J. Aust. Math. Soc. **99** (2015), 12–29
doi:10.1017/S1446788714000652

THE CLASS OF (2, 2)-GROUPS WITH HOMOCYCLIC REGULATOR QUOTIENT OF EXPONENT p^3 HAS BOUNDED REPRESENTATION TYPE

DAVID M. ARNOLD, ADOLF MADER, OTTO MUTZBAUER and EBRU SOLAK✉

(Received 10 September 2013; accepted 21 October 2014; first published online 4 December 2014)

Communicated by B. Martin

Abstract

The class of almost completely decomposable groups with a critical typeset of type $(2, 2)$ and a homocyclic regulator quotient of exponent p^3 is shown to be of bounded representation type. There are only 16 isomorphism at p types of indecomposables, all of rank 8 or lower.

2010 *Mathematics subject classification*: primary 20K15; secondary 20K25, 20K35, 15A21, 16G60.

Keywords and phrases: almost completely decomposable group, indecomposable, bounded representation type.

1. Introduction

In spite of Kaplansky's observation that the theory of torsion-free abelian groups seems to consist of a collection of counterexamples to anything one might think of, there is a well-developed theory of completely decomposable and almost completely decomposable groups, [2, 8].

A torsion-free abelian group A is an additive subgroup of a $\mathbb{Q}$-vector space V where $\mathbb{Q}$ is the field of rational numbers. By $\mathbb{Q}A$ we mean the subvector space spanned by A, and the rank $\operatorname{rk} A$ of A is defined by $\operatorname{rk} A = \dim \mathbb{Q}A$.

Completely decomposable groups are direct sums of groups of rank 1, and *almost completely decomposable groups* are abelian groups that contain a finite-rank completely decomposable subgroup of finite index. Every almost completely decomposable group G contains a canonical completely decomposable fully invariant subgroup $\mathrm{R}(G)$, the *regulator* of G. In this paper we deal exclusively with almost completely decomposable groups with p-primary *regulator quotient* $G/\mathrm{R}(G)$, the so-called p-local case. While there is ample evidence that classification of almost completely decomposable groups up to isomorphism is hopeless, there is a weakening of isomorphism that serves very well. Given a prime p, two almost completely decomposable groups G and H are *isomorphic at p* if there exist $f : G \to H$, $g : H \to G$, and an integer n prime to p with $fg = n$ and $gf = n$. Note that for p-local almost completely decomposable groups isomorphism at p is equivalent to near-isomorphism.

By well-known theorems of Arnold [1] and Faticoni and Schultz [7], the p-local almost completely decomposable groups with isomorphism at p form a Remak–Krull–Schmidt category, that is, decompositions with indecomposable summands are unique up to isomorphism at p and two groups that are isomorphic at p have identical decompositions up to isomorphism at p. Also a p-local group is indecomposable as an abelian group if and only if it is indecomposable in the isomorphism at p category. This means that p-local almost completely decomposable groups are classified up to isomorphism at p once the indecomposable groups are determined. This is completely analogous to the situation of finite abelian groups, but while the indecomposable finite abelian are easily found, the same is not true for almost completely decomposable groups. It is necessary to restrict oneself to special classes of local almost completely decomposable groups, and then two things can happen. If the class has *unbounded representation type*, that is, it contains indecomposable groups of arbitrarily large ranks, then a survey of the indecomposable groups in the class is considered hopeless. The other possibility, *bounded representation type*, is that there are up to isomorphism at p only finitely many indecomposable groups. In this case one wishes to compile a complete list of isomorphism at p classes of indecomposable groups and so achieve a classification of the class up to isomorphism at p.

A type τ is *p-reduced* if $pA \neq A$ for any group of rank 1 and of type τ. A *(2, 2)-group* is an almost completely decomposable group G with regulator $R = \mathrm{R}(G) = R_{\tau_1} \oplus R_{\tau_2} \oplus R_{\tau_3} \oplus R_{\tau_4}$ where the τ_i are p-reduced types such $\tau_1 < \tau_2$, $\tau_3 < \tau_4$, τ_i incomparable with τ_j for $i \leq 2 < j$, and R_{τ_i} is a finite direct sum of groups of rank 1 and of type τ_i.

A $(2,2)$-group with regulator quotient of exponent p^m is called *$(2,2)$-p^m-group*. A $(2,2)$-p^m-group with a homocyclic regulator quotient is called a $(2,2)$-p^m-hc-*group*.

This paper is devoted to a classification of indecomposable $(2,2)$-p^3-hc-groups, thereby confirming bounded representation type in this case. More precisely, we present a complete collection of isomorphism at p types of indecomposables. They are of rank $4, 5, 6, 8$. The groups are encoded by (integral) 'coordinate matrices', and the proof includes finding a normal form for coordinate matrices (see Section 3).

The $(2,2)$-p^m-groups for $m \leq 2$ have finite representation type, and the indecomposable groups of these classes are described in [9].

The class of $(2,2)$-p^m-groups for $m \geq 3$ has unbounded representation type; see [3]. The class of $(2,2)$-p^m-hc-groups for $m \geq 5$ has unbounded representation type [6]. The case of $(2,2)$-p^4-hc is open and is likely to be rather complicated with many indecomposable groups. In general, if the regulator quotient is not assumed to be homocyclic, then there will also be more indecomposable groups.

2. Coordinate matrices

The goal of this section is to describe a p-reduced, p-local $(2,2)$-group by means of an integer matrix, the 'coordinate matrix'. The coordinate matrix is obtained by means of 'bases' of $R = \mathrm{R}(G)$ and G/R. We start with the general concept.

DEFINITION 2.1. Let G be a p-reduced, p-local almost completely decomposable group with regulator R and p-basis $(x_1, \ldots, x_m)$ of R. Let $r = \mathrm{rk}(G/R)$. A matrix $\delta = [\delta_{i,j}]$ is a *coordinate matrix* of G modulo R if δ is integral, there is a basis $(\epsilon_1, \ldots, \epsilon_r)$ of G/R, there are representatives $g_i \in G$ of ϵ_i, and there is a p-basis $(x_1, \ldots, x_m)$ of R such that

$$g_i = p^{-k_i}\Big(\sum_{j=1}^{m} \delta_{i,j} x_j\Big)$$

where $\langle \epsilon_i \rangle \cong \mathbb{Z}_{p^{k_i}}$.

Coordinate matrices always exist and are uniquely determined by the bases (x_i) and (g_i); see [4, Lemmas 6 and 7]. Two p-reduced, p-local almost completely decomposable groups are isomorphic at p if and only if their coordinate matrices are equivalent via an equivalence relation defined by certain row and column operations listed below; see [4, Theorem 12].

Moreover, if the exponent of the regulator quotient of G is p^k, then any entry a_{ij} in the coordinate matrix δ may be replaced by an integer congruent to a_{ij} modulo p^k.

Let G be a $(2,2)$-group of rank m with regulator $R = R_{\tau_1} \oplus R_{\tau_2} \oplus R_{\tau_3} \oplus R_{\tau_4}$. We indicate a purification by the subscript '$*$'. Then the ordered set $(x_{1,1}, \ldots, x_{1,r_1}, x_{2,1}, \ldots, x_{2,r_2}, x_{3,1}, \ldots, x_{3,r_3}, x_{4,1}, \ldots, x_{4,r_4}) = (x_1, \ldots, x_m)$, where $x_i \notin pR$, is called a *p-basis* of R if $R = \bigoplus_{i,j} \langle x_{i,j} \rangle_*$, where $R_{\tau_i} = \bigoplus_{j=1}^{r_i} \langle x_{i,j} \rangle_*$. The choice of the p-basis divides the coordinate matrix in four blocks α_1, α_2, β_1, β_2 of sizes $r \times r_i$, $i = 1,2,3,4$. We write $\delta = [\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$ and $\alpha = [\alpha_1 \mid \alpha_2]$ and $\beta = [\beta_1 \mid \beta_2]$.

REMARK 2.2. We call transformations of rows and of columns of a coordinate matrix of a homocyclic $(2,2)$-group G *allowed* if the transformed coordinate matrix is the coordinate matrix of a group H where G and H are isomorphic at p. Then the following row and column operations on the coordinate matrix of a homocyclic $(2,2)$-group are allowed.

(1) Any multiple of a row may be added to any row below or above it.
(2) Any row or column may be multiplied by an integer relatively prime to p.
(3) Any multiple of a column of α_1 may be added to another column of $[\alpha_1 \parallel \alpha_2]$ and any multiple of a column of α_2 may be added to another column of α_2.
(4) Any multiple of a column of β_1 may be added to another column of $[\beta_1 \parallel \beta_2]$ and any multiple of a column of β_2 may be added to another column of β_2.

We now state the regulator criterion in [4, Lemma 13], in the special case of $(2,2)$-groups.

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

LEMMA 2.3. *Let G be a $(2,2)$-group. Then G has a regulating regulator. Let $r = \operatorname{rank}(G/R)$. The completely decomposable subgroup $R = R_{\tau_1} \oplus R_{\tau_2} \oplus R_{\tau_3} \oplus R_{\tau_4}$ of finite index in G is the regulator of G if and only if $R_{\tau_1} \oplus R_{\tau_2}$ and $R_{\tau_3} \oplus R_{\tau_4}$ are pure in G, and this holds if and only if α and β of a coordinate matrix $[\alpha \parallel \beta]$, relative to any p-basis of R (chosen as above), both have p-rank r.*

An almost completely decomposable group is called *clipped* if it has no summand of rank 1. An integral matrix $A = [a_{i,j}]$ is called *p-reduced* if:

(1) there is at most one 1 in a row and column and all other entries are in $p\mathbb{Z}$;
(2) if an entry 1 of A is at the position (i_s, j_s), then $a_{i_s,j} = 0$ for all $j > j_s$ and $a_{i,j_s} = 0$ for all $i < i_s$, and $a_{i_s,j}, a_{i,j_s} \in p\mathbb{Z}$ for all $j < j_s$ and all $i > i_s$.

Thus in a p-reduced matrix, the entries to the left of and below a 1 are in $p\mathbb{Z}$. By elementary row transformations upward and elementary column transformations to the right A can be transformed into a p-reduced matrix; see [4, Lemma 14].

3. Standard coordinate matrices

By Arnold's theorem two torsion-free groups of finite rank which are isomorphic at p have (up to isomorphism at p of summands) the same decomposition; see [1, Corollary 12.9]. Hence, given a coordinate matrix we may transform the matrix by allowed row and column transformations listed above and arrive at a coordinate matrix of the same group or of a group that is isomorphic at p to the original group. If we arrive at a matrix that shows that the group to which it belongs does or does not decompose, then the original group does or does not decompose.

For the convenience of the reader we gather together techniques, language conventions and standard conclusions.

- The term *line* means a row or a column.
- A *pivot* denotes a nonzero entry that will be used to annihilate either in its column or in its row.
- Block names, like pC, are placeholders only and are reused again and again with changing values.
- By '$p^l \in A$' we mean an entry in the matrix A that is in $p^l\mathbb{Z} \setminus p^{l+1}\mathbb{Z}$. Since we may multiply a line by a unit modulo p we can ignore unit factors in our situation.
- The matrix $A = [a_{i,j}]$ has a *cross at* (i_0, j_0) if $a_{i_0,j_0} \neq 0$ and $a_{i_0,j} = 0$, $a_{i,j_0} = 0$ for all $i \neq i_0$ and $j \neq j_0$. The entry a_{i_0,j_0} is called *cross entry*.
- By '$x \in A$ leads to a cross' we mean that this entry x can be used as a pivot in its row and its column to produce a cross by allowed line transformations, that is, x is afterward the cross entry.
- We apply transformations to annihilate entries. While doing this, some other entries that were originally zero may become nonzero; those entries are called *fill-ins*. Mostly we want to change certain submatrices either to a 0-matrix or to a matrix of the ('normed') form p^hI, $h \geq 0$. The phrase 'we can annihilate' tacitly

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

includes that the occurring fill-ins can be removed by subsequent transformations and the previously 'normed' blocks are reestablished. Note that sometimes fill-ins occur that have a prefactor p^3, hence can be and are replaced by 0, because we deal with groups with regulator quotient of exponent p^3.

- The phrase 'we form the Smith normal form of A' means that by a sequence of arbitrary elementary row and column transformations we obtain a diagonal matrix with diagonal entries 0 or p^l with exponents in increasing order from the top down. It is tacitly included that all the line transformations are allowed and that destroyed 'normed' blocks can be reestablished.
- If blocks split into subblocks, then if possible we keep the original name to avoid overwhelming indexing. Forming the Smith normal form, in general, splits blocks.

A matrix is *decomposed* if it is of the form $\left[\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right]$. Here either one of the matrices A, B is allowed to have no rows or no columns, that is, the decomposed matrices include the special cases $[0\ B], \left[\begin{smallmatrix} 0 \\ B \end{smallmatrix}\right]$, $[A\ 0], \left[\begin{smallmatrix} A \\ 0 \end{smallmatrix}\right]$. A matrix A is called *decomposable* if there are row and column permutations that transform it to a decomposed form, that is, there are permutation matrices P, Q such that PAQ is decomposed.

Lemma 3.1. *A p-local, p-reduced almost completely decomposable group with an inverted forest as critical typeset is decomposable if and only if there exists a decomposable coordinate matrix.*

Proof. If the p-local, p-reduced almost completely decomposable group $G = H \oplus L$ is decomposable, then H, L may be assumed to be given by coordinate matrices δ^H, δ^L. Since the critical typeset of G is an inverted forest, we have $\mathrm{R}(G) = \mathrm{R}(H) \oplus \mathrm{R}(L)$; see [5, Lemma 3.1]. Thus G has the coordinate matrix $\delta^G = \delta^H \oplus \delta^L$. Hence a decomposable group G has a decomposable coordinate matrix.

Conversely, a group G with a decomposable coordinate matrix obviously is decomposable. □

We establish kind of a standard form for coordinate matrices of $(2, 2)$-p^3-hc-groups without summands of rank 3 or lower. This is no restriction in the context of decomposition, because such summands are not $(2, 2)$-groups.

Proposition 3.2. *A* $(2, 2)$-p^3-hc-*group G without summands of rank* 3 *or lower has a coordinate matrix*

$$[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{ccc|ccc||c|c} I_A & 0 & 0 & 0 & 0 & 0 & A & A' \\ \hline 0 & pI_B & 0 & I_B & 0 & 0 & B & B' \\ \hline 0 & 0 & p^2 I_C & 0 & I_C & 0 & pC & C' \\ \hline 0 & 0 & 0 & 0 & 0 & I_D & pD & D' \end{array}\right] \tag{3.1}$$

$$\underbrace{\qquad\qquad}_{\alpha_1}\quad\underbrace{\qquad\qquad}_{\alpha_2}\quad\underbrace{\quad}_{\beta_1}\quad\underbrace{\quad}_{\beta_2}$$

such that:

(1) *the sizes of the identity matrices I_A, I_B, I_C, I_D all are isomorphism at p invariants of G and the sum of the sizes of I_A, I_B, I_C, I_D is the rank of the regulator quotient;*

(2) *the submatrix $[\beta_1 \mid \beta_2]$ is p-reduced and the submatrix of β_2, obtained by omitting the* 0*-rows, is the identity matrix—in particular, the blocks A', B', C', D' are completely determined by β_1;*

(3) *β_1 has no* 0*-line and a cross in β_1 has a cross entry $\neq 1$ in B or in C, and such a cross displays a summand of rank* 4.

Proof. As G is clipped neither α nor β can contain a 0-column. The regulator criterion implies that neither α nor β can have a 0-row. All elementary row and column transformations are allowed in α_1, hence α_1 may be assumed to be in Smith normal form. Moreover, we may assume α to be p-reduced, hence there are 0-rows in α_2 to the right of I_A. The regulator criterion requires that the submatrix of α_2, obtained by omitting its 0-rows, can be transformed to the identity matrix by column transformations in α_2, hence without changing α_1. The claimed form of α is now established.

(1) It can be shown that the sizes of the identity matrices I_A, I_B, I_C, I_D all are isomorphism at p invariants of G.

(2) Row transformations upward in α create fill-ins in α that can be removed by suitable allowed column transformations in α. Hence β may be transformed by row transformations upward and the usual allowed column transformations. So we may assume that β is in p-reduced form. Using the same arguments as for α we show that there are zeros to the right of any 1 in β_1 and the submatrix remaining after omitting all zero rows from β_2 may be changed to the identity matrix. This can be done without changing β_1 or α.

(3) Clearly, β_1 has no 0-line, because there are no summands of rank 3 or lower.

It remains to show, in particular, to complete the proof of (3), that the entries of β_1 in C and D all are in $p\mathbb{Z}$. A $1 \in D$ leads to a cross in β. Hence the entries of D all are in $p\mathbb{Z}$. Also a $1 \in C$ leads to a cross in β_1 using the fact that the entries in D are in $p\mathbb{Z}$. In both cases there are summands of rank 3 or lower. All other locations of a cross in β_1 display summands of rank 3 or lower if the cross entry is 1, and summands of rank 4 if the cross entry is not a unit modulo p. □

Note that in order to keep the matrices from growing too large, we often use the abbreviated 'normal' form of the coordinate matrix in Equation (3.1),

$$[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{c|c||c|c} I_A & 0 & A & A' \\ \hline pI_B & I_B & B & B' \\ \hline p^2 I_C & I_C & pC & C' \\ \hline 0 & I_D & pD & D' \end{array}\right].$$

The following example illustrates the special form that is claimed to exist in Proposition 3.2.

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

Example 3.3. Let G be a $(2,2)$-p^3-hc-group with a coordinate matrix

$$[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \left[\begin{array}{ccccc|cccc||ccccc|cccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & p & 1 & 0 & 0 & 0 \\
0 & 0 & p & 0 & 0 & 1 & 0 & 0 & 0 & p^2 & p & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & p & 0 & 0 & 1 & 0 & 0 & 0 & 0 & p^2 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & p^2 & 0 & 0 & 1 & 0 & p^2 & 0 & 0 & p & 0 & 0 & 0 & 1 & 0 \\
\multicolumn{5}{c|}{\underbrace{0\quad 0\quad 0\quad 0\quad 0}_{\alpha_1}} & \multicolumn{4}{c||}{\underbrace{0\quad 0\quad 0\quad 1}_{\alpha_2}} & \multicolumn{5}{c|}{\underbrace{0\quad 0\quad 0\quad p\quad p^2}_{\beta_1}} & \multicolumn{4}{c}{\underbrace{0\quad 0\quad 0\quad 1}_{\beta_2}}
\end{array}\right]
\begin{array}{l} A^1 \\ A^2 \\ B^1 \\ B^2 \\ C \\ D \end{array}$$

Here the cross in β_1 with a cross point in the B^2-row leads to a direct summand of rank 4.

A coordinate matrix of a $(2,2)$-p^3-hc-group as in Proposition 3.2 is called *standard*. Note that in a standard coordinate matrix the form of α_2, β_2 is completely determined by α_1, β_1, respectively.

Line transformations of β_1 are called *α-allowed* if after executing such, α can be returned to its previous form by column transformations of α. All column transformations of β_1 are automatically α-allowed.

The following row transformations are α-allowed.

(1) Any line may be multiplied by a unit.
(2) Any row transformation may be applied to A, B, C, D.
(3) Any multiple of a row in D may be added to any other row.
(4) Any multiple of a row in C may be added to any row in $C \cup B \cup A$.
(5) Any multiple of a row in B may be added to any row in $B \cup A$.
(6) Any p-multiple of a row in A may be added to a row in B.
(7) Any p-multiple of a row in B may be added to a row in C.
(8) Any p-multiple of a row in C may be added to a row in D.
(9) Any p^2-multiple of a row in A may be added to a row in C.
(10) Any p^2-multiple of a row in B may be added to a row in D.

Proposition 3.4. *A $(2,2)$-p^3-hc-group is decomposable if and only if there exists a standard coordinate matrix* $[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$ *with decomposable* β_1.

Proof. We proceed as in Lemma 3.1 and specify to standard coordinate matrices. Let G be a $(2,2)$-p^3-hc-group. If $G = H \oplus L$, then H, L may be assumed to be given by standard coordinate matrices $[\alpha_1^H \mid \alpha_2^H \parallel \beta_1^H \mid \beta_2^H]$ and $[\alpha_1^L \mid \alpha_2^L \parallel \beta_1^L \mid \beta_2^L]$, respectively. Thus G has the coordinate matrix

$$[\alpha^G \parallel \beta^G] = [\alpha_1^G \mid \alpha_2^G \parallel \beta_1^G \mid \beta_2^G] = [\alpha_1^H \oplus \alpha_1^L \mid \alpha_2^H \oplus \alpha_2^L \parallel \beta_1^H \oplus \beta_1^L \mid \beta_2^H \oplus \beta_2^L].$$

We rearrange $[\alpha_1^H \oplus \alpha_1^L \mid \alpha_2^H \oplus \alpha_2^L]$ by row and column permutations to obtain α^G in standard form. This permutes the rows of β^G. Note that β_2^G can be transformed into standard form without changing α^G, β_1^G, respectively. Since $\beta_1^G = \beta_1^H \oplus \beta_1^L$ is decomposed, the resulting β_1^G in standard form is decomposable. Thus for a decomposable group G there exists a standard coordinate matrix with decomposable β_1^G.

Conversely, if G has a standard coordinate matrix with decomposable β_1, then G obviously is decomposable. □

Example 3.5. Let G be a $(2,2)$-p^3-hc-group with a coordinate matrix

$$[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2] = \begin{array}{c}\left[\begin{array}{cc|c||cc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & p & 1 & 0 & p^2 & 1 \end{array}\right]\begin{array}{c} A \\ B \end{array} \\ \begin{array}{cccc} \underbrace{\qquad}_{\alpha_1} & \underbrace{\qquad}_{\alpha_2} & \underbrace{\qquad}_{\beta_1} & \underbrace{\qquad}_{\beta_2} \end{array}\end{array}$$

The matrix β_1 is decomposed, more precisely it has crosses, hence G is decomposable. In fact G is the sum of groups of rank 2 and rank 4, according to the rows A and B.

4. Indecomposable $(2,2)$-p^3-hc-groups

Example 4.1. It is easy to list all indecomposable $(2,2)$-p^3-hc-groups of rank 4, because there are only the following three possibilities for α, namely $[p \mid 1]$, $[p^2 \mid 1]$ and I_2, the identity matrix of size 2. Moreover, β_1 is a column. Hence in the first two cases for α we obtain $\beta_1 = [p]$ or $= [p^2]$. If $\alpha = I_2$, then $\beta_1 = \left[\begin{smallmatrix}1\\p\end{smallmatrix}\right]$ or $= \left[\begin{smallmatrix}1\\p^2\end{smallmatrix}\right]$. These six examples are pairwise not isomorphic at p and form a complete collection of indecomposable $(2,2)$-p^3-hc-groups of rank 4. They are listed below, (1)–(6).

We next list the isomorphism at p classes of indecomposable $(2,2)$-p^3-hc-groups. We define the type of a group G using the invariants of their standard coordinate matrix and β_1. The notation is self-explanatory. G is of type $(\mathrm{rk}G/R, \mathrm{rk}G, x, \beta_1)$, where x is a part of $\left[\begin{smallmatrix}A\\B\\C\\D\end{smallmatrix}\right]$ indicating which block rows are present:

(1) $\left[\begin{array}{c|c||c|c} p & 1 & p & 1\end{array}\right]$ of type $\left(1, 4, [B], \left[p\right]\right)$;

(2) $\left[\begin{array}{c|c||c|c} p & 1 & p^2 & 1\end{array}\right]$ of type $\left(1, 4, [B], \left[p^2\right]\right)$;

(3) $\left[\begin{array}{c|c||c|c} p^2 & 1 & p & 1\end{array}\right]$ of type $\left(1, 4, [C], \left[p\right]\right)$;

(4) $\left[\begin{array}{c|c||c|c} p^2 & 1 & p^2 & 1\end{array}\right]$ of type $\left(1, 4, [C], \left[p^2\right]\right)$;

(5) $\left[\begin{array}{c|c||c|c} 1 & 0 & 1 & 0 \\ 0 & 1 & p & 1\end{array}\right]$ of type $\left(2, 4, \begin{bmatrix}A\\D\end{bmatrix}, \begin{bmatrix}1\\p\end{bmatrix}\right)$;

(6) $\left[\begin{array}{c|c||c|c} 1 & 0 & 1 & 0 \\ 0 & 1 & p^2 & 1\end{array}\right]$ of type $\left(2, 4, \begin{bmatrix}A\\D\end{bmatrix}, \begin{bmatrix}1\\p^2\end{bmatrix}\right)$;

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

(7) $\begin{bmatrix} 1 & \mid & 0 & \| & p & \mid & 1 & 0 \\ 0 & \mid & 1 & \| & p^2 & \mid & 0 & 1 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} A \\ D \end{bmatrix}, \begin{bmatrix} p \\ p^2 \end{bmatrix}\right)$;

(8) $\begin{bmatrix} 1 & \mid & 0 & \| & 1 & 0 & \mid & 0 \\ 0 & \mid & 1 & \| & p & p^2 & \mid & 1 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} A \\ D \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ p & p^2 \end{bmatrix}\right)$;

(9) $\begin{bmatrix} 1 & 0 & \mid & 0 & \| & 1 & \mid & 0 \\ 0 & p^2 & \mid & 1 & \| & p & \mid & 1 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} 1 \\ p \end{bmatrix}\right)$;

(10) $\begin{bmatrix} 1 & 0 & \mid & 0 & \| & p & \mid & 1 & 0 \\ 0 & p^2 & \mid & 1 & \| & p^2 & \mid & 0 & 1 \end{bmatrix}$ of type $\left(2, 6, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} p \\ p^2 \end{bmatrix}\right)$;

(11) $\begin{bmatrix} 1 & 0 & \mid & 0 & \| & 1 & 0 & \mid & 0 \\ 0 & p^2 & \mid & 1 & \| & p & p^2 & \mid & 1 \end{bmatrix}$ of type $\left(2, 6, \begin{bmatrix} A \\ C \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ p & p^2 \end{bmatrix}\right)$;

(12) $\begin{bmatrix} p & \mid & 1 & 0 & \| & 1 & \mid & 0 \\ 0 & \mid & 0 & 1 & \| & p & \mid & 1 \end{bmatrix}$ of type $\left(2, 5, \begin{bmatrix} B \\ D \end{bmatrix}, \begin{bmatrix} 1 \\ p \end{bmatrix}\right)$;

(13) $\begin{bmatrix} p & \mid & 1 & 0 & \| & p & \mid & 1 & 0 \\ 0 & \mid & 0 & 1 & \| & p^2 & \mid & 0 & 1 \end{bmatrix}$ of type $\left(2, 6, \begin{bmatrix} B \\ D \end{bmatrix}, \begin{bmatrix} p \\ p^2 \end{bmatrix}\right)$;

(14) $\begin{bmatrix} p & \mid & 1 & 0 & \| & 1 & 0 & \mid & 0 \\ 0 & \mid & 0 & 1 & \| & p & p^2 & \mid & 1 \end{bmatrix}$ of type $\left(2, 6, \begin{bmatrix} B \\ D \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ p & p^2 \end{bmatrix}\right)$;

(15) $\begin{bmatrix} 1 & 0 & \mid & 0 & 0 & \| & p & 0 & \mid & 1 & 0 \\ 0 & p & \mid & 1 & 0 & \| & 0 & 1 & \mid & 0 & 0 \\ 0 & 0 & \mid & 0 & 1 & \| & p^2 & p & \mid & 0 & 1 \end{bmatrix}$ of type $\left(3, 8, \begin{bmatrix} A \\ B \\ D \end{bmatrix}, \begin{bmatrix} p & 0 \\ 0 & 1 \\ p^2 & p \end{bmatrix}\right)$;

(16) $\begin{bmatrix} 1 & 0 & \mid & 0 & 0 & \| & 1 & 0 & \mid & 0 & 0 \\ 0 & p^2 & \mid & 1 & 0 & \| & p & 0 & \mid & 1 & 0 \\ 0 & 0 & \mid & 0 & 1 & \| & p & p^2 & \mid & 0 & 1 \end{bmatrix}$ of type $\left(3, 8, \begin{bmatrix} A \\ C \\ D \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ p & 0 \\ p & p^2 \end{bmatrix}\right)$.

By [4, Theorem 12] two groups are isomorphic at p if and only if their coordinate matrices can be transformed into one another by allowed transformations. If a group G is given by a standard coordinate matrix $[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$, then, by Proposition 3.4, to show the indecomposability of G we have to do the following:

(1) apply elementary row transformations to β_1 such that the standard form of α can be reestablished by allowed column transformations of α;

(2) apply elementary column transformations to β_1.

If β_1 never changes to a decomposable matrix, then G is indecomposable.

To decide if two groups given by β_1 and β_1' are not isomorphic at p apply the transformations (1) and (2) above, and show that β and β' never can be transformed into each other.

Both tasks are easily shown for the given small matrices β_1 in the list.

In the next theorem we show that this list is complete.

Theorem 4.2. *There are precisely the* 16 *isomorphism at p types in the list above of indecomposable* $(2,2)$*-groups with homocyclic regulator quotient of exponent* p^3.

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

PROOF. Let G be an indecomposable $(2, 2)$-group with homocyclic regulator quotient of exponent p^3. Then G cannot have summands of rank 3 because G would have to be equal to such a summand but $(2, 2)$-groups have ranks 4 or lower. The case of groups of rank 4 is easily disposed of (see (1)–(6) in the list; see also Example 4.1). We therefore exclude also summands of rank 4. Then the standard coordinate matrix $[\alpha_1 \mid \alpha_2 \parallel \beta_1 \mid \beta_2]$ given in Proposition 3.2 has additional properties: first, there is never a cross in β_1; and second, at least two of the blocks A, B, C, D are present because the presence of just one block allows the transformation of β_1 to Smith normal form and then is decomposed. Moreover, it is impossible that only one block row is present, because this would allow transformation of β_1 to Smith normal form, that is, the group would then be decomposable.

In order to apply Proposition 3.4 to determine decompositions of G, we must start with a coordinate matrix in standard form and must preserve the standard form throughout. Therefore we must restrict ourselves to α-allowed line transformations of β_1.

The matrix in Equation (3.1) incorporates all possibilities where block rows as well as block columns may be absent. We label the block rows by A through D and speak of presence or absence of such rows. Note that the presence of A includes the presence of the first block column because I is a square matrix. In general, block rows and block columns that intersect in matrices I, pI, p^2I always are simultaneously present or not.

Our final goal is to obtain a block matrix for β_1 such that all blocks are 0 or p^hI, $h \geq 0$. This is done by creating more and more blocks of the form 0 or p^hI, $h \geq 0$, in β_1. In the process the block structure of β_1 gets refined caused by the splitting of block lines when introducing the Smith normal form of other blocks.

Existing blocks p^hI must be preserved by later transformations, and existing blocks 0 must be reestablished if they get changed by 'fill-ins'.

By Proposition 3.4 it is essential to apply only those allowed row transformations that allow the standard form of α to be reestablished by column transformations.

If it is not obvious how to clear fill-ins, we indicate how they can be turned into 0-blocks again by allowed transformations. For blocks with unspecified content we use placeholder names and reuse the names even if the unspecified content changes.

(a) *Smith Normal Forms for* A, B, *and consequences.*

We show that

$$[\alpha \parallel \beta_1] = \left[\begin{array}{c|c||ccccccc} I_A & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 \\ I_A & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & I & 0 & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & I_C & pC_1 & p^2C_2 & p^2C_3 & p^2C_4 & 0 & 0 & p^2C_7 \\ 0 & I_D & pD_1 & p^2D_2 & p^2D_3 & p^2D_4 & pD_5 & p^2D_6 & p^2D_7 \end{array}\right]. \quad (4.1)$$

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

Note that β is completely determined by β_1 and that β_2 is, up to 0-rows, the identity matrix. Thus for decomposability questions β_2 is irrelevant.

Starting with Equation (3.1), we first form the Smith normal form for B and then the Smith normal form of the part X of A above the 0-part of B, and we get, after deleting a 0-row and rearranging columns,

$$[\alpha \parallel \beta_1] = \left[\begin{array}{c|c||ccccccc} I_A & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & 0 & 0 & pI & 0 & 0 & 0 & A_1 & A_2 \\ I_A & 0 & 0 & 0 & p^2I & 0 & 0 & A_3 & A_4 \\ I_A & 0 & 0 & 0 & 0 & 0 & 0 & A_5 & A_6 \\ pI_B & I_B & 0 & 0 & 0 & 0 & I & 0 & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & I_C & pC_1 & pC_2 & pC_3 & pC_4 & pC_5 & pC_6 & pC_7 \\ 0 & I_D & pD_1 & pD_2 & pD_3 & pD_4 & pD_5 & pD_6 & pD_7 \end{array}\right].$$

Above I in B we annihilate in A, and with I in A we annihilate the row.

Next we show that the blocks $A_1, \ldots, A_6$ are either not present or 0. As mentioned before, we use only allowed row transformations with the additional property that there are allowed column transformations that reestablish the standard form of α.

Note that due to the presence of p^2I in B we may assume that the nonzero entries of A_6 are either units or are in $p\mathbb{Z} \setminus p^2\mathbb{Z}$. If there is a unit in A_6 then we obtain a cross in $\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right]$. By rearranging rows and columns the cross may be incorporated in the existing I in A, thereby enlarging it. Thus we may assume that all entries of A_6 are in $p\mathbb{Z}$.

But a $p \in A_6$ allows us to annihilate in p^2I below and the fill-ins below A_5 can be annihilated by the present pI just above in B. This creates a 0-row in β_1. Thus $A_6 = 0$.

All nonzero entries of A_5 are units due to the presence of pI below. But a unit in A_5 allows us to annihilate in pI below, and since already $A_6 = 0$, a 0-row is created in β_1. This shows that the A_5/A_6-row is not present.

Note that due to the presence of pI in A all entries of A_1 and A_2 are units or 0, and due to the presence of pI in B all entries of A_3 are units or 0. A unit in A_2 leads to a cross in $\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right]$. We incorporate the crosses in the I in A, thereby enlarging it, and may assume that $A_2 = 0$. The same holds for A_3, that is, we may assume that $A_3 = 0$. Now we consider A_4. All nonzero entries in $p^2\mathbb{Z}$ can be annihilated due to the presence of p^2I below. All other nonzero entries of A_4 lead to a cross in $\left[\begin{smallmatrix}A\\B\end{smallmatrix}\right]$ with cross entry 1 or p. We incorporate the crosses with cross entry 1 in the I in A, thereby enlarging it, and we incorporate the crosses with cross entry p in the pI in A. Thus we may assume that $A_4 = 0$.

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

Next we show that we may assume $A_1 = 0$. The Smith normal form of A_1 is $\left[\begin{smallmatrix} I & 0 \\ 0 & 0 \end{smallmatrix}\right]$ as mentioned before. We annihilate first with I in A_1 downward pI in B, and then we annihilate pI in A. There is a block $-p^2I$ filled in B below the original pI in A. This block column has the same properties in $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ as the block column of p^2I with p^2I in B. Hence we may enlarge this block column, causing $A_1 = 0$. Thus we get

$$[\alpha \parallel \beta_1] = \left[\begin{array}{c|c||ccccccc} I_A & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 \\ I_A & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & I & 0 & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & 0 & pI & 0 \\ pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & I_C & pC_1 & pC_2 & pC_3 & pC_4 & pC_5 & pC_6 & pC_7 \\ 0 & I_D & pD_1 & pD_2 & pD_3 & pD_4 & pD_5 & pD_6 & pD_7 \end{array}\right].$$

The submatrix pC_5 can be annihilated due to the presence of I in B above, so $pC_5 = 0$. A $p \in pD_4$ leads to a cross. So we write p^2D_4. In turn we get crosses and, as for D_4, we get

$$p^2D_4, p^2C_4, p^2D_3, p^2C_3, p^2D_7, p^2C_7, p^2D_2, p^2C_2, p^2D_6, p^2C_6. \qquad (*)$$

The arguments for the different statements in (*) are all very similar, therefore we prove exemplarily one of these statements, say p^2C_2.

We assume that we have already obtained $p^2D_4, p^2C_4, p^2D_3, p^2C_3, p^2D_7, p^2C_7, p^2D_2$. If $p \in pC_2$, then we annihilate with this p in its column and afterwards we annihilate in its row to obtain a cross in β_1 with this p as cross entry. The row transformations to annihilate the column change α. But the fill-ins in the last row have the prefactor p^3 and can be replaced by 0, and fill-ins in the block row of A containing pI can be cleared by allowed column transformations of α.

Given the properties denoted in (*), we may use pI above p^2C_6 to get $p^2C_6 = 0$. Thus β_1 has the form as in Equation (4.1).

(b) *Treatment of D.*

We deal with the interference of D_1 and D_5. There is no $p^2 \in pD_5$ due to the presence of I in B above and there is no 0-column. Now a $p \in pD_1$ allows us to annihilate in pD_5 and a $p \in pD_5$ allows us to annihilate p^2s in pD_1. Fill-ins can be deleted by I in B and by I in A, respectively. Hence a $p \in D_1$ continues to a 0-row in pD_5 and a $p^2 \in pD_1$ can only survive if this row continues to a 0-row in pD_5. Thus forming first the Smith normal form of pD_1 and afterwards the Smith normal form of the relevant parts of pD_5, we get

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

$$[\alpha \,\|\, \beta_1] = \left[\begin{array}{c|c||ccccccccc}
I_A & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 \\
pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\
pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\
p^2I_C & I_C & pC_{11} & pC_{12} & pC_{13} & p^2C_2 & p^2C_3 & p^2C_4 & 0 & 0 & p^2C_7 \\
0 & I_D & pI & 0 & 0 & p^2D_{21} & p^2D_{31} & p^2D_{41} & 0 & p^2D_{61} & p^2D_{71} \\
0 & I_D & 0 & p^2I & 0 & p^2D_{22} & p^2D_{32} & p^2D_{42} & 0 & p^2D_{62} & p^2D_{72} \\
0 & I_D & 0 & 0 & 0 & p^2D_{23} & p^2D_{33} & p^2D_{43} & 0 & p^2D_{63} & p^2D_{73} \\
0 & I_D & 0 & 0 & 0 & p^2D_{24} & p^2D_{34} & p^2D_{44} & pI & p^2D_{64} & p^2D_{74}
\end{array}\right].$$

First, we annihilate with pI in D_1 in its column and then in its row. Fill-ins in C can be deleted by pI in B, and fill-ins in A can be deleted by pI in A and B, respectively. Thus $pC_{11} = 0$, $p^2D_{21} = 0$ and $p^2D_{61} = 0$. Second, we annihilate with pI in pD_5 the block p^2D_{64}. Fill-ins can be deleted by pI in B.

Now we deal with $D_{42}, D_{43}, D_{32}, D_{33}, D_{73}, D_{72}$, in this sequence. A $p^2 \in p^2D_{42}$ or a $p^2 \in p^2D_{43}$ leads to a cross in β_1, regardless with which we start. So we may assume that $p^2D_{42} = 0$, $p^2D_{43} = 0$. In turn and by the same arguments $p^2D_{32} = 0$, $p^2D_{33} = 0$, and $p^2D_{72} = 0$, $p^2D_{73} = 0$.

Further, a $p^2 \in p^2D_{22}$ or a $p^2 \in p^2D_{23}$ leads to $\beta_1' = \left[\begin{smallmatrix} p & | & A \\ p^2 & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 5, \left[\begin{smallmatrix} A \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} p \\ p^2 \end{smallmatrix}\right]\right)$; see (7) in the list. Hence, omitting all those summands, we may assume that $p^2D_{22} = 0$ and $p^2D_{23} = 0$. Thus we get

$$[\alpha \,\|\, \beta_1] = \left[\begin{array}{c|c||ccccccccc}
I_A & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 & 0 \\
I_A & 0 & 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 & 0 \\
pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 & 0 \\
pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & pI & 0 \\
pI_B & I_B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\
p^2I_C & I_C & 0 & pC_{12} & pC_{13} & p^2C_2 & p^2C_3 & p^2C_4 & 0 & 0 & p^2C_7 \\
0 & I_D & pI & 0 & 0 & 0 & p^2D_{31} & p^2D_{41} & 0 & 0 & p^2D_{71} \\
0 & I_D & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & p^2D_{62} & 0 \\
0 & I_D & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2D_{63} & 0 \\
0 & I_D & 0 & 0 & 0 & p^2D_{24} & p^2D_{34} & p^2D_{44} & pI & 0 & p^2D_{74}
\end{array}\right].$$

Now we deal with the p^2D_{62}-column. A $p^2 \in p^2D_{63}$ leads to $\beta_1' = \left[\begin{smallmatrix} p & | & B \\ p^2 & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 6, \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} p \\ p^2 \end{smallmatrix}\right]\right)$; see (13) in the list. Hence, omitting all those summands, we may assume that $p^2D_{63} = 0$. But this displays a 0-row in D. Thus the p^2D_{63}-row is not present. In turn $p^2 \in p^2D_{62}$ leads to the same type of a summand. Thus also the p^2D_{62}-row is not present. But then there is a cross in the D_6-column. So this column is not present and consequently the pI-row in B is not present. So we get

$$[\alpha \,\|\, \beta_1] = \left[\begin{array}{cc|cccccccc} I_A & | \; 0 & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 \\ pI_B & | \; I_B & 0 & 0 & 0 & 0 & 0 & 0 & I & 0 \\ pI_B & | \; I_B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & | \; I_C & 0 & pC_{12} & pC_{13} & p^2C_2 & p^2C_3 & p^2C_4 & 0 & p^2C_7 \\ 0 & | \; I_D & pI & 0 & 0 & 0 & p^2D_{31} & p^2D_{41} & 0 & p^2D_{71} \\ 0 & | \; I_D & 0 & p^2I & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & | \; I_D & 0 & 0 & 0 & p^2D_{24} & p^2D_{34} & p^2D_{44} & pI & p^2D_{74} \end{array}\right].$$

By I in A above, all entries of pC_{12} and pC_{13} are in $(p\mathbb{Z} \setminus p^2\mathbb{Z}) \cup \{0\}$. Thus with a $p \in pC_{13}$ we annihilate in pC_{12}. Fill-ins in A can be deleted by I in A again. Hence a surviving $p \in C_{12}$ allows us to annihilate in p^2I below in D. All fill-ins are in $p^3\mathbb{Z}$ and can be replaced by 0. But this creates a 0-row in D. Consequently, $pC_{12} = 0$ and this displays a summand with $\beta_1' = \left[\begin{smallmatrix} 1 & | & A \\ p^2 & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 4, \left[\begin{smallmatrix} A \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ p^2 \end{smallmatrix}\right]\right)$; see (6) in the list of rank 4. Hence, omitting all those summands, the pC_{12}-column is not present and in turn the rows in A, D that have I, p^2I in the pC_{12}-column also are not present. Hence we get

$$[\alpha \,\|\, \beta_1] = \left[\begin{array}{cc|ccccccc} I_A & | \; 0 & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & 0 & pI & 0 & 0 & 0 & 0 \\ I_A & | \; 0 & 0 & 0 & 0 & p^2I & 0 & 0 & 0 \\ pI_B & | \; I_B & 0 & 0 & 0 & 0 & 0 & I & 0 \\ pI_B & | \; I_B & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & | \; I_C & 0 & pC_{13} & p^2C_2 & p^2C_3 & p^2C_4 & 0 & p^2C_7 \\ 0 & | \; I_D & pI & 0 & 0 & p^2D_{31} & p^2D_{41} & 0 & p^2D_{71} \\ 0 & | \; I_D & 0 & 0 & p^2D_{24} & p^2D_{34} & p^2D_{44} & pI & p^2D_{74} \end{array}\right].$$

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

A $p^2 \in p^2D_{44}$ leads to $\beta_1' = \left[\begin{smallmatrix} 0 & 1 & | & B \\ p^2 & p & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 6, \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ p & p^2 \end{smallmatrix}\right]\right)$; see (14) in the list. Fill-ins in C and D can be deleted by I in B and by pI in D, respectively. Hence, omitting all those summands, we may assume that $p^2D_{44} = 0$. Then a $p^2 \in p^2D_{34}$ leads to $\beta_1' = \left[\begin{smallmatrix} p^2 & 0 & | & A \\ 0 & 1 & | & B \\ p^2 & p & | & D \end{smallmatrix}\right]$ and we get a 0-row in β_1. Thus we may assume that $p^2D_{34} = 0$. In turn a $p^2 \in p^2D_{74}$ leads to $\beta_1' = \left[\begin{smallmatrix} 1 & 0 & | & B \\ 0 & p^2 & | & B \\ p & p^2 & | & D \end{smallmatrix}\right]$ and we get a 0-row in β_1. Thus $p^2D_{74} = 0$. Consequently, a $p^2 \in p^2D_{24}$ leads to $\beta_1' = \left[\begin{smallmatrix} p & 0 & | & A \\ 0 & 1 & | & B \\ p^2 & p & | & D \end{smallmatrix}\right]$, a summand of type $\left(3, 8, \left[\begin{smallmatrix} A \\ B \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} p & 0 \\ 0 & 1 \\ p^2 & p \end{smallmatrix}\right]\right)$; see (15) in the list. Fill-ins in C can be deleted by I in B. Hence, omitting all those summands, we may assume that $p^2D_{24} = 0$. But this displays $\beta_1' = \left[\begin{smallmatrix} 1 & | & B \\ p & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 5, \left[\begin{smallmatrix} B \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ p \end{smallmatrix}\right]\right)$; see (12) in the list. Hence, omitting all those summands, the last row of D and thus also the I-row in B are no longer present. Thus we get

$$[\alpha \parallel \beta_1] = \begin{bmatrix} I_A & | & 0 & \| & I & 0 & 0 & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & I & 0 & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & 0 & pI & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & 0 & 0 & p^2I & 0 & 0 \\ pI_B & | & I_B & \| & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & | & I_C & \| & 0 & pC_1 & p^2C_2 & p^2C_3 & p^2C_4 & p^2C_7 \\ 0 & | & I_D & \| & pI & 0 & 0 & p^2D_3 & p^2D_4 & p^2D_7 \end{bmatrix}.$$

Now we show that

$$D' = [p^2D_3 \mid p^2D_4 \mid p^2D_7] = [0 \mid pI, 0 \mid 0].$$

A 0-row in D' leads to $\beta_1' = \left[\begin{smallmatrix} 1 & | & A \\ p & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 4, \left[\begin{smallmatrix} A \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ p \end{smallmatrix}\right]\right)$; see (5) in the list of rank 4. Hence, omitting all those summands, there is no 0-row in D'.

Now $p^2 \in D_4$ allows us to annihilate in p^2D_3 and in p^2D_7, and a surviving $p^2 \in D_3$ allows us to annihilate in p^2D_7. Fill-ins in A can be deleted by p^2I in B. Thus a surviving $p^2 \in D_3$ or a surviving $p^2 \in D_7$ is the only nonzero entry in its row in D'. So such a surviving $p^2 \in D_3$ or $p^2 \in D_7$ allows us to annihilate in p^2I above in A and in p^2I above in B, respectively. Fill-ins in A and in B can be deleted by I above in A. This creates a 0-row in A and in B, respectively, a contradiction. Hence $p^2D_3 = 0$ and $p^2D_7 = 0$.

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

Finally, the Smith normal form of p^2D_4 is $[pI, 0]$, using again the fact that D' has no 0-row. Thus we get

$$[\alpha \,\|\, \beta_1] = \begin{bmatrix} I_A & | & 0 & \| & I & 0 & 0 & 0 & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & I & 0 & 0 & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & 0 & pI & 0 & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & 0 & 0 & p^2I & 0 & 0 & 0 \\ pI_B & | & I_B & \| & 0 & 0 & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & | & I_C & \| & 0 & pC_1 & p^2C_2 & p^2C_3 & p^2C_{41} & p^2C_{42} & p^2C_7 \\ 0 & | & I_D & \| & pI & 0 & 0 & 0 & p^2I & 0 & 0 \end{bmatrix}.$$

(c) *Treatment of C.*

Recall that all entries of pC_1 are in $(p\mathbb{Z} \setminus p^2\mathbb{Z}) \cup \{0\}$ and that pC_1 has no 0-column. Hence the Smith normal form of pC_1 is $\left[\begin{smallmatrix} pI \\ 0 \end{smallmatrix}\right]$. Moreover, a $p^2 \in p^2C_{42}$ continues to a nonzero row in pC_1 because otherwise this p^2 allows us to annihilate the whole C-row and this leads to a cross. So, since p^2C_{42} has no 0-column, forming the Smith normal form of the nonzero part of p^2C_{42}, we get $[pC_1 \| p^2C_{42}] = \left[\begin{smallmatrix} pI & 0 & \| & p^2I \\ 0 & pI & \| & 0 \\ 0 & 0 & \| & 0 \end{smallmatrix}\right]$.

Consequently a $p^2 \in p^2C_{42}$ leads to $\beta_1' = \left[\begin{smallmatrix} 1 & 0 & | & A \\ p & p^2 & | & C \end{smallmatrix}\right]$ and a summand of type $\left(2, 6, \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ p & p^2 \end{smallmatrix}\right]\right)$; see (11) in the list. Hence, omitting all those summands, we get $p^2C_{42} = 0$ and the p^2C_{42}-column is no longer present.

A $p^2 \in p^2C_3$ allows us to annihilate in $[p^2C_2, p^2C_4, p^2C_7]$. Fill-ins in A can be deleted by pI in A, by p^2I in D and by p^2I in B, respectively. Thus the row of this p^2 continues to a nonzero row of pC_1 and, finally, to a 0-row in A. Fill-ins in A can be deleted by I in A again. Thus $p^2C_3 = 0$, causing crosses. So the p^2C_3-column and the p^2I-row in A are not longer present. Hence we get

$$[\alpha \,\|\, \beta_1] = \begin{bmatrix} I_A & | & 0 & \| & I & 0 & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & I & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & 0 & pI & 0 & 0 \\ pI_B & | & I_B & \| & 0 & 0 & 0 & 0 & p^2I \\ p^2I_C & | & I_C & \| & 0 & pC_1 & p^2C_2 & p^2C_4 & p^2C_7 \\ 0 & | & I_D & \| & pI & 0 & 0 & p^2I & 0 \end{bmatrix}.$$

A $p^2 \in C_7$ allows us to annihilate in $[p^2C_2, p^2C_4]$ and in p^2I above. Fill-ins can be deleted by pI in A, by p^2I in D and by pI in A, respectively. This creates a 0-row in B. Thus $p^2C_7 = 0$ and this displays a cross. So the p^2C_7-column and in turn the p^2I-row

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

in B are not present. Recall that the Smith normal form of pC_1 is $\left[\begin{smallmatrix} pI \\ 0 \end{smallmatrix}\right]$. So we get

$$[\alpha \,\|\, \beta_1] = \begin{bmatrix} I_A & | & 0 & \| & I & 0 & 0 & 0 \\ I_A & | & 0 & \| & 0 & I & 0 & 0 \\ I_A & | & 0 & \| & 0 & 0 & pI & 0 \\ p^2 I_C & | & I_C & \| & 0 & pI & p^2 C_{21} & p^2 C_{41} \\ p^2 I_C & | & I_C & \| & 0 & 0 & p^2 C_{22} & p^2 C_{42} \\ 0 & | & I_D & \| & pI & 0 & 0 & p^2 I \end{bmatrix}.$$

We annihilate with pI in C the block p^2C_{21}, and we annihilate with p^2I in D the block p^2C_{41}. There is a sequence of fill-ins that can be removed by pI and by I in A, respectively. But this leads to $\beta_1' = \left[\begin{smallmatrix} 1 & | & A \\ p & | & C \end{smallmatrix}\right]$ and a summand of type $\left(2, 5, \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 \\ p \end{smallmatrix}\right]\right)$; see (9) in the list. Hence, omitting all those summands, we get that the pI-column with pI in C is not present and in turn also the I-row in A with I above this pI is not present and $p^2C_{41} = 0$. Thus we get

$$[\alpha \,\|\, \beta_1] = \begin{bmatrix} I_A & | & 0 & \| & I & 0 & 0 \\ I_A & | & 0 & \| & 0 & pI & 0 \\ p^2 I_C & | & I_C & \| & 0 & p^2 C_2 & p^2 C_4 \\ 0 & | & I_D & \| & pI & 0 & p^2 I \end{bmatrix}.$$

There is no 0-row in $[p^2C_2, p^2C_4]$. A $p^2 \in p^2C_4$ allows us to annihilate in p^2C_2. Fill-ins in D can be deleted by pI to the left in D and fill-ins in A can be deleted by pI in A. A 0-column of p^2C_4 leads to $\beta_1' = \left[\begin{smallmatrix} 1 & 0 & | & A \\ p & p^2 & | & D \end{smallmatrix}\right]$ and a summand of type $\left(2, 5, \left[\begin{smallmatrix} A \\ D \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ p & p^2 \end{smallmatrix}\right]\right)$; see (8) in the list. Hence, omitting all those summands, we get that p^2C_4 has no 0-column. Clearly, there is no 0-column in $[p^2C_2, p^2C_4]$ to avoid a cross. So forming the Smith normal form of p^2C_4, deleting in p^2C_2 and then forming the Smith normal form of the nonzero rest of p^2C_2, we get

$$[\alpha \,\|\, \beta_1] = \begin{bmatrix} I_A & | & 0 & \| & I & 0 & 0 \\ I_A & | & 0 & \| & 0 & pI & 0 \\ p^2 I_C & | & I_C & \| & 0 & 0 & p^2 I \\ p^2 I_C & | & I_C & \| & 0 & p^2 I & 0 \\ 0 & | & I_D & \| & pI & 0 & p^2 I \end{bmatrix}.$$

Now taking the first column of β_1 we read off $\beta_1' = \begin{bmatrix} 1 & 0 & | & A \\ 0 & p^2 & | & C \\ p & p^2 & | & D \end{bmatrix}$ and a summand of type $\left(3, 8, \begin{bmatrix} A \\ C \\ D \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & p^2 \\ p & p^2 \end{bmatrix}\right)$; see (16) in the list (after obvious changes). Then we take the second column and read off $\beta_1' = \left[\begin{smallmatrix} p & | & A \\ p^2 & | & C \end{smallmatrix}\right]$ and a summand of type $\left(2, 6, \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right], \left[\begin{smallmatrix} p \\ p^2 \end{smallmatrix}\right]\right)$; see (10) in the list. This ends the proof. □

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press

References

[1] D. M. Arnold, *Finite Rank Torsion Free Abelian Groups and Rings*, Lecture Notes in Mathematics, 931 (Springer, Berlin, 1982).

[2] D. M. Arnold, *Abelian Groups and Representations of Partially Ordered Sets*, CMS Advanced Books in Mathematics (Springer, New York, 2000).

[3] D. M. Arnold, A. Mader, O. Mutzbauer and E. Solak, 'Almost completely decomposable groups of unbounded representation type', *J. Algbra* **349** (2012), 50–62.

[4] D. M. Arnold, A. Mader, O. Mutzbauer and E. Solak, 'Indecomposable (1, 3)-groups and a matrix problem', *Czechoslovak. Math. J.* **63** (2013), 307–355.

[5] D. M. Arnold, A. Mader, O. Mutzbauer and E. Solak, 'Representations of posets and decompositions of torsion-free Abelian groups', *Conference Proceedings, International Conference on Algebra* in Honor of P. Smith and J. Clark, Palestine, *J. Math.* **3**(Spec 1) (2014), 320–342.

[6] D. M. Arnold, A. Mader, O. Mutzbauer and E. Solak, 'Z_{p^m}-representations of finite posets', submitted (2014).

[7] T. Faticoni and P. Schultz, 'Direct decompositions of almost completely decomposable groups with primary regulating index', in: *Groups and Modules* (Marcel Dekker, New York, 1996), 233–242.

[8] A. Mader, *Almost Completely Decomposable Groups*, Algebra, Logic and Applications Series, 13 (Gordon and Breach, Amsterdam, 2000).

[9] E. Solak, 'Classification of a class of torsion-free abelian groups', Rendiconti del Seminario Matematico della Universita di Padova, to appear.

DAVID M. ARNOLD, Department of Mathematics, Baylor University,
Waco, TX 76798-7328, USA
e-mail: David_Arnold@baylor.edu

ADOLF MADER, Department of Mathematics, University of Hawaii,
2565 McCarthy Mall, Honolulu, HI 96822, USA
e-mail: adolf@math.hawaii.edu

OTTO MUTZBAUER, Universität Würzburg, Mathematics Institute,
Emil-Fischer-Str. 30, 97074 Würzburg, Germany
e-mail: mutzbauer@mathematik.uni-wuerzburg.de

EBRU SOLAK, Department of Mathematics, Middle East Technical University,
Üniversiteler Mah., Dumlupınar Bulvarı, No 1, 06800 Ankara, Turkey
e-mail: esolak@metu.edu.tr

https://doi.org/10.1017/S1446788714000652 Published online by Cambridge University Press