. Then $\wp^{-1}$ represents the Steinitz class of the extension L of K.

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Proof. We present one of several possible proofs. We have an exact sequence of $\mathcal{O}_K$ modules

$$0 \to \mathcal{O}_K^n \to \mathcal{O}_L \to \mathcal{O}_K/\wp \to 0.$$

So, we have the equality in the K-group $K_0[\mathcal{O}_K]$:

$$[\mathcal{O}_L] = n[\mathcal{O}_K] + [\mathcal{O}_K] - [\wp].$$

On the other hand,

$$[\mathcal{O}_L] = (n-1)[\mathcal{O}_K] + \mathrm{St}_{L/K}\,.$$

Therefore,

$$\mathrm{St}_{L/K} = 2[\mathcal{O}_K] - [\wp].$$

From the exact sequence,

$$0 \to \mathcal{O}_K \to \wp^{-1} \to \frac{\wp^{-1}}{\mathcal{O}_K} \cong \frac{\mathcal{O}_K}{\wp} \to 0,$$

we have $2[\mathcal{O}_K] - [\wp] = [\wp^{-1}]$. Therefore,

$$\mathrm{St}_{L/K} = [\wp^{-1}]$$

as objects in $K_0[\mathcal{O}_K]$. However, $\mathrm{St}_{L/K}$ and $\wp^{-1}$ are projective modules over $\mathcal{O}_K$, and as noted in Lemma 1, two projective modules which are equal in the K-group are actually isomorphic, proving the lemma.

§3. The function field case

Let X be a projective variety over an algebraically closed field k in which n is invertible. From the Kummer sequence in étale topology

$$0 \to \mathbf{Z}/n\mathbf{Z} \to \mathcal{O}^* \xrightarrow{n} \mathcal{O}^* \to 0,$$

it follows that the isomorphism classes of line bundles of order n is represented by $H^1_{\text{ét}}(X, \mathbf{Z}/n) \cong \mathrm{Hom}[\pi_1(X), \mathbf{Z}/n\mathbf{Z}]$. Moreover, this identification is functorial. It follows that given a line bundle of finite order, say n, on a variety X, there exists a variety Y, together with a finite unramified map to X of degree n, with the property that for any variety Z together with a map to X, the pull back of the line bundle on X is trivial on Z if and only if the mapping from Z to X factors through the mapping from Y to

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X. The variety Y is a degree n unramified cover of X where n is the order of the line bundle on X constructed using the Riemann existence theorem, and the above property of Y proved using the *lifting criterion* in covering spaces.

From the above consideration, there is an unramified covering of degree n associated to a line bundle of order n. This has a concrete construction too which we describe now, cf. Exercise 2.7 of Chapter 4 of Hartshorne's book [H] for $n = 2$.

Let $\mathcal{L}$ be a line bundle of order n over a scheme X. Let $\phi : \mathcal{L}^n \cong \mathbf{1}$ be a fixed isomorphism of $\mathcal{L}^n$ with the trivial line bundle on X. We will use this data to construct an unramified cover of X of degree n. Let $\mathcal{A} = \mathbf{1} \oplus \mathcal{L} \oplus \cdots \oplus \mathcal{L}^{n-1}$. This is a sheaf of $\mathcal{O}_X$-algebras on the scheme X defined by the obvious law of multiplication from $\mathcal{L}^i \times \mathcal{L}^j \to \mathcal{L}^{i+j}$ if $i + j \leq n - 1$. If $i + j \geq n$, define the law of multiplication $\mathcal{L}^i \times \mathcal{L}^j \to \mathcal{L}^{i+j-n}$ by using the isomorphism of $\mathcal{L}^{i+j}$ with $\mathcal{L}^{i+j-n}$ via ϕ. By taking the **Spec** of the $\mathcal{O}_X$ algebra $\mathcal{A}$, we get a degree n unramified covering $X_{\mathcal{L}}$ of X whenever n is invertible on the scheme X. This follows from the fact that if n is invertible in a ring R, then for any invertible element r of R,

$$\frac{R[x]}{(x^n - r)}$$

is an unramified extension of the ring R.

It is easy to see that the line bundle $\mathcal{L}$ becomes trivial when pulled back to $X_{\mathcal{L}}$, and from earlier remarks it follows that if X is a projective variety over an algebraically closed field k in which n is invertible, $X_{\mathcal{L}}$ has the universal property that for any variety Z together with a map to X such that $\mathcal{L}$ pulled back to Z is trivial, the mapping from Z to X factors through $X_{\mathcal{L}}$.

Example 1. We show by an example that such an $X_{\mathcal{L}}$ does not exist for invertible ideals in the ring of integers of number fields. More precisely, we find two degree 2 unramified extensions of a number field K and a non-principal ideal in K which becomes principal in both of them. For this, let $K = \mathbf{Q}(\sqrt{-21})$, and $L_1 = K(\sqrt{-3}), L_2 = K(\sqrt{-1})$. The ideal class group of K is $\mathbf{Z}/2 \oplus \mathbf{Z}/2$, generated by primes in K above 2 and 3. As we will see in more detail in Example 2, all the ideals in K become principal in L_1, and the prime ideal above 2 in K becomes principal in L_2. So, the prime ideal above 2 in K becomes principal in two distinct quadratic unramified extensions.

PROPOSITION 1. *Let k be a field, and X a projective variety over k. Let Y be an unramified abelian Galois covering of X with Galois group G with the order of G invertible in k. Assume that for every integer n for which G has an element of order n, k^* also has an element of order n. The mapping from the abelian variety $\mathrm{Pic}^0(X)$ to $\mathrm{Pic}^0(Y)$ obtained by pull back of line bundles has finite kernel, say H. The sum of elements in H is a line bundle on X of order ≤ 2 which represents the Steinitz module $\mathrm{St}_{Y/X}$.*

Proof. Since Y is an unramified Galois covering of X with Galois group G, the sheaf $\mathcal{O}_Y$ thought of as an $\mathcal{O}_X$-module is a regular representation of G over $\mathcal{O}_X$. Since the order of G is invertible in k, we have a canonical direct sum decomposition of $\mathcal{O}_Y$, thought of as a vector bundle on X, as a direct sum of line bundles

$$\mathcal{O}_Y = \sum_{\alpha \in \hat{G}} \mathcal{L}_\alpha,$$

where $\hat{G}$ is the group of homomorphisms of G into k^* and $\mathcal{L}_\alpha$ is the α eigenspace of the action of G on $\mathcal{O}_Y$. We check that the line bundles $\mathcal{L}_\alpha$ on X become trivial when pulled back to Y. We do this by proving that the vector bundle $\mathcal{O}_Y$ over X becomes trivial when pulled back to Y, and is in fact isomorphic to $\mathcal{O}_Y[G]$ as G-bundles. To prove this claim, represent the scheme X locally as $\mathrm{Spec}(A)$ and its inverse image in Y as $\mathrm{Spec}(B)$. This gives B the structure of an étale algebra over A with Galois group G, i.e. we have an isomorphism

$$B \otimes_A B \cong \sum_{g \in G} B = B[G],$$

given by $b \otimes 1 \to \sum_{g \in G} g \cdot b$, proving our claim. Clearly we have an isomorphism

$$k[G] \cong \sum_{\chi \in \hat{G}} k,$$

given by $g \to \sum_{\chi \in \hat{G}} \chi(g)$. Tensoring this isomorphism by $\mathcal{O}_Y$, we have an isomorphism

$$\mathcal{O}_Y[G] \cong \sum_{\chi \in \hat{G}} \mathcal{O}_Y.$$

This proves that the line bundles $\mathcal{L}_\chi$ are trivial when pulled back to Y.

Since the order of $\hat{G}$ is the same as the order of G, it follows that we have at least as many line bundles on X as the order of G which become trivial when pulled back to Y. From the functorial identification of line bundles of finite order on X with $\mathrm{Hom}[\pi_1(X), \mathbf{Q}/\mathbf{Z}]$ it follows that the line bundles on X which become trivial on Y (which corresponds to a normal subgroup of $\pi_1(X)$ with quotient G) are in bijective correspondence with the homomorphisms from G to $\mathbf{Q}/\mathbf{Z}$ which has the same order as the order of G. So we have found all the line bundles on X which become trivial on Y. Since the determinant line bundle associated to $\mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_n$ is $\prod \mathcal{L}_i$, this completes the proof of the proposition.

§4. The number field case

We will mostly be working in the situation where $\mathcal{O}_L$ is unramified over $\mathcal{O}_K$, in which case, as noted earlier, since the discriminant ideal is trivial, the Steinitz class is an element of order ≤ 2 in the class group of K. One knows by class field theory that unramified abelian extensions are in bijective correspondence with the subgroups of the class group Cl_K of K; a subgroup H of the class group Cl_K of K corresponds to an unramified abelian extension of K with Galois group Cl_K / H. We would have liked to describe the Steinitz class in terms of the pair (Cl_K, H). However, examples below suggest that the Steinitz class of the corresponding unramified extension cannot be described in terms of the pair (Cl_K, H) alone.

As we have seen above, in the function field case, there is a strong link between the Steinitz class $\mathrm{St}_{L/K}$ and the ideals which capitulate in the extension. We begin by recalling a few well-known results on capitulation.

4.1. Capitulation

Let L be a finite extension of a global field K. An ideal $\mathcal{A}$ in $\mathcal{O}_K$ is called a capitulation ideal for the extension L of K if $\mathcal{A}^e$, the ideal in $\mathcal{O}_L$ generated by $\mathcal{A}$ is principal. There is still no satisfactory understanding of which ideals capitulate even for unramified abelian extensions. In this paragraph we recall a result of Iwasawa [I] and deduce some consequences. We also refer to the paper of Cornell and Rosen [CR] for related matters.

Proposition 2. *Let L be a finite unramified Galois extension of a global field K. Let E_L denote the units of $\mathcal{O}_L$ if L is a number field, and globally invertible functions on the smooth projective curve X_L if L is a function field. Then the elements in the class group of $\mathcal{O}_K$ if K is a number field and elements in the Picard group of X_K, if K is a function field,*

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which become trivial when extended to $\mathcal{O}_L$ *(resp. under pull back to* X_L*) is a subgroup of the class group of* $\mathcal{O}_K$ *(resp. Picard group of* X_K*) isomorphic to* $H^1(\mathrm{Gal}(L/K), E_L)$*.*

Proof. We will write the proof assuming that K is a number field. The same proof works for function field too.

Let $\mathcal{A}$ be an ideal in $\mathcal{O}_K$ which becomes principal in L, say $\mathcal{A}^e = (x)$. Since $\mathcal{A} = \mathcal{A}^\sigma = (x^\sigma) = (x)$, $x^\sigma = u_\sigma \cdot x$ for a unit u_σ in E_L where σ is an arbitrary element of the Galois group of L over K. Clearly $\sigma \to u_\sigma$ is a 1-cocycle of $\mathrm{Gal}(L/K)$ with values in E_L. So, every ideal in K which becomes principal in L gives rise to an element in $H^1(\mathrm{Gal}(L/K), E_L)$. Conversely, if $\phi = \{\phi_\sigma\} \in H^1(\mathrm{Gal}(L/K), E_L)$, then by Hilbert's theorem 90, ϕ becomes trivial in $H^1(\mathrm{Gal}(L/K), L^*)$. So, one can write $\phi_\sigma = t^\sigma t^{-1}$ for some $t \in L^*$. Since ϕ_σ is a unit, the fractional ideal generated by t, $t\mathcal{O}_\mathcal{L}$, is Galois invariant. Since L is unramified over K, it is easy to see that all Galois invariant ideals in $\mathcal{O}_L$ come from ideals in $\mathcal{O}_K$, completing the proof of the proposition.

Corollary 2. *The order of any ideal in the ideal class group of* $\mathcal{O}_K$ *which becomes principal in* $\mathcal{O}_L$ *divides the degree of the field extension* $[L : K]$*.*

Remark 2. We remark that Hilbert's Satz 94 (as generalized in [Su]) says that for an unramified, abelian extension L/K, the order of the subgroup of Cl_K which capitulates in L is divisible by the degree of the extension of L over K.

Corollary 3. *If* H *is the Hilbert class field of* K *with* E *as the group of units, then*

$$H^1(\mathrm{Gal}(H/K), E_H) \cong \mathrm{Gal}(H/K) \cong \mathrm{Cl}_K\,.$$

Proof. This follows from Proposition 2 together with the following well known theorems:

(a) The Galois group of the Hilbert class field H of K is canonically isomorphic to the class group of K.

(b) The principal ideal theorem which states that every ideal in K becomes principal in L.

Remark 3. Corollary 3 is in fact equivalent to the principal ideal theorem!

Corollary 4. *Let L be an unramified abelian extension of K of degree m and let the class group of K be of order mn with $(m,n)=1$. Let M and N be the unique subgroups of orders m and n in the class group Cl_K of K. We have $\mathrm{Gal}(L/K) \cong \mathrm{Cl}_K/N \cong M$. Then the ideals in the class group of K which become principal in L are precisely those which correspond to the elements in the subgroup M of Cl_K.*

Proof. The ideals in the class group of K which become principal in L are certainly annihilated when multiplied by m. All such elements of the ideal class group are contained in M. On the other hand, if the ideals in the class group of K which become principal in L were a proper subgroup of M then the cardinality of the set of ideals in the class group of L which come from K will be of the form $a \cdot n, a > 1, (a,n) = 1$. By the principal ideal theorem, this subgroup of order $a \cdot n$ in the class group of L will have to become principal in the Hilbert class field of K, contradicting Corollary 2.

Corollary 5. *Assume that K and L are function fields of curves such that the constant functions in K form an algebraically closed field of characteristic $p \geq 0$. Then the line bundles on X_K which become trivial on X_L are in bijective correspondence with $H^1(\mathrm{Gal}(L/K), k^*) \cong \mathrm{Hom}(\mathrm{Gal}(L/K), k^*)$. If k has characteristic $p > 0$, k^* has no non-trivial elements of order p and therefore no element of order p on the Picard group of X_K can become trivial on X_L for any unramified extension L of K. Also, if $\mathrm{Gal}(L/K)$ is a p group, the mapping from $\mathrm{Pic}(X_K)$ to $\mathrm{Pic}(X_L)$ is injective.*

Remark 4. Unlike in the function field case, it may not even be true that for an invertible ideal in the ring of integers of order n in the class group of a number field, there is an unramified, abelian extension of degree n in which the extension of the invertible ideal becomes trivial. There is certainly an extension of order n (which may even be chosen to be abelian if the nth roots of unity are contained in the number field) such that the extension of the invertible ideal is trivial, but one may not be able to have it unramified and abelian. We elaborate this point. Given a number field K containing the nth roots of unity, and an ideal I of $\mathcal{O}_K$ of order n in the class group, we can construct an extension L of K with the property that L is

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abelian over K, and I capitulates in the extension L/K. To construct L we simply put $I^n = (\alpha)$, for $\alpha \in \mathcal{O}_K$, and define L to be K adjoined an nth root of α. This is an abelian extension in which I capitulates. It is unramified at all finite places not dividing n. Note that the choice of α is ambiguous up to units, and that L depends on the choice. However, even with this freedom of multiplying α by a unit, we may not be able to construct an unramified abelian extension in which I capitulates. For instance if $K = \mathbf{Q}(\sqrt{-p})$, with p a prime > 0, and is congruent to 1 modulo 4, the ideal I above 2 in K is non-trivial in the ideal class group of K with $I^2 = (2)$. The only units in K being ± 1, the extensions L constructed above are $K(\sqrt{2})$ and $K(\sqrt{-2})$, neither of which is unramified over K. It is true on the other hand that the ideal I does capitulate in the unramified quadratic extension of K, but its construction does not follow any general rule. By Hilbert's Satz 94 (cf. [Su]), if the n-primary part of the class group of K is cyclic then one can always construct an unramified abelian extension of degree n in which any given element of order n in the class group of K capitulates. There is an example in [HS] of imaginary quadratic extensions with class group isomorphic to $\mathbf{Z}/3 \oplus \mathbf{Z}/3$, such that there is an element of order 3 in the class group which does not capitulate in any unramified abelian extension of degree 3.

4.2. Steinitz modules

As noted during the course of the proof of Proposition 1, if L is an unramified Galois extension with Galois group G of a number field K with ring of integers B in L and A in K, one has the isomorphism

$$B \otimes_A B \cong \sum_{g \in G} B.$$

This implies that the Steinitz module for the extension L of K becomes trivial when extended to L. We note this in the following proposition.

PROPOSITION 3. *If L is an unramified Galois extension of a number field K then the Steinitz module of the extension L of K becomes trivial when extended to the ring of integers of L.*

This proposition when combined with Corollary 2 yields the following. We are grateful to Marcin Mazur for pointing this corollary to us as a consequence of the methods in [Na] for calculating the Steinitz module.

COROLLARY 6. *Let L be an unramified Galois extension of K of odd degree. Then the Steinitz module of L over K is trivial.*

Partly the difficulty in understanding Steinitz modules lies in the fact that the ideals which capitulate are not well understood. However, we have seen in Corollary 4 that there is one situation in which the ideals which capitulate in an extension of number fields is well understood. So, keeping in mind the situation of curves, we may ask the question:

QUESTION 1. *Let K be a number field with Cl_K as its class group. Let L be an unramified abelian extension of K whose Galois group is identified to Cl_K/H by the class field theory. Suppose that the orders of H and Cl_K/H are coprime. Then $\mathrm{Cl}_K = H \times H'$ for a unique subgroup H' of Cl_K. Then is the Steinitz module the sum of elements in H'?*

We will see in the next section that even under such restrictive conditions, the question has negative answer.

We ask another general question but for which we have no answer.

QUESTION 2. *Which elements of order 2 in the class group of K arise as the Steinitz module of a degree 2 unramified extension of K? We note that by class field theory, there are exactly as many elements of order 2 in the class group of K as the number of unramified abelian extensions of K of degree 2.*

4.3. Quadratic fields

We will look at Question 1 in the simplest case of unramified quadratic extensions of quadratic fields, and show that it fails even in this case.

The counter-example to Question 1 is provided by looking at $K = \mathbf{Q}(\sqrt{-p})$ where $p \equiv 1 \bmod 4$ is a prime in $\mathbf{Z}$ which is > 0. By genus theory, there is a unique copy of $\mathbf{Z}/2$ in the class group of K, and the unique unramified degree 2 extension of K is obtained by attaching $\sqrt{p}$. In the next lemma we will show that $\mathrm{St}_{K(\sqrt{p})/K}$ is the prime ideal in K over p. Since the prime ideal in K above p is principal, this gives a negative answer to Question 1.

LEMMA 4. *Let $K = \mathbf{Q}(\sqrt{D})$ be a quadratic extension of $\mathbf{Q}$ with D square-free. Let $p \equiv 1 \bmod 4$ be a prime dividing D, and let $L = K(\sqrt{p})$ be a quadratic unramified extension of K. Then the Steinitz class $\mathrm{St}_{L/K}$ of L over K is the prime ideal in K above p.*

Proof. We will compute the Steinitz class $\mathrm{St}_{L/K}$ using Lemma 3. The computation of the ring of integers of L is given in exercise 42 on page 51 of [M]. We write $D = p \cdot q$, and divide the proof of the lemma into two cases.

Case 1 : $q \equiv 2, 3 \bmod 4$. In this case, the ring of integers of L is the free $\mathbf{Z}$ module generated by

$$1, (1 + \sqrt{p})/2, \sqrt{q}, (\sqrt{q} + \sqrt{D})/2.$$

It follows that the module generated by 1 and $(1+\sqrt{p})/2$ over $\mathbf{Z}[\sqrt{D}]$, which is the ring of integers of K, is of index p in the ring of integers of L. By Lemma 3, the Steinitz class $\mathrm{St}_{L/K}$ is the class of the prime ideal in K lying above p.

Case 2 : $q \equiv 1 \bmod 4$. In this case the ring of integers of L is the free $\mathbf{Z}$ module generated by

$$1, (1 + \sqrt{p})/2, (1 + \sqrt{p} + \sqrt{D} + p\sqrt{q})/4, (1 + \sqrt{q})/2.$$

We denote these generators by a, b, c, d. The free $\mathcal{O}_K$ module generated by $1, (1 + \sqrt{p})/2$ is the free $\mathbf{Z}$ module generated by

$$1, (1 + \sqrt{p})/2, (1 + \sqrt{p} + \sqrt{D} + p\sqrt{q})/4, (1 + \sqrt{D})/2.$$

We denote these generators by a, b, c, d'. Because $d' = 2c - b + a(1+p)/2 - pd$, it has index p in the ring of integers of L. In this case again we conclude that $\mathrm{St}_{L/K}$ is the prime ideal in K above p.

4.4. Another example

We present one more calculation of Steinitz module and of capitulation, one in which the Steinitz module is non-trivial, but the sum of capitulating ideals is trivial.

Let $K = \mathbf{Q}(\sqrt{-21}), \mathcal{O}_K = \mathbf{Z}(\sqrt{-21})$. The class number of K can be seen to be 4, and the Hilbert class field of K seen to be $K(\sqrt{-3}, \sqrt{-7})$. We take $L = K(\sqrt{-3})$. It follows from exercise 42(c), page 51, of Marcus' book that the ring of integers in L is the free $\mathbf{Z}$ module generated by

$$1, \frac{1 + \sqrt{-3}}{2}, \sqrt{7}, \frac{\sqrt{7} + \sqrt{-21}}{2}.$$

It is easy to see that the free submodule of $\mathcal{O}_L$ generated by 1 and $\frac{1+\sqrt{-3}}{2}$ over $\mathbf{Z}[\sqrt{-21}]$ is of index 3 in $\mathcal{O}_L$: $\mathbf{Z}[\sqrt{-21}](1, \frac{1+\sqrt{-3}}{2})$ is a free $\mathbf{Z}$ module generated by $1, \frac{1+\sqrt{-3}}{2}, \sqrt{-21}, \frac{\sqrt{-21}+3\sqrt{7}}{2}$, or by, $1, \frac{1+\sqrt{-3}}{2}, 3\sqrt{7}, \frac{\sqrt{-21}+3\sqrt{7}}{2}$. It has index 3 in the submodule generated by $1, \frac{1+\sqrt{-3}}{2}, \sqrt{7}, \frac{\sqrt{-21}+3\sqrt{7}}{2}$ which is the same as that generated by $1, \frac{1+\sqrt{-3}}{2}, \sqrt{7}, \frac{\sqrt{7}+\sqrt{-21}}{2}$ which is $\mathcal{O}_L$. From Lemma 3, the Steinitz module for $\mathcal{O}_L$ is the prime ideal in $\mathcal{O}_K$ above 3.

The class group of $\mathbf{Q}(\sqrt{-21})$ can be checked to be $\mathbf{Z}/2 \oplus \mathbf{Z}/2$. The primes 2 and 3 are ramified, say

$$(2) = \wp_2^2$$
$$(3) = \wp_3^2.$$

So, $\wp_2$ and $\wp_3$ are elements of order 2 in the class group of K. These are linearly independent as $\mathrm{Nm}(\wp_2\wp_3) = 6$ is not the norm of any integral element of $\mathbf{Z}[\sqrt{-21}]$.

Claim 1. *$\wp_2$ and $\wp_3$ become principal in $\mathbf{Z}[\sqrt{-21}, \sqrt{-3}]$.*

Proof. We first prove that $\wp_2$ becomes principal in $\mathcal{O}_L$. For this we note that $(3+\sqrt{7})^2 = (2)$ as ideals in $\mathbf{Z}[\sqrt{7}]$, and therefore as ideals in $\mathcal{O}_L$. Since $\wp_2^2 = (2)$ as ideals in $\mathbf{Z}[\sqrt{-21}]$, so also as ideals in $\mathcal{O}_L$. Therefore we have

$$2 = (3+\sqrt{7})^2 = \wp_2^2.$$

By unique factorization of ideals,

$$(3+\sqrt{7}) = \wp_2$$

as ideals in $\mathcal{O}_L$. This proves that $\wp_2$ becomes principal in $\mathcal{O}_L$.

We now check that $\wp_3$ also becomes principal in $\mathcal{O}_L$. For this it suffices to observe that

$$(3) = (\sqrt{-3})^2 = \wp_3^2,$$

and again $\wp_3 = (\sqrt{-3})$ as ideals in $\mathcal{O}_L$.

Remark 5. The extension $\mathbf{Q}(\sqrt{-21}, \sqrt{-3})$ of $\mathbf{Q}(\sqrt{-21})$ gives an example of a situation in which all the ideals of a number field become principal in a proper subfield of the Hilbert class field. First example of this kind was constructed by Iwasawa.

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References

[CR] G. Cornell and M. Rosen, *Cohomological analysis of class group extension problem*, Queen's papers in pure and applied Mathematics edited by P. Ribenboim, **54** (1979), 287–308.

[H] R. Hartshorne, Algebraic Geometry, Graduate texts in Mathematics, Springer-Verlag.

[FST] A. Frohlich, J-P. Serre and J. Tate, *A different with an odd class*, J. de Crelle, **209** (1962), 6–7.

[I] K. Iwasawa, *A note on the group of units in an algebraic number fields*, J. Math. Pures Applied, **35** (1956), 189–192.

[M] D.A. Marcus, Number Fields, Universitext, Springer-Verlag.

[HS] F-P. Heider and B. Schmithals, *Zur Kapitulation der Idealklassen in unverzweigten primzyklischen Erweiterungen*, J. Reine Agnew. Math., **336** (1982), 1–25.

[K] H. Kisilevsky, *Olga Tausky-Todd's work in Class field theory*, Pacific J. of Mathematics, Olga Taussky-Todd memorial issue (1997), 219–224.

[Mi] K. Miyake, *Algebraic investigations of Hilbert's theorem* 94*, the principal ideal theorem, and the capitulation problem*, Expositiones Mathematicae, **7** (1989), 289–346.

[Na] W. Narkiewicz, Elementary and Analytic theory of algebraic numbers, 2nd edition, Springer-Verlag, PWN-Polish scientific publishers.

[Su] H. Suzuki, *A generalization of Hilbert's theorem* 94, Nagoya Math. J., **121** (1991), 161-169.

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