On certain discontinuous wave functions

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1. Among the many solutions of the wave equation investigated by the late Professor Bateman there is one type which has so far received little attention.

In the simplest case let $f(x, y)$ be a function defined in the whole x, y plane, put $\sigma^2 = c^2t^2 - z^2$, and consider

$$\begin{aligned} W &= \frac{1}{2\pi}\int_0^{2\pi} f(x + \sigma\cos\lambda,\ y + \sigma\sin\lambda)\, d\lambda \qquad &\text{when } \sigma^2 \geqq 0 \\ &= 0 &\text{when } \sigma^2 < 0. \end{aligned} \tag{1}$$

An elementary calculation shows that [1]

$$f_{xx} + f_{yy} + f_{zz} - c^{-2}f_{tt} = \sigma^{-1}\frac{d}{d\lambda}(-f_x\sin\lambda + f_y\cos\lambda), \tag{2}$$

and it follows that W satisfies the wave equation

$$W_{xx} + W_{yy} + W_{zz} - c^{-2}W_{tt} = 0 \tag{3}$$

if f is a continuous function of its two variables with continuous partial derivatives of the first and second orders. Bateman's contention is that W satisfies the wave equation under milder restrictions on f.

At first (2) would seem to suggest that the continuity of, at any rate, the first order derivatives is essential for W to be a wave function: such a conclusion, however, would be incorrect. If f, or any of its relevant partial derivatives, is discontinuous along certain curves in the x, y plane, the integral must be broken up into several parts, at the points where curves of discontinuity intersect the circle with radius σ, centre (x, y). These part-integrals have variable limits, and the contributions of these limits to the partial derivatives cancel the contribution arising from the differentiation of the integrand.

2. Let us consider the integral

$$V = \int_\alpha^\beta f(x + \sigma\cos\lambda,\ y + \sigma\sin\lambda)\, d\lambda \tag{4}$$

in which α and β may depend on x, y, and $\sigma = (c^2t^2 - z^2)^{\frac{1}{2}}$. We shall

[1] Suffixes x, y, z, t indicate partial differentiations.

write f briefly for $f(x+\sigma\cos\lambda,\ y+\sigma\sin\lambda)$ and denote partial derivatives of f with respect to its two variables by f_1, f_2, f_{11} etc., with the understanding that in expressions outside the integral sign the limits must be substituted for λ; we write out explicitly the contribution of the upper limit and indicate by dots that a similar contribution of the lower limit should be subtracted.

If f together with its partial derivatives is continuous in the interval under consideration, we have

$$V_x = \int f_1 d\lambda + \beta_x f - \ldots$$

and $$V_{xx} = \int f_{11} d\lambda + \beta_{xx} f + \beta_x(2-\sigma\beta_x\sin\beta)f_1 + \sigma\beta_x^2 f_2\cos\beta - \ldots$$

with a corresponding expression for V_{yy}. Also

$$V_\sigma = \int (f_1\cos\lambda + f_2\sin\lambda)d\lambda + \beta_\sigma f - \ldots$$

$$V_{\sigma\sigma} = \int (f_{11}\cos^2\lambda + 2f_{12}\cos\lambda\sin\lambda + f_{22}\sin^2\lambda)d\lambda + \beta_{\sigma\sigma} f +$$
$$+\beta_\sigma(2\cos\beta - \sigma\beta_\sigma\sin\beta)f_1 + \beta_\sigma(2\sin\beta+\sigma\beta_\sigma\cos\beta)f_2 - \ldots$$

so that

$$\begin{aligned} V_{xx}+V_{yy}+V_{zz}-c^{-2}V_{tt} &= V_{xx}+V_{yy}-V_{\sigma\sigma}-\sigma^{-1}V_\sigma \\ &= (\beta_{xx}+\beta_{yy}-\beta_{\sigma\sigma}-\sigma^{-1}\beta_\sigma)f+ \\ &+\{2(\beta_x-\beta_\sigma\cos\beta-\sigma^{-1}\sin\beta)-\sigma(\beta_x^2+\beta_y^2-\beta_\sigma^2-\sigma^{-2})\sin\beta\}f_1 + \\ &+\{2(\beta_y-\beta_\sigma\sin\beta+\sigma^{-1}\cos\beta)+\sigma(\beta_x^2+\beta_y^2-\beta_\sigma^2-\sigma^{-2})\cos\beta\}f_2 - \ldots \end{aligned} \qquad (5)$$

since the contribution of the integrals is

$$\int\{f_{11}\sin^2\lambda - 2f_{12}\cos\lambda\sin\lambda + f_{22}\cos^2\lambda - \sigma^{-1}(f_1\cos\lambda + f_2\sin\lambda)\}d\lambda$$
$$= \int \frac{d}{d\lambda}\left\{\sigma^{-1}(-f_1\sin\lambda + f_2\cos\lambda)\right\}d\lambda = \sigma^{-1}(-f_1\sin\beta + f_2\cos\beta) - \ldots$$

3. The limits of integration are determined by the curves of discontinuities so that β will be a root of an equation of the form

$$h(x+\sigma\cos\beta,\ y+\sigma\sin\beta) = 0.$$

Assuming that h has partial derivatives of the first and second orders,

$$(1-\sigma\beta_x\sin\beta)h_1 + \sigma\beta_x h_2\cos\beta = 0$$

with two similar relations obtained by differentiation of $h=0$ with respect to y and σ. Eliminating $h_1 : h_2$ from any two of these three relations, we have

$$\sin\beta - \sigma\beta_x + \sigma\beta_\sigma\cos\beta = 0 \qquad (6)$$

$$\cos\beta + \sigma\beta_y - \sigma\beta_\sigma\sin\beta = 0 \qquad (7)$$

$$1 - \sigma\beta_x\sin\beta + \sigma\beta_y\cos\beta = 0 \qquad (8)$$

The combination $-\sigma(\beta_x + \beta_\sigma\cos\beta)\,(6) + \sigma(\beta_y + \beta_\sigma\sin\beta)\,(7) - (8)$ results in

$$\sigma^2(\beta_x^2 + \beta_y^2 - \beta_\sigma^2) - 1 = 0,$$

and the combination $-(6)_x - \cos\beta\,(6)_\sigma + (7)_y + \sin\beta\,(7)_\sigma + \beta_\sigma\,(8)$ in

$$\sigma(\beta_{xx} + \beta_{yy} - \beta_{\sigma\sigma}) - \beta_\sigma = 0$$

and the right-hand side of (5) is seen to vanish identically.

Thus we have proved that W, being a sum of integrals of the form (4), is certainly a solution of the wave equation if f is continuous and possesses continuous first and second order partial derivatives except at a finite number of "smooth" curves, *i.e.* curves with a continuously turning tangent. Even an infinity of such curves is admissible provided that they are placed so that any circle cuts only a finite number of them. For $t = 0$, W vanishes outside the plane $z = 0$ and is equal to f in that plane; $W_t = 0$ everywhere at $t = 0$.

4. Bateman's more interesting results refer to the corresponding problem for a spherical surface rather than a plane.

Let f be a function defined on the sphere S with radius a, centre at the origin, P any point at distance r from the origin, C the locus of all points on S whose distance from P is ct, $\bar{f}$ the mean value of f over C, and

$$W = \frac{a}{r}\bar{f} \quad \text{if } |a - ct| \leqq r \leqq a + ct, \quad \text{and} = 0 \text{ otherwise.}$$

Of this function Bateman says[1] "in all cases that have been examined W has been found to be a solution of the wave equation ... and to satisfy the initial conditions (for $t = 0$) $W = f$ when P is on S, $W = 0$ when P is not on S, $W_t = 0$ everywhere."

From Bateman's work it follows that if (r, θ, ϕ) are spherical

[1] H. Bateman, *Ann. of Maths.* (2) 31, 158-162 (1930) (where the factor a is omitted), and *Partial Differential Equations of Mathematical Physics* (1932) p. 189 Examples.

polar coordinates, $\Sigma Y_n(\theta, \phi)$ the expansion of f in spherical surface harmonics, and $\cos\gamma = (a^2 + r^2 - c^2t^2)/(2ar)$, then

$$W = \frac{a}{r}\Sigma P_n(\cos\gamma)\,Y_n(\theta,\phi) \qquad (|a - ct| \leqq r \leqq a + ct)$$

and hence that W satisfies the wave equation provided that it is permissible to perform the partial differentiations term-by-term.

An alternative representation of W is as follows. Let f be given as a function of the stereographic coordinates $\xi = \tan\frac{1}{2}\theta\cos\phi$, $\eta = \tan\frac{1}{2}\theta\sin\phi$, so that $f = f(\xi, \eta)$. Then

$$W = \frac{a}{2\pi r}\int_0^{2\pi} f(X, Y)\frac{(\cos\theta + \cos\gamma)\,d\lambda}{1 + \cos\theta\cos\gamma + \sin\theta\sin\gamma\cos(\phi - \lambda)}$$

with $\quad X = \dfrac{\sin\theta\cos\phi + \sin\gamma\cos\lambda}{\cos\theta + \cos\gamma}, \quad Y = \dfrac{\sin\theta\sin\phi + \sin\gamma\sin\lambda}{\cos\theta + \cos\gamma}$

for $|a - ct| \leqq r \leqq a + ct$, and this integral may serve to prove Bateman's result for functions f which have discontinuities along certain curves on S. That W remains a wave function for such functions, Bateman conjectured and made plausible by considering special examples.

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