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ADMISSION CONTROL FOR MULTIDIMENSIONAL WORKLOAD INPUT WITH HEAVY TAILS AND FRACTIONAL ORNSTEIN–UHLENBECK PROCESS

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Abstract

The infinite source Poisson arrival model with heavy-tailed workload distributions has attracted much attention, especially in the modeling of data packet traffic in communication networks. In particular, it is well known that under suitable assumptions on the source arrival rate, the centered and scaled cumulative workload input process for the underlying processing system can be approximated by fractional Brownian motion. In many applications one is interested in the stabilization of the work inflow to the system by modifying the net input rate, using an appropriate admission control policy. In this paper we study a natural family of admission control policies which keep the associated scaled cumulative workload input asymptotically close to a prespecified linear trajectory, uniformly over time. Under such admission control policies and with natural assumptions on arrival distributions, suitably scaled and centered cumulative workload input processes are shown to converge weakly in the path space to the solution of a d-dimensional stochastic differential equation driven by a Gaussian process. It is shown that the admission control policy achieves moment stabilization in that the second moment of the solution to the stochastic differential equation (averaged over the d-stations) is bounded uniformly for all times. In one special case of control policies, as time approaches ∞, we obtain a fractional version of a stationary Ornstein–Uhlenbeck process that is driven by fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$.

Keywords: Poisson random measure; Gaussian random measure; self-similarity; heavy-tailed distribution; fractional Brownian motion; fractional Ornstein–Uhlenbeck process; admission control

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1. Introduction

This paper is motivated by an arrival model for data traffic in communication networks considered by Kurtz [6]. We will introduce the model in a more accessible form in the special, though quite general, case of interest here. Let N_0 be a counting process where $N_0(t)$, $t \geq 0$, represents the number of source activations by time t. The jth source activated at time U_j brings a unit rate workload into the system which lasts for a random length of time τ_j, where

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$\{\tau_j\}_{j\geq 1}$ are independent and identically distributed (i.i.d.) with distribution ν on $[0,\infty)$. At time t, the cumulative work input in the system from the source j is, thus, $\tau_j \wedge (t-U_j)_+$, where $a\wedge b = \min\{a,b\}$ and $x_+ = \max\{x,0\}$. The rate at which jumps of $N_0(t)$ occur is given as $\lambda(t, W_0(t))$, where $W_0(t)$ is the total cumulative work input in the system at time t from all the sources and λ is a strictly positive function on $\mathbb{R}_+\times\mathbb{R}_+$.

A precise mathematical definition of the coupled stochastic processes N_0, W_0 is given as follows. Let $0<S_1<S_2<\cdots$ be the jump times of unit-rate Poisson process N and $\{\tau_j\}_{j\geq 1}$ be an i.i.d. sequence, independent of $\{S_j\}_{j\geq 1}$, with common distribution ν. Let $\lambda\colon \mathbb{R}_+\times\mathbb{R}_+\to(0,\infty)$ be a continuous function. Let ξ be the Poisson random measure on $\mathbb{R}_+\times\mathbb{R}_+$ with intensity measure $\eta = m\times\nu$, where m denotes the Lebesgue measure on $[0,\infty)$, defined as $\xi=\sum_{j=1}^{\infty}\delta_{(S_j,\tau_j)}$. Define N_0, W_0 through the system of equations

$$N_0(t) = N(\Lambda_0(t)), \tag{1.1}$$

$$W_0(t) = \sum_{j\colon S_j\leq\Lambda_0(t)} \tau_j\wedge(t-\gamma_0(S_j)) = \int_{[0,\Lambda_0(t)]\times[0,\infty)} r\wedge(t-\gamma_0(s))\xi(\mathrm{d}s,\mathrm{d}r), \tag{1.2}$$

$$\Lambda_0(t) = \int_0^t \lambda(s, W_0(s))\,\mathrm{d}s, \qquad \gamma_0(t) = \Lambda_0^{-1}(t).$$

Note that the above set of equations recursively defines the stochastic processes (N_0, W_0) from one jump instant $\gamma_0(S_j)$ of $N_0(t)$ to the next.

Setting the filtration $\mathcal{F}_u = \sigma\{S_j,\tau_j\colon S_j\leq u\}$, it is easy to see that $\{\gamma_0(u)\}_{u\geq 0}$ is an $\{\mathcal{F}_u\}$-adapted process and, thus, for any $t\geq 0$, $\Lambda_0(t)=\gamma_0^{-1}(t)$ is a bounded $\{\mathcal{F}_u\}$-stopping time. In particular, $N_0(t)-\Lambda_0(t)$ is $\{\mathcal{G}_t\}$-martingale, where $\mathcal{G}_t = \mathcal{F}_{\Lambda_0(t)}$, $t\geq 0$. Thus, N_0 is a counting process with $\{\mathcal{G}_t\}$-stochastic intensity $\lambda(t, W_0(t))$.

The key results of [6] are the law of large numbers and the central limit theorem for the scaled system (X_n, Y_n), where

$$X_n(t) = \frac{1}{n}N_n(t), \qquad Y_n(t) = \frac{1}{n}W_n(t), \tag{1.3}$$

and (N_n, W_n) are defined as in (1.1) and (1.2) except on replacing N_0 with a Poisson process with rate n and λ with the function $\lambda_n(t, w(t)) = n\lambda(t, n^{-1}w(t))$. Under suitable assumptions, (X_n, Y_n) converges in probability to (X, Y), satisfying

$$X(t) = \int_0^t \lambda(s, Y(s))\,\mathrm{d}s, \qquad Y(t) = \int_0^t \mu(t-s)\lambda(s, Y(s))\,\mathrm{d}s,$$

where $\mu(t) = \mathbb{E}\{\tau_i\wedge t\}$ (Theorem 2.1 of [6]). Also, under suitable assumptions (that include the differentiability of the function $y\mapsto\lambda(t,y)$), the scaled and centered process $(\tilde{X}_n,\tilde{Y}_n) = \sqrt{n}(X_n - X, Y_n - Y)$ converges in distribution to $(\tilde{X},\tilde{Y})$, satisfying

$$\tilde{X}(t) = \Xi(B(t)) + \int_0^t \lambda_y(s, Y(s))\tilde{Y}(s)\,\mathrm{d}s,$$

$$\tilde{Y}(t) = \int_{B(t)} (r\wedge(t-\gamma(s)))\Xi(\mathrm{d}s,\mathrm{d}r) + \int_0^t \mu(t-s)\lambda_y(s, Y(s))\tilde{Y}(s)\,\mathrm{d}s, \tag{1.4}$$

where $B(t) = [0,\Lambda(t)]\times[0,\infty)$, $\Lambda(t) = \int_0^t\lambda(z, Y(z))\,\mathrm{d}z$, $\gamma(t) = \Lambda^{-1}(t)$, $\lambda_y(t,y) = (\partial/\partial y)\lambda(t,y)$, and Ξ is a Gaussian random measure with the control measure

$$\mathbb{E}\{|\Xi(\mathrm{d}s,\mathrm{d}r)|^2\} = \mathrm{d}s\,\nu(\mathrm{d}r), \tag{1.5}$$

where ν is the distribution of τ_i (Theorem 2.2 of [6]).

One special case of τ_i is particularly interesting in the context of modeling data traffic in modern communication networks. This is the case where τ_i are heavy-tailed with the distribution

$$\nu(\mathrm{d}r) = (\beta - 1)\theta(\theta r + 1)^{-\beta}\,\mathrm{d}r, \qquad r \geq 0, \tag{1.6}$$

where $\beta \in (2, 3)$ is the tail index and $\theta \in (0, \infty)$ is a scale parameter. When λ is the constant function and (1.6) is assumed, it is well known that, under suitable assumptions and proper scaling, the cumulative workload input process converges to fractional Brownian motion. A version of this fact appears in Section 4 of [6], see also [3]–[5], [7], [8], and [10].

For the convergence to fractional Brownian motion, in a scaled system, it is also necessary to rescale τ_is or the measure $\nu(\mathrm{d}r)$ in (1.6). One way to see this is to observe that without rescaling, the Gaussian random measure Ξ in (1.4) and (1.5) is not self-similar in the variable r. For this reason (see, e.g. Section 4 of [6] for the case when λ is constant), it is natural to scale the measures as

$$\nu_n(\mathrm{d}r) = (\beta - 1)n\theta(n\theta r + 1)^{-\beta}\,\mathrm{d}r \tag{1.7}$$

or, equivalently, replace τ_i by τ_i/n.

This case is not included in Theorems 2.1 and 2.2 of [6] which, although allowing for state dependent λ, treat the scaled system (1.3) that has no scaling in the intensity measure ν, and, hence, the key convergence results of [6] for nonconstant λ cannot be applied with (1.7). In fact, as already suggested by the result in Section 4 of [6] for the constant λ case, dealing with (1.7) for nonconstant λ is expected to be more involved. For example, a natural normalization in this case is no longer $\sqrt{n}$.

Models where λ is a function of the state process are natural when one considers control mechanisms for regulating the amount of work input in the system. A common form of a control policy that aims to appropriately balance long processing delays with low processor utilization consists of suitably decreasing the input rate when the workload input in the system is very high and increasing the rate when it drops too low. Study of the asymptotic behavior of the workload input process with heavy-tailed session length distributions under such state feedback control mechanisms is the subject of this paper. We will consider a scaled multidimensional system where the session lengths are distributed according to ν_n as in (1.7), and establish limit theorems for settings where λ is state dependent. We are particularly interested in the design of control policies that keep the net workload input (asymptotically) close to a prespecified linear trajectory such that the variability (suitably scaled) is bounded uniformly in time. The slope of the linear trajectory represents the system processing rate and, thus, such control policies yield uniform in time reliability bounds on probabilities of processor underutilization and overload.

Let us now describe briefly our model and the results we have established. We suppose that a system consists of d-processing stations, and that work arrives to each station (independently of others) as before. The function λ controlling the arrival rate, however, now depends on the average total workload input across all the stations. More specifically, denoting the total cumulative workload input at the ith station by $y_i(t)$ and their average $\bar{y}(t) = (1/d)\sum_i y_i(t)$, we suppose that $\lambda = f(t, \bar{y}(t))$. Even more specifically, we will work with a special λ having the form

$$\lambda = f(t, \bar{y}(t)) = \exp\{-g(\bar{y}(t) - bt)\} \tag{1.8}$$

for some $b > 0$ and function g. The constant b represents the processing rate at each station, although processing of work is not explicitly included in our model and plays no role in the analysis. The function g will satisfy the following assumption.

Assumption 1.1. *It holds that $g(0) = 0$. The function g is twice differentiable and its first and second derivatives g' and g'' satisfy*

$$0 < \ell \le g'(x) \le L \quad \textit{for all } x \in \mathbb{R}$$

and

$$|g''(x)| \le L \quad \textit{for all } x \in \mathbb{R}$$

for some $\ell, L \in (0, \infty)$.

The above assumption will be taken to hold throughout this work and will not be explicitly noted in the statements of various results. Note that under this assumption, g is a strictly increasing function, and $g(u) > 0$ if $u > 0$ and $g(u) < 0$ if $u < 0$. From the properties of g, we see that the function λ in (1.8) has a natural physical interpretation: the rate of session arrivals at the ith station increases when $\bar{y}(t)$ drops below bt, while it decreases when $\bar{y}(t)$ exceeds bt. We will refer to g as an *admission control policy*.

Our scaled system will be characterized by independent Poisson random measures $\xi_{n,i}$ having common intensity measure $n^{\alpha} m \times \nu_n$ where ν_n is as in (1.7) and the cumulative workload input process $Y_{n,i}(t)$, unlike (1.3) will now be normalized by a factor of $n^{\alpha-1}$, rather than n (see (2.4)). We will assume that

$$\alpha \in (\beta - 1, \min\{3\beta - 5, 5 - \beta\}). \tag{1.9}$$

The reason for such choice of α and for the normalization $n^{\alpha-1}$ will be given below (see Remark 2.2).

Precise evolution equations for $Y_{n,i}$ are given in Section 2. We now give a brief description of our main results. In Theorem 2.1 we prove a law of large numbers result stating that, as $n \to \infty$,

$$\boldsymbol{Y}_n = (Y_{n,1}, \dots, Y_{n,d})^{\top}$$

converges in probability in $D_{\mathbb{R}_+^d}[0, \infty)$ to a continuous (nonrandom) trajectory $\boldsymbol{U} = (U, \dots, U)^{\top}$, where U is characterized as the unique solution of an ordinary differential equation (ODE) (see (2.6)), and a rate of convergence is given as well. The solution U has the property that $\sup_{t \ge 0} |U(t) - bt| < \infty$. In fact, with a particular choice of b, namely $b = 1/\theta(\beta - 2)$, we have $U(t) = bt$ for all t.

Next, we study the fluctuations of $\boldsymbol{Y}_n$. In Theorem 2.2 we show that a suitably centered and normalized form of $\boldsymbol{Y}_n$, denoted as $\boldsymbol{Z}_n$ (see (2.10)), converges in distribution in $D_{\mathbb{R}^d}[0, \infty)$ to the solution $\boldsymbol{Z} = (Z_1, \dots, Z_d)^{\top}$ of a d-dimensional stochastic differential equation (SDE) (see (2.11)), driven by d independent Gaussian processes R_i, $i = 1, \dots, d$. The moment stabilization property of the admission control policy is demonstrated in Theorem 2.3, which states that $\sup_{t \ge 0} \mathbb{E}|\bar{Z}(t)|^2 < \infty$, where $\bar{Z} = (1/d) \sum_{i=1}^{d} Z_i$.

We remark that in the $b = 1/\theta(\beta - 2)$ case, one can achieve the law of large number limit of bt by simply taking the admission control policy to be $g \equiv 0$ (this function obviously does not satisfy Assumption 1.1). However, in the $g \equiv 0$ case, the limit process obtained from the fluctuation central limit theorem will have variance that increases to ∞ as $t \to \infty$.

Finally, we show that in one particular case, the average of the limit process $\bar{Z}$ is driven by a Gaussian H-self-similar process $\bar{R}$ with $H = (4 - \beta)/2 > \frac{1}{2}$. The driving process $\bar{R}$ is not fractional Brownian motion since it does not have stationary increments. This is directly related to the fact that the limit process $\bar{Z}$ satisfies $\bar{Z}(0) = 0$ and, hence, is not stationary. The process $\bar{Z}(T + \cdot)$ is expected to become stationary as $T \to \infty$. Similarly, the driving process

$\bar{R}$ is expected to have stationary increments in the long run (i.e. $\bar{R}(T+\cdot)-\bar{R}(T)$ approaches a process with stationary increments, as $T\to\infty$). We study the asymptotic behavior of the process $\bar{Z}(T+\cdot)$ as $T\to\infty$ in Theorem 2.4. For simplicity we restrict ourselves here to the $b=1/\theta(\beta-2)$ case. It is shown that, as $T\to\infty$, the process $\bar{Z}(T+\cdot)$ converges in distribution in $C_{\mathbb{R}}[0,\infty)$ to a stationary Ornstein–Uhlenbeck process driven by fractional Brownian motion with Hurst parameter $H=(4-\beta)/2>\frac{1}{2}$.

The paper is organized as follows. We state all the results in Section 2. Section 3 contains the proofs of Proposition 2.1 and Theorem 2.1. The proof of the central limit theorem will be provided in Section 4. In Section 5 we represent the limit (centered) station average process $\bar{Z}$ as an integral with respect to a Gaussian process and provide the proof of Theorem 2.3 on the moment stabilization property of the admission control policy g. Section 6 is devoted to the study of the asymptotic behavior, as $T\to\infty$, of the process $\bar{Z}$ obtained from the central limit theorem, and the proof of Theorem 2.4 is given.

The following notation will be used. We denote the set of nonnegative integers by $\mathbb{N}$ and nonnegative reals by $\mathbb{R}_+$. For a Polish space S, $C_S[0,\infty)$ (respectively $D_S[0,\infty)$) will denote the space of continuous (respectively right-continuous with left limits (RCLL)) functions endowed with the local uniform (respectively Skorokhod) topology. We denote C by the generic constants in $(0,\infty)$ whose value may change from one proof to the next.

2. Model formulation and main results

We begin in this section with the evolution equations for the unscaled system.

2.1. Unscaled system

Let $\xi_{0,i}$, $i=1,\dots,d$, be independent Poisson random measures on $[0,\infty)\times[0,\infty)$ with common intensity measure $\eta=m\times\nu$, where m denotes the Lebesgue measure on $[0,\infty)$ and ν is given in (1.6). Then, $\xi_{0,i}$ can be represented as

$$\xi_{0,i}=\sum_{j=1}^{\infty}\delta_{(S_{i,j},\tau_{i,j})},$$

where $0<S_{i,1}<S_{i,2}<\cdots$ are the jump times of independent unit rate Poisson processes for $i=1,\dots,d$, and $\tau_{i,j}$ are i.i.d. with distribution ν. These Poisson random measures will be the building blocks for our counting processes N_i with desired stochastic intensities.

Let $f\colon\mathbb{R}_+\times\mathbb{R}\to\mathbb{R}_+$ be a function of the form

$$f(t,y)=\exp\{-g(y-bt)\},$$

where $g\colon\mathbb{R}\to\mathbb{R}$ is a function satisfying Assumption 1.1. Let $\boldsymbol{X}_0=(X_{0,1},\dots,X_{0,d})^\top$ and $\boldsymbol{Y}_0=(Y_{0,1},\dots,Y_{0,d})^\top$ be $\mathbb{N}^d$- and $\mathbb{R}^d_+$-valued RCLL processes given through the following system of equations:

$$X_{0,i}(t)=N_{0,i}(t)=\xi_{0,i}(B_0(t))=\sum_{j=1}^{\infty}\mathbf{1}_{\{S_{i,j}\le\Lambda_0(t)\}},$$

$$Y_{0,i}(t)=\int_{B_0(t)}r\wedge(t-\gamma_0(s))\xi_{0,i}(\mathrm{d}s,\mathrm{d}r)=\sum_{j\colon S_{i,j}\le\Lambda_0(t)}\tau_{i,j}\wedge(t-\gamma_0(S_{i,j})),\qquad(2.1)$$

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where

$$\Lambda_0(t) = \int_0^t f(s, \bar{Y}_0(s))\, \mathrm{d}s, \ \bar{Y}_0(t) = \frac{1}{d}\sum_{i=1}^{d} Y_{0,i}(t), \qquad \gamma_0(t) = \Lambda_0^{-1}(t) \tag{2.2}$$

and

$$B_0(t) = [0, \Lambda_0(t)] \times [0, \infty).$$

Note that from Assumption 1.1, Λ_0 is continuous and strictly increasing. Therefore, γ_0 is well defined and continuous as well. Here, $\gamma_0(S_{i,j})$ is the jth activation time at the ith station, that is, the jth jump time of $N_{0,i}(t)$. For $\gamma_0(S_{i,j}) \le t$, $t - \gamma_0(S_{i,j})$ is the amount of time up to t since the jth session activation at the ith station and $T_{i,j} = \gamma_0(S_{i,j}) + \tau_{i,j}$ is the end time of the jth session at the ith station. Thus, $\tau_{i,j} \wedge (t - \gamma_0(S_{i,j}))$ is the work input by the jth activated source at the ith station, up to time t.

From Assumption 1.1 it follows that

$$f(t, y) = \exp\{-g(y - bt)\} \le \exp\{-g(-bt)\} \quad \text{for all } y \ge 0. \tag{2.3}$$

In particular, f is a strictly positive function that is locally bounded, namely

$$\sup_{t\in[0,T],\, y\in\mathbb{R}_+} f(t, y) < \infty \quad \text{for all } T > 0.$$

From this it follows that there is a unique solution to the system of equations (2.1) and (2.2). Furthermore, γ_0 is a $\{\mathcal{F}_u\}$-adapted process, where

$$\mathcal{F}_u = \sigma\{\xi_i(A) \colon A \in \mathcal{B}([0, u] \times [0, \infty)), i = 1, \dots, d\}.$$

Consequently, for any $t \ge 0$, $\Lambda_0(t) = \gamma_0^{-1}(t)$ is a bounded $\{\mathcal{F}_u\}$-stopping time and, therefore,

$$N_{0,i}(t) - \Lambda_0(t) = \tilde{\xi}_{0,i}([0, \Lambda_0(t)] \times [0, \infty))$$

is a $\{\mathcal{G}_t\} = \{\mathcal{F}_{\Lambda_0(t)}\}$-martingale, where $\tilde{\xi}_{0,i} = \xi_{0,i} - \eta$ is the compensated Poisson random measure associated with $\xi_{0,i}$, $i = 1, \dots, d$.

2.2. Scaled workload and main results

We now introduce the scaled system. Roughly speaking, the scaling corresponds to replacing $\tau_{i,j}$ with $\tau_{i,j}/n$, $S_{i,j}$ with $S_{i,j}/n^\alpha$ and dividing the cumulative workload input processes by $n^{\alpha-1}$. More precisely, for each fixed $n \in \mathbb{N}$, let $\xi_{n,1}, \dots, \xi_{n,d}$ be independent Poisson random measures on $[0, \infty) \times [0, \infty)$ with common intensity measure

$$\eta_n(\mathrm{d}s, \mathrm{d}r) = n^\alpha\, \mathrm{d}s\, \nu_n(\mathrm{d}r),$$

where ν_n is introduced in (1.7). Define, for $i = 1, \dots, d$,

$$\begin{aligned} X_{n,i}(t) &= \frac{1}{n^\alpha} N_{n,i}(t) = \frac{1}{n^\alpha}\xi_{n,i}(B_n(t)), \\ Y_{n,i}(t) &= \frac{1}{n^{\alpha-1}} \int_{B_n(t)} r \wedge (t - \gamma_n(s))\xi_{n,i}(\mathrm{d}s, \mathrm{d}r), \end{aligned} \tag{2.4}$$

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where

$$\Lambda_n(t) = \int_0^t f(s, \bar{Y}_n(s))\,\mathrm{d}s, \qquad \bar{Y}_n(t) = \frac{1}{d}\sum_{i=1}^{d} Y_{n,i}(t), \qquad \gamma_n(t) = \Lambda_n^{-1}(t), \tag{2.5}$$

and

$$B_n(t) = [0, \Lambda_n(t)] \times [0, \infty).$$

As for the unscaled system, we see that the solution $(\boldsymbol{X}_n, \boldsymbol{Y}_n)^\top$ of the system (2.4) exists and is unique on $[0, \infty)$ for each n, where $\boldsymbol{X}_n = (X_{n,1}, \dots, X_{n,d})^\top$ and $\boldsymbol{Y}_n = (Y_{n,1}, \dots, Y_{n,d})^\top$; and, moreover, $\boldsymbol{X}_n, \boldsymbol{Y}_n \in D_{\mathbb{R}_+^d}[0, \infty)$.

Consider the ODE

$$\dot{U}(t) = af(t, U(t)), \qquad U(0) = 0, \qquad t \geq 0, \tag{2.6}$$

where

$$a = (\beta - 1)\theta \int_0^\infty r(\theta r + 1)^{-\beta}\,\mathrm{d}r = (\beta - 1)\theta \int_0^\infty n^2 r(n\theta r + 1)^{-\beta}\,\mathrm{d}r = \frac{1}{\theta(\beta - 2)}.$$

The following proposition will be proved in Section 3.

Proposition 2.1. *There is a unique continuous function U that solves (2.6). The solution satisfies*

$$\sup_{t \geq 0} |U(t) - bt| < \infty.$$

In the $b = a$ case, we have $U(t) = bt$, for all $t \geq 0$.

Remark 2.1. As an immediate consequence of the above proposition we have that $f(t, U(t)) = \exp\{-g(U(t) - bt)\}$ is bounded above and bounded below away from 0, namely,

$$0 < \inf_{t \geq 0}\{f(t, U(t))\} \leq \sup_{t \geq 0}\{f(t, U(t))\} < \infty.$$

Denote

$$f_y(t, y) = \frac{\partial}{\partial y} f(t, y) = -\exp\{-g(y - bt)\}g'(y - bt).$$

Since g' is bounded from below and above, it follows from the above proposition that $f_y(t, U(t))$ is also bounded below and bounded above away from 0.

Let

$$\boldsymbol{X} = (X_1, \dots, X_d)^\top = a^{-1}(U, U, \dots, U)^\top, \qquad \boldsymbol{Y} = (Y_1, \dots, Y_d)^\top = (U, U, \dots, U)^\top. \tag{2.7}$$

The following is the first main result of this paper.

Theorem 2.1. *As $n \to \infty$, $(\boldsymbol{X}_n, \boldsymbol{Y}_n)^\top \to (\boldsymbol{X}, \boldsymbol{Y})^\top$ in $D_{\mathbb{R}_+^{2d}}[0, \infty)$, in probability. Furthermore, for any $t > 0$ and any $q \in [0, \beta - 2)$,*

$$\sup_{0 \leq s \leq t} n^q |\bar{Y}_n(s) - U(s)| \to 0, \tag{2.8}$$

in probability, as $n \to \infty$.

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Let $V \in C_{\mathbb{R}}[0, \infty)$ be given as the solution of

$$V(t) = a \int_0^t f_y(s, U(s))V(s)\,\mathrm{d}s - a\theta^{2-\beta} \int_0^t \frac{f(s, U(s))}{(t-s)^{\beta-2}}\,\mathrm{d}s. \tag{2.9}$$

From Remark 2.1 the solution V of the above linear equation exists and is unique.

Define

$$Z_{n,i}(t) = n^{(\alpha+\beta-3)/2}\biggl(Y_{n,i}(t) - Y_i(t) - \frac{V(t)}{n^{\beta-2}}\biggr), \qquad i = 1, \dots, d. \tag{2.10}$$

Our next result provides the limiting behavior of the processes $\mathbf{Z}_n = (Z_{n,1}, \dots, Z_{n,d})^\top$. Note that $Y_i(t) + V(t)/n^{\beta-2}$ is not the expectation of $Y_{n,i}$, and, hence, $Z_{n,i}$ in the above equation is not the conventional centered process of $Y_{n,i}$. However, from Proposition 2.1 and Theorem 2.1, one can show $Y_i(t) = \lim_{n\to\infty} \mathbb{E}\{Y_{n,i}(t)\}$. Also, as n increases to ∞, the term $V(t)/n^{\beta-2}$ tends to zero. Thus, the next result can be regarded as a central limit theorem for the scaled and (nearly) centered process $\boldsymbol{Y}_n$.

Theorem 2.2. *As $n \to \infty$, $\mathbf{Z}_n$ converges in distribution in $D_{\mathbb{R}^d}[0, \infty)$ to $\mathbf{Z} = (Z_1, \dots, Z_d)^\top$, where $\boldsymbol{Z}$ satisfies*

$$Z_i(t) = \int_{B(t)} r \wedge (t - \gamma(s))\Sigma_i(\mathrm{d}s, \mathrm{d}r) + a \int_0^t f_y(s, U(s))\bar{Z}(s)\,\mathrm{d}s, \qquad i = 1, \dots, d, \tag{2.11}$$

where

$$\Lambda(t) = \int_0^t f(s, U(s))\,\mathrm{d}s, \qquad \gamma(t) = \Lambda^{-1}(t), \tag{2.12}$$

$$B(t) = [0, \Lambda(t)] \times [0, \infty), \qquad \bar{Z}(t) = \frac{1}{d}\sum_{i=1}^{d} Z_i(t),$$

and $\Sigma_1, \dots, \Sigma_d$ are independent Gaussian random measures on $[0, \infty) \times [0, \infty)$ with common control measure $\mathrm{d}s(\beta-1)\theta^{1-\beta}r^{-\beta}\,\mathrm{d}r$.

Integrals with respect to Gaussian random measures characterized by a control measure are defined, for example, in Chapter 3 of [12].

Remark 2.2. When $\lambda \equiv 1$ (constant) or $f \equiv 1$, note that $\Lambda_n(t) = t$ and $\gamma_n(t) = t$ in (2.5), $B_n(t) = [0, t] \times [0, \infty)$ and, hence,

$$Y_{n,i}(t) = \frac{1}{n^{\alpha-1}} \int_0^t \int_0^\infty r \wedge (t-s)\xi_{n,i}(\mathrm{d}s, \mathrm{d}r)$$

in (2.4). After the change of variables $s \to s/n$ and $r \to r/n$, this can be written as

$$\begin{aligned} Y_{n,i}(t) &= \frac{1}{n^\alpha} \int_0^{nt} \int_0^\infty r \wedge (nt-s)\xi_{n,i}\biggl(\mathrm{d}\Bigl(\frac{s}{n}\Bigr), \mathrm{d}\Bigl(\frac{r}{n}\Bigr)\biggr) \\ &= \frac{1}{n^\alpha} \int_0^{nt} \int_0^\infty r \wedge (nt-s)\zeta_{n,i}(\mathrm{d}s, \mathrm{d}r), \end{aligned} \tag{2.13}$$

where $\zeta_{n,i}$ is a Poisson random measure with intensity measure $n^{\alpha-1}\,\mathrm{d}s\,(\beta-1)\theta(\theta r+1)^{-\beta}\,\mathrm{d}r$. Written as (2.13), $n^\alpha Y_{n,i}$ can be interpreted as the cumulative workload input in the system

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scaled in time by n and where heavy-tailed workloads are associated with sources arriving at Poisson rate $\lambda_n = n^{\alpha-1}$. This is the view taken, for example, in [4] and [8]. It is well known that, after proper normalization and centering, the total workload input converges to fractional Brownian motion in the so-called fast regime, that is, when

$$\frac{\lambda_n}{n^{(\beta-1)-1}} = \frac{n^{\alpha-1}}{n^{\beta-2}} = n^{\alpha-\beta+1} \to \infty.$$

This holds when $\alpha - \beta + 1 > 0$, which is a part of our assumption (1.9). It is also known that the normalization of the right-hand side of (2.13) (to the central limit theorem) is

$$\frac{n^{\alpha}}{(\lambda_n n^{3-(\beta-1)})^{1/2}} = \frac{n^{\alpha}}{n^{(\alpha-\beta+3)/2}} = n^{(\alpha+\beta-3)/2},$$

which coincides with that used in (2.10).

Remark 2.3. Let $\boldsymbol{Z}^* = (Z_1^*, \dots, Z_d^*)^{\top}$ be given as the solution of

$$Z_i^*(t) = R_i^*(t) + a \int_0^t f_y(s, U(s)) \bar{Z}^*(s)\, \mathrm{d}s, \tag{2.14}$$

where

$$R_i^*(t) = \int_0^t \int_0^{\infty} (f(s, U(s)))^{1/2} (r \wedge (t-s)) \Sigma_i(\mathrm{d}s, \mathrm{d}r), \qquad i = 1, \dots, d,$$

and

$$\bar{Z}^*(t) = \frac{1}{d} \sum_{i=1}^{d} Z_i^*(t).$$

One can check that $\boldsymbol{R}^* = (R_1^*, \dots, R_d^*)^{\top}$ and $\boldsymbol{R} = (R_1, \dots, R_d)^{\top}$ have the same distribution, where

$$R_i(t) = \int_{B(t)} r \wedge (t - \gamma(s)) \Sigma_i(\mathrm{d}s, \mathrm{d}r), \qquad i = 1, \dots, d.$$

Consequently, $\boldsymbol{Z}$ and $\boldsymbol{Z}^*$ are equal in law and, thus, (2.14) gives an alternative representation for the weak limit of $\boldsymbol{Z}_n$ as $n \to \infty$.

The following result shows the moment stabilization property of the admission control policy g.

Theorem 2.3. *The following uniform moment bound holds:*

$$\sup_{t \geq 0} \mathbb{E}\{|\bar{Z}(t)|^2\} \leq \frac{2\theta^{1-\beta}}{d(\beta-2)(3-\beta)(a\mu)^{4-\beta}} \Gamma(4-\beta),$$

where $\mu := \inf_{s \geq 0}\{-f_y(s, U(s))\} \in (0, \infty)$ *and* $\Gamma(\cdot)$ *is the gamma function.*

Remark 2.4. The case when there is no admission control corresponds to $g \equiv 0$. Although the function $g \equiv 0$ does not satisfy Assumption 1.1, it can be shown along similar lines that in this case Theorem 2.1 holds with $U(t) = at$, and, therefore, $\sup_{t \geq 0} |U(t) - bt|$ will be finite if and only if $b = a$. Furthermore, Theorem 2.2 will hold as well (when $b = a$), but the moment stabilization property in Theorem 2.3 fails.

Finally, we consider the asymptotic behavior of $\bar{Z}(T+\cdot)$ as $T \to \infty$. Here, we restrict ourselves to the $b = a$ case. Then from Proposition 2.1, (2.11), and (2.12), the limit process in Theorem 2.2 can be written as

$$Z_i(t) = \int_0^t \int_0^\infty r \wedge (t-s) \Sigma_i(\mathrm{d}s, \mathrm{d}r) - \kappa \int_0^t \bar{Z}(s)\,\mathrm{d}s, \qquad i = 1, \ldots, d,$$

where $\kappa = ag'(0) \in (0, \infty)$.

Let $B_H = \{B_H(t),\ t \geq 0\}$ be a standard fractional Brownian motion with Hurst parameter $H = (4-\beta)/2 \in (\frac{1}{2}, 1)$, namely, B_H is a mean zero Gaussian process with covariance

$$\mathbb{E}\{B_H(t)B_H(s)\} = \tfrac{1}{2}(t^{2H} + s^{2H} - |t-s|^{2H}).$$

Let $Z_\infty(0)$ be a normal random variable with mean zero and variance

$$\sigma_0^2 := \mathbb{E}\{|Z_\infty(0)|^2\} = \frac{\theta^{1-\beta}}{d(\beta-2)} \int_0^\infty \int_0^\infty \mathrm{e}^{-\kappa v}\mathrm{e}^{-\kappa u}|u-v|^{2-\beta}\,\mathrm{d}u\,\mathrm{d}v < \infty,$$

and let $(B_H, Z_\infty(0))$ be jointly Gaussian, and the covariance function of B_H and $Z_\infty(0)$ be

$$\operatorname{cov}(B_H(t), Z_\infty(0)) = \frac{\theta^{1-\beta}}{\sigma d\,(\beta-2)} \int_0^t \int_0^\infty \mathrm{e}^{-\kappa v}(u+v)^{2-\beta}\,\mathrm{d}v\,\mathrm{d}u,$$

where $\sigma = \sqrt{2\theta^{1-\beta}/d\,(\beta-2)(3-\beta)(4-\beta)}$. Let Z_∞ be the fractional Ornstein–Uhlenbeck process given as the unique solution of

$$Z_\infty(t) = Z_\infty(0) - \kappa \int_0^t Z_\infty(s)\,\mathrm{d}s + \sigma B_H(t). \tag{2.15}$$

Theorem 2.4. *Let $b = a$ and let $\mathbf{Z}$ be as in Theorem 2.2. Then, as $T \to \infty$, $\bar{Z}(T+\cdot)$ converges in distribution in $C_{\mathbb{R}}[0,\infty)$ to Z_∞, given by (2.15). Moreover, the process Z_∞ is stationary.*

3. Law of large numbers

In this section we will prove Proposition 2.1 and Theorem 2.1.

Proof of Proposition 2.1. Consider the ODE

$$\dot{u}(t) = a\exp\{-g(u(t))\} - b, \qquad u(0) = 0, \qquad t \geq 0. \tag{3.1}$$

Clearly, a differentiable function u solves (3.1) if and only if $U(t) = u(t) + bt$ solves (2.6). From Assumption 1.1, the function $h(x) = a\exp\{-g(x)\} - b$, $x \in \mathbb{R}$, is locally Lipschitz. For each $n \in \mathbb{N}$, define $h_n(x) = h((x \wedge n) \vee (-n))$, $x \in \mathbb{R}$. Since h_n is a Lipschitz function, for any $n \in \mathbb{N}$, the ODE

$$\dot{u}(t) = h_n(u(t)), \qquad u(0) = 0, \qquad t \geq 0 \tag{3.2}$$

has a unique solution u_n. Let K be the unique solution of the equation

$$a\exp\{-g(K)\} - b = 0,$$

That is, $g(K) = \log(a/b)$.

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Then, for all $n > |K|$, if $b > a$, $u_n(t) \leq 0$ for all t, and $u_n(t)$ decreases to $K \in (-\infty, 0)$; if $b < a$, $u_n(t) \geq 0$ for all t, and $u_n(t)$ increases to $K \in (0, \infty)$; and, finally, if $b = a$, $u_n(t) = 0$ for all t. Consequently, for any $n > |K|$,

$$\sup_{t\geq 0} |u_n(t)| \leq |K| \tag{3.3}$$

and u_n solves (3.1). This proves the existence of solutions.

Now consider uniqueness. Let $\tilde{u}$ be another solution of (3.1). Let $\tau = \inf\{t : |\tilde{u}(t)| \geq |K| + 1\}$. From the unique solvability of (3.2) for any $n \geq |K| + 1$, $\tilde{u}(t) = u_n(t)$ for all $t \in [0, \tau)$. From (3.3) we now see that $\tau = \infty$. This proves the unique solvability of (3.1) and, consequently, that of (2.6). Also, as noted above,

$$\sup_{t\geq 0} |U(t) - bt| = \sup_{t\geq 0} |u(t)| \leq |K|$$

and $U(t) - bt = u(t) = 0$ for all t if $b = a$. The result follows.

Next, we present the proof of Theorem 2.1.

Proof of Theorem 2.1. Let $\tilde{\xi}_{n,i} = \xi_{n,i} - \eta_n$ be the compensated Poisson random measure associated with $\xi_{n,i}$, $i = 1, \ldots, d$. Rewrite $\boldsymbol{X}_n$ and $\boldsymbol{Y}_n$ as

$$\begin{aligned}
X_{n,i}(t) &= \frac{1}{n^{\alpha}} \tilde{\xi}_{n,i}(B_n(t)) + \frac{1}{n^{\alpha}} \eta_n(B_n(t)) = \frac{1}{n^{\alpha}} \tilde{\xi}_{n,i}(B_n(t)) + \Lambda_n(t), \\
Y_{n,i}(t) &= \frac{1}{n^{\alpha-1}} \int_{B_n(t)} r \wedge (t - \gamma_n(s)) \tilde{\xi}_{n,i}(\mathrm{d}s, \mathrm{d}r) \\
&\quad + \frac{1}{n^{\alpha-1}} \int_{B_n(t)} r \wedge (t - \gamma_n(s)) n^{\alpha} (\beta - 1) n\theta (n\theta r + 1)^{-\beta} \, \mathrm{d}s \, \mathrm{d}r.
\end{aligned} \tag{3.4}$$

By the change of variables $s = \int_0^v f(u, \bar{Y}_n(u)) \, \mathrm{d}u = \Lambda_n(v)$, the second term on the right-hand side of (3.4) equals

$$n^2 \theta (\beta - 1) \int_0^t \int_0^{\infty} f(v, \bar{Y}_n(v)) (r \wedge (t - v)) (n\theta r + 1)^{-\beta} \, \mathrm{d}r \, \mathrm{d}v. \tag{3.5}$$

Consider the inner integral in (3.5). For $0 \leq v < t$, by simple calculation, we see that

$$\begin{aligned}
&\theta(\beta - 1) n^2 \int_0^{\infty} (r \wedge (t - v)) (n\theta r + 1)^{-\beta} \, \mathrm{d}r \\
&\quad = \theta(\beta - 1) n^2 \bigg[\int_0^{t-v} r (n\theta r + 1)^{-\beta} \, \mathrm{d}r + \int_{t-v}^{\infty} (t - v)(n\theta r + 1)^{-\beta} \, \mathrm{d}r \bigg] \\
&\quad = \frac{1}{\theta(\beta - 2)} - \frac{1}{\theta(\beta - 2)(n\theta(t - v) + 1)^{\beta - 2}} \\
&\quad = a \bigg(1 - \frac{1}{(n\theta(t - v) + 1)^{\beta - 2}} \bigg).
\end{aligned}$$

Therefore, for each $i = 1, \ldots, d$,

$$\begin{aligned}
Y_{n,i}(t) &= \frac{1}{n^{\alpha-1}} \int_{B_n(t)} r \wedge (t - \gamma_n(s)) \tilde{\xi}_{n,i}(\mathrm{d}s, \mathrm{d}r) \\
&\quad + \int_0^t a f(s, \bar{Y}_n(s)) \bigg(1 - \frac{1}{(n\theta(t - s) + 1)^{\beta - 2}} \bigg) \, \mathrm{d}s.
\end{aligned}$$

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Recall the definition of X_i and Y_i from (2.7). Then

$$X_{n,i}(t) - X_i(t) = \frac{1}{n^\alpha}\tilde{\xi}_{n,i}(B_n(t)) + \int_0^t f(s, \bar{Y}_n(s))\,\mathrm{d}s - \int_0^t f(s, U(s))\,\mathrm{d}s, \tag{3.6}$$

$$\begin{aligned} Y_{n,i}(t) - Y_i(t) = {} & \frac{1}{n^{\alpha-1}}\int_{B_n(t)} r \wedge (t - \gamma_n(s))\tilde{\xi}_{n,i}(\mathrm{d}s, \mathrm{d}r) \\ & + \int_0^t a[f(s, \bar{Y}_n(s)) - f(s, U(s))]\,\mathrm{d}s \\ & - a\int_0^t \frac{f(s, \bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s. \end{aligned} \tag{3.7}$$

Let us first show (2.8). Summing over i and normalizing by d, we have

$$\bar{Y}_n(t) - U(t) = \frac{1}{d}\sum_{i=1}^{d}(Y_{n,i}(t) - Y_i(t)) = a\int_0^t [f(s, \bar{Y}_n(s)) - f(s, U(s))]\,\mathrm{d}s + \mathcal{S}_n(t), \tag{3.8}$$

where

$$\begin{aligned} \mathcal{S}_n(t) &= \frac{1}{d}\sum_{i=1}^{d} A_{n,i}(t) - a\int_0^t \frac{f(s, \bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s \\ &= \bar{A}_n(t) - a\int_0^t \frac{f(s, \bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s, \end{aligned} \tag{3.9}$$

and for $i = 1, \ldots, d$,

$$A_{n,i}(t) = \frac{1}{n^{\alpha-1}}\int_{B_n(t)} r \wedge (t - \gamma_n(s))\tilde{\xi}_{n,i}(\mathrm{d}s, \mathrm{d}r).$$

We will now argue that

$$n^q \mathcal{S}_n \to 0 \text{ in probability in } D_{\mathbb{R}^d}[0, \infty) \quad \text{for all } q \in [0, \beta - 2). \tag{3.10}$$

Consider first the second term on the right-hand side of (3.9). Since $\bar{Y}_n(t) \geq 0$ for all $t \geq 0$ and $\beta \in (2, 3)$, from (2.3), we have

$$\begin{aligned} \int_0^t \frac{f(s, \bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s &\leq \frac{1}{n^{\beta-2}\theta^{\beta-2}}\int_0^t \frac{\exp\{-g(-bs)\}}{(t-s)^{\beta-2}}\,\mathrm{d}s \\ &\leq \frac{(\sup_{0\leq s\leq t}\exp\{-g(-bs)\})t^{3-\beta}}{n^{\beta-2}\theta^{\beta-2}(3-\beta)}. \end{aligned}$$

Consequently, for every $t > 0$,

$$\lim_{n\to\infty} n^q \int_0^t \frac{f(s, \bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s = 0 \quad \text{almost surely for all } q \in [0, \beta - 2). \tag{3.11}$$

Thus, in order to prove (3.10), it suffices to show that

$$\sup_{0\leq s\leq t} n^q |\boldsymbol{A}_n(s)| \to 0 \quad \text{in probability, as } n \to \infty, \tag{3.12}$$

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for every $q \in [0, \beta - 2)$ and for every $t > 0$, where $\boldsymbol{A}_n = (A_{n,1}, \ldots, A_{n,d})^\top$. The proof of (3.12) can be found below, following the proof of this theorem.

From Assumption 1.1, we have that $y \mapsto f(t, y)$ is a Lipschitz function on $\mathbb{R}_+$, uniformly in t on compact intervals, since

$$\sup_{y\in\mathbb{R}_+} |f_y(t, y)| = \sup_{y\in\mathbb{R}_+} |-\exp\{-g(y - bt)\}g'(y - bt)| \le L\exp\{-g(-bt)\} \tag{3.13}$$

for all $t \in [0, \infty)$. The convergence in (2.8) now follows by an application of Gronwall's lemma to (3.8).

Finally, we argue that

$$(\boldsymbol{X}_n, \boldsymbol{Y}_n)^\top \to (\boldsymbol{X}, \boldsymbol{Y})^\top \text{ in } D_{\mathbb{R}_+^{2d}}[0, \infty), \quad \text{in probability, as } n \to \infty. \tag{3.14}$$

For $n \in \mathbb{N}$, define the filtration $\{\mathcal{F}_u^n\}$ as

$$\mathcal{F}_u^n = \sigma\{\xi_n^i(A) \colon A \in \mathcal{B}([0, u] \times [0, \infty)), i = 1, \ldots, d\}.$$

Then, for each $i = 1, \ldots, d$, $\tilde{\xi}_{n,i}([0, u] \times [0, \infty))$ is an $\{\mathcal{F}_u^n\}$-martingale. As an analogue of the unscaled process in Section 2.1, γ_n is a continuous, strictly increasing $\{\mathcal{F}_u^n\}$-adapted process. Consequently, for every $t \ge 0$, $\Lambda_n(t) = \gamma_n^{-1}(t)$ is a $\{\mathcal{F}_u^n\}$-stopping time. Therefore,

$$M_{n,i}^{(1)}(t) = \tilde{\xi}_{n,i}([0, \Lambda_n(t)] \times [0, \infty)) = \tilde{\xi}_{n,i}(B_n(t)) \tag{3.15}$$

is a $\{\mathcal{G}_t^n\}$-martingale, where $\mathcal{G}_t^n = \mathcal{F}_{\Lambda_n(t)}^n$. By Doob's maximal inequality, for some $C > 0$,

$$\begin{aligned}
\mathbb{P}\bigg\{\sup_{0\le s\le t} \frac{1}{n^\alpha}|M_{n,i}^{(1)}(t)| \ge \epsilon\bigg\} &\le \frac{C\mathbb{E}|M_{n,i}^{(1)}(t)|^2}{n^{2\alpha}\epsilon^2} \\
&= \frac{C\mathbb{E}\{\int_0^t f(s, \bar{Y}_n(s))\,\mathrm{d}s\}}{n^\alpha\epsilon^2} \\
&\le \frac{C\int_0^t \exp\{-g(-bs)\}\,\mathrm{d}s}{n^\alpha\epsilon^2}.
\end{aligned} \tag{3.16}$$

Combining (3.15) and (3.16) we have, as $n \to \infty$,

$$\mathbb{P}\bigg\{\sup_{0\le s\le t} \frac{1}{n^\alpha}|\tilde{\xi}_{n,i}(B_n(t))| \ge \epsilon\bigg\} \to 0. \tag{3.17}$$

Thus, the first term on the right-hand side of (3.6) converges to the zero process, uniformly on compacts, in probability, as $n \to \infty$. Now (3.14) follows on combining (3.13), (3.17), (3.11), and (3.12) (with $q = 0$), and applying Gronwall's lemma to (3.6) and (3.7).

Proof of (3.12). Recall that, for each $i = 1, \ldots, d$, $\tilde{\xi}_{n,i}([0, u] \times [0, \infty))$ is an $\{\mathcal{F}_u^n\}$-martingale and for every $t \ge 0$, $\Lambda_n(t) = \gamma_n^{-1}(t)$ is a $\{\mathcal{F}_u^n\}$-stopping time. Observe that

$$A_{n,i}(t) = \mathcal{U}_{n,i}^{(1)}(\Lambda_n(t)),$$

where, for $i = 1, \ldots, d$,

$$\mathcal{U}_{n,i}^{(1)}(u) = \frac{1}{n^{\alpha-1}} \int_{[0,u]\times[0,\infty)} r \wedge (t - \gamma_n(s))_+ \tilde{\xi}_{n,i}(\mathrm{d}s, \mathrm{d}r).$$

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Note that $\mathcal{U}_{n,i}^{(1)}(u)$ is a $\{\mathcal{F}_u^n\}$-martingale with predictable quadratic variation process

$$\frac{1}{n^{2(\alpha-1)}}\int_{[0,u]\times[0,\infty)}(r\wedge(t-\gamma_n(s))_+)^2\eta_n(\mathrm{d}s,\mathrm{d}r).$$

Using the change of variables $s=\int_0^v f(u,\bar{Y}_n(u))\,\mathrm{d}u=\Lambda_n(v)$, we have, for each $t>0$,

$$\begin{aligned}\mathbb{E}\{|\boldsymbol{A}_n(t)|^2\} &= \mathbb{E}\biggl\{\sum_{i=1}^d|\mathcal{U}_{n,i}^{(1)}(\Lambda_n(t))|^2\biggr\}\\ &= \frac{1}{n^{2\alpha-2}}\mathbb{E}\biggl\{\sum_{i=1}^d\int_{[0,\Lambda_n(t)]\times[0,\infty)}(r\wedge(t-\gamma_n(s)))^2\eta_n(\mathrm{d}s,\mathrm{d}r)\biggr\}\\ &= \frac{n\theta(\beta-1)d}{n^{\alpha-2}}\mathbb{E}\biggl\{\int_0^t\int_0^\infty f(v,\bar{Y}_n(v))(r\wedge(t-v))^2(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\biggr\}.\end{aligned}\tag{3.18}$$

Splitting the integral over $[0,\infty)$ as $[0,t-v]\cup(t-v,\infty)$ and making the substitution $nr\mapsto r$, (3.18) can be bounded by

$$\begin{aligned}n^{-\alpha}\theta(\beta-1)d\sup_{0\le s\le t}\{\exp\{-g(-bs)\}\}\biggl[&\int_0^t\int_0^{n(t-v)}r^2(\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\\ &+\int_0^t\int_{n(t-v)}^\infty(n(t-v))^2(\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\biggr].\end{aligned}\tag{3.19}$$

Integrating the first term on the right-hand side of (3.19), we see that

$$\frac{1}{n^\alpha}\int_0^t\int_0^{n(t-v)}r^2(\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\le\frac{n^{3-\beta-\alpha}t^{4-\beta}}{\theta^\beta(3-\beta)(4-\beta)}.$$

Since $\beta-2<(\beta+\alpha-3)/2$ (or, equivalently, $\alpha>\beta-1$), we obtain, for all $t\ge 0$,

$$\frac{n^{2q}}{n^\alpha}\int_0^t\int_0^{n(t-v)}r^2(\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\to 0\quad\text{for all } q\in[0,\beta-2).\tag{3.20}$$

Also, the integration of the second term on the right-hand side of (3.19) shows that

$$\frac{1}{n^\alpha}\int_0^t\int_{n(t-v)}^\infty(n(t-v))^2(\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\le\frac{n^{3-\beta-\alpha}t^{4-\beta}}{\theta^\beta(\beta-1)(4-\beta)}.$$

Thus, we have

$$\frac{n^{2q}}{n^\alpha}\int_0^t\int_{n(t-v)}^\infty(n(t-v))^2(\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\to 0\quad\text{for all } q\in[0,\beta-2).\tag{3.21}$$

Combining (3.18)–(3.21) we conclude that

$$\lim_{n\to\infty}n^{2q}\mathbb{E}\{|\boldsymbol{A}_n(t)|^2\}=0\quad\text{for all } q\in[0,\beta-2).\tag{3.22}$$

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We argue next that $n^q \boldsymbol{A}_n = n^q(A_{n,1}, \ldots, A_{n,d})^\top$ converges to the zero process in $D_{\mathbb{R}^d}[0,\infty)$, in probability. In view of (3.22), it suffices to check that $\{n^q \boldsymbol{A}_n\}$ is tight. To prove tightness we will use a standard tightness criterion. Namely, we will show that, for each fixed $T > 0$ there exists $C_T > 0$ such that for $0 \le h \le 1$ and $h \le t \le T$,

$$n^{4q}\mathbb{E}\{|\boldsymbol{A}_n(t+h) - \boldsymbol{A}_n(t)|^2 |\boldsymbol{A}_n(t) - \boldsymbol{A}_n(t-h)|^2\} \le C_T h^2. \tag{3.23}$$

The above inequality, together with the relative compactness of $n^q \boldsymbol{A}_n(t)$ for each $t \ge 0$ (which follows from (3.22)), yields tightness of $\{n^q \boldsymbol{A}_n\}$ (cf. Theorems 3.8.6 and 3.8.8 of [2]).

Now fix $T > 0$. In order to show (3.23), it is sufficient to prove that, for any $0 \le h \le 1$ and $0 \le t \le T$,

$$n^{4q}\mathbb{E}\{|\boldsymbol{A}_n(t+h) - \boldsymbol{A}_n(t)|^4\} \le C_T h^2. \tag{3.24}$$

In the following, we use $C_T > 0$ to denote a generic constant depending on T, θ, and β whose value may vary from line to line. For $r, s, h, t \in \mathbb{R}$, denote

$$\vartheta_n^{h,t}(r,s) = r \wedge (t + h - \gamma_n(s))_+ - r \wedge (t - \gamma_n(s))_+.$$

Define, for $i = 1, \ldots, d$,

$$\mathcal{U}_{n,i}^{(2)}(u) = \frac{1}{n^{\alpha-1}} \int_{[0,u]\times[0,\infty)} \vartheta_n^{h,t}(r,s)\tilde{\xi}_{n,i}(\mathrm{d}s, \mathrm{d}r). \tag{3.25}$$

Observe that $\mathcal{U}_{n,i}^{(2)}(u)$ is a $\{\mathcal{F}_u^n\}$-martingale with quadratic variation process

$$\frac{1}{n^{2(\alpha-1)}} \int_{[0,u]\times[0,\infty)} (\vartheta_n^{h,t}(r,s))^2 \xi_n(\mathrm{d}s, \mathrm{d}r).$$

Since $\gamma_n(s) \le t$ if and only if $s \le \Lambda_n(t)$, we have

$$A_{n,i}(t+h) - A_{n,i}(t) = \mathcal{U}_{n,i}^{(2)}(\Lambda_n(t+h)).$$

Recalling that $\Lambda_n(t+h)$ is a $\{\mathcal{F}_u^n\}$-stopping time, we have by the Burkholder–Davis–Gundy inequality (cf. Theorem IV.3.46 of [11]) that for some $C > 0$,

$$\begin{aligned}\mathbb{E}\{|A_{n,i}(t+h) - A_{n,i}(t)|^4\} &= \mathbb{E}\{|\mathcal{U}_{n,i}^{(2)}(\Lambda_n(t+h))|^4\} \\ &\le \frac{C}{n^{4(\alpha-1)}}\mathbb{E}\bigg\{\bigg(\int_{[0,\Lambda_n(t+h)]\times[0,\infty)} [\vartheta_n^{h,t}(r,s)]^2 \xi_{n,i}(\mathrm{d}s, \mathrm{d}r)\bigg)^2\bigg\}.\end{aligned}$$

Writing $\xi_{n,i} = \tilde{\xi}_{n,i} + \eta_n$, the above can be bounded by

$$\begin{aligned}&\frac{2C}{n^{4(\alpha-1)}}\mathbb{E}\bigg\{\int_0^{\Lambda_n(t+h)} \int_0^\infty [\vartheta_n^{h,t}(r,s)]^4 n^\alpha \theta(\beta-1)n(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg\} \\ &\quad + \frac{2C}{n^{4(\alpha-1)}}\mathbb{E}\bigg\{\bigg(\int_0^{\Lambda_n(t+h)} \int_0^\infty [\vartheta_n^{h,t}(r,s)]^2 n^\alpha \theta(\beta-1)n(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2\bigg\}.\end{aligned} \tag{3.26}$$

Denote, for $r, s, h, t \in \mathbb{R}$,

$$\tilde{\vartheta}_n^{h,t}(r,s) = r \wedge (t+h-s)_+ - r \wedge (t-s)_+.$$

By a change of variables, the first term on the right-hand side of (3.26) equals

$$\frac{2C}{n^{4(\alpha-1)}}\mathbb{E}\bigg\{\int_0^{t+h}\int_0^{\infty} f(s,\bar{Y}_n(s))[\tilde{\vartheta}_n^{h,t}(r,s)]^4 n^{\alpha}\theta(\beta-1)n(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg\}.$$

Using the estimate $f(s,\bar{Y}_n(s))) \le \sup_{0\le s\le T+1}\{\exp\{-g(-bs)\}\} < \infty$ and (A.1) from Lemma A.1 in Appendix A, the above can be bounded by

$$C_T n^{5-\beta-3\alpha}h^{6-\beta}. \tag{3.27}$$

For the second term on the right-hand side of (3.26), by a change of variables once more, equals

$$\frac{2C}{n^{4(\alpha-1)}}\mathbb{E}\bigg\{\bigg(\int_0^{t+h}\int_0^{\infty} f(s,\bar{Y}_n(s))[\tilde{\vartheta}_n^{h,t}(r,s)]^2 n^{\alpha}\theta(\beta-1)n(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2\bigg\},$$

which can similarly be bounded by

$$C_T n^{6-2\beta-2\alpha}h^{2(4-\beta)}, \tag{3.28}$$

using (A.2) in Lemma A.1. Observing that

$$\beta-2 < \tfrac{1}{4}\min\{\beta+3\alpha-5, 2\beta+2\alpha-6\} \quad \text{and} \quad \min\{6-\beta, 2(4-\beta)\} > 2,$$

and combining (3.27) and (3.28), we conclude that (3.24) holds for every $q \in [0, \beta-2)$. This completes the proof of (3.12).

4. Central limit theorem

In this section we prove Theorem 2.2. From (2.9) and (3.7), we can write (2.10) as

$$\begin{aligned}
Z_{n,i}(t) &= \int_{B_n(t)} (r\wedge(t-\gamma_n(s)))\Sigma_{n,i}(\mathrm{d}s,\mathrm{d}r) \\
&\quad + a\int_0^t n^{(\alpha+\beta-3)/2}\bigg[f(s,\bar{Y}_n(s)) - f(s,U(s)) - f_y(s,U(s))\frac{V(s)}{n^{\beta-2}}\bigg]\mathrm{d}s \\
&\quad - n^{(\alpha+\beta-3)/2}\int_0^t \frac{af(s,\bar{Y}_n(s))\ \mathrm{d}s}{(n\theta(t-s)+1)^{\beta-2}} + n^{(\alpha-\beta+1)/2}\int_0^t \frac{a\theta^{2-\beta}f(s,U(s))\,\mathrm{d}s}{(t-s)^{\beta-2}},
\end{aligned} \tag{4.1}$$

where, with $\sigma_n = n^{(\alpha-\beta+1)/2}$,

$$\Sigma_{n,i}(A) = \frac{n^{(\alpha+\beta-3)/2}}{n^{\alpha-1}}\tilde{\xi}_{n,i}(A) = \sigma_n^{-1}\tilde{\xi}_{n,i}(A)$$

is a random signed measure on $[0,\infty)\times[0,\infty)$, $i=1,\dots,d$. Note that

$$\mathrm{var}(\Sigma_{n,i}(A)) = n^{\beta-1}m\times\nu_n(A), \qquad i=1,\dots,d, \ \text{for } A\in\mathcal{B}(\mathbb{R}_+^2)$$

with $m\times\nu_n(A) < \infty$. Note also that

$$\begin{aligned}
f(s,\bar{Y}_n(s)) &- f(s,U(s)) - f_y(s,U(s))\frac{V(s)}{n^{\beta-2}} \\
&= (\bar{Y}_n(s)-U(s))\int_0^1 [f_y(s,U(s)+x(\bar{Y}_n(s)-U(s))) - f_y(s,U(s))]\,\mathrm{d}x \\
&\quad + \bigg(\bar{Y}_n(s)-U(s)-\frac{V(s)}{n^{\beta-2}}\bigg)f_y(s,U(s)).
\end{aligned}$$

Thus, the middle term on the right-hand side of (4.1) equals

$$a\int_0^t n^{(\alpha+\beta-3)/2}(\bar{Y}_n(s)-U(s))\int_0^1 [f_y(s,U(s)+x(\bar{Y}_n(s)-U(s)))-f_y(s,U(s))]\,\mathrm{d}x\,\mathrm{d}s$$
$$+\,a\int_0^t f_y(s,U(s))\bar{Z}_n(s)\,\mathrm{d}s,$$

where, recall, $\bar{Z}_n(s)=(1/d)\sum_{i=1}^d Z_{n,i}(s)$. Let

$$R_{n,i}(t)=\int_{B_n(t)}(r\wedge(t-\gamma_n(s)))\Sigma_{n,i}(\mathrm{d}s,\mathrm{d}r),\qquad i=1,\dots,d, \tag{4.2}$$

$$C_n(t)=a\int_0^t n^{(\alpha+\beta-3)/2}(\bar{Y}_n(s)-U(s)) \times\int_0^1 [f_y(s,U(s)+x(\bar{Y}_n(s)-U(s)))-f_y(s,U(s))]\,\mathrm{d}x\,\mathrm{d}s, \tag{4.3}$$

$$D_n(t)=n^{(\alpha-\beta+1)/2}\int_0^t\frac{a\theta^{2-\beta}f(s,U(s))\,\mathrm{d}s}{(t-s)^{\beta-2}}-n^{(\alpha+\beta-3)/2}\int_0^t\frac{af(s,\bar{Y}_n(s))\,\mathrm{d}s}{(n\theta(t-s)+1)^{\beta-2}}. \tag{4.4}$$

Letting $\mathcal{R}_n(t)=(\mathcal{R}_{n,1}(t),\dots,\mathcal{R}_{n,d}(t))^\top$, where $\mathcal{R}_{n,i}(t)=R_{n,i}(t)+C_n(t)+D_n(t)$, we can rewrite equation (4.1) as

$$Z_{n,i}(t)=\mathcal{R}_{n,i}(t)+a\int_0^t f_y(s,U(s))\bar{Z}_n(s)\,\mathrm{d}s,\qquad i=1,\dots,d. \tag{4.5}$$

Proof of Theorem 2.2. Define $\psi\colon D_{\mathbb{R}^d}[0,\infty)\to D_{\mathbb{R}^d}[0,\infty)$ by

$$[\psi(x)]_i(t)=x_i(t)+a\int_0^t f_y(s,U(s))\overline{\psi(x)}(s)\,\mathrm{d}s,\qquad i=1,\dots,d,\ x\in D_{\mathbb{R}^d}[0,\infty),$$

where $\overline{\psi(x)}=(1/d)\sum_{i=1}^d[\psi(x)]_i$. Then, ψ is a continuous mapping from $D_{\mathbb{R}^d}[0,\infty)$ to $D_{\mathbb{R}^d}[0,\infty)$. Also, from (4.5) we see that $\boldsymbol{Z}_n=(Z_{n,1},\dots,Z_{n,d})^\top=\psi(\mathcal{R}_n)$.

Combining Lemmas 4.2, 4.3, and 4.1 below, we see that $\mathcal{R}_n$ converges to $\boldsymbol{R}=(R_1,\dots,R_d)^\top$ in distribution in $D_{\mathbb{R}^d}[0,\infty)$, where

$$R_i(t)=\int_{B(t)}(r\wedge(t-\gamma(s)))\Sigma_i(\mathrm{d}s,\mathrm{d}r),\qquad i=1,\dots,d, \tag{4.6}$$

and Σ_i, $i=1,\dots,d$, is as in Theorem 2.2. The result now follows from the continuous mapping theorem.

The next three lemmas were used in the proof of Theorem 2.2 above.

Lemma 4.1. *Let $\boldsymbol{R}_n=(R_{n,1},\dots,R_{n,d})^\top$ and $\boldsymbol{R}=(R_1,\dots,R_d)^\top$ be as given by (4.2) and (4.6), respectively. As $n\to\infty$, $\boldsymbol{R}_n$ converges to $\boldsymbol{R}$ in distribution in $D_{\mathbb{R}^d}[0,\infty)$.*

Proof. Let $\tilde{\boldsymbol{R}}_n=(\tilde{R}_{n,1},\dots,\tilde{R}_{n,d})^\top$, where

$$\tilde{R}_{n,i}(t)=\int_{B(t)}[r\wedge(t-\gamma(s))]\Sigma_{n,i}(\mathrm{d}s,\mathrm{d}r),\qquad t\geq 0.$$

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Since $\Lambda_n(t)$ is an $\{\mathcal{F}_u^n\}$-stopping time for each $t \geq 0$, $\mathbf{1}_{[0,\Lambda_n(t)]}(s)[r \wedge (t - \gamma_n(s))]$ is $\mathcal{F}_s^n$-predictable. Thus, applying the isometry property of the stochastic integral and recalling the definition of ν_n in (1.7), we obtain

$$
\begin{aligned}
&\mathbb{E}\{R_{n,i}(t) - \tilde{R}_{n.i}(t)\}^2 \\
&\quad = \mathbb{E}\int_0^\infty \int_0^\infty (\mathbf{1}_{[0,\Lambda_n(t)]}(s)[r \wedge (t - \gamma_n(s))] \\
&\qquad\qquad\qquad - \mathbf{1}_{[0,\Lambda(t)]}(s)[r \wedge (t - \gamma(s))])^2 n^{\beta-1}\nu_n(\mathrm{d}r)\,\mathrm{d}s \\
&\quad \leq 2\mathbb{E}\bigg\{\int_0^\infty \int_0^\infty \mathbf{1}_{[0,\Lambda_n(t)]}(s)[r \wedge (t - \gamma_n(s)) - r \wedge (t - \gamma(s))]^2 \frac{\theta(\beta-1)n^\beta}{(n\theta r + 1)^\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg\} \\
&\qquad + 2\mathbb{E}\{|\Lambda_n(t) - \Lambda(t)|\} \int_0^\infty (r \wedge t)^2 \theta(\beta-1)n^\beta(n\theta r + 1)^{-\beta}\,\mathrm{d}r. \qquad (4.7)
\end{aligned}
$$

Now we consider the first term on the right-hand side of (4.7). From the definitions of γ_n and γ we see, for any $s \geq 0$,

$$
s = \int_0^{\gamma_n(s)} f(z, \bar{Y}_n(z)\,\mathrm{d}z = \int_0^{\gamma(s)} f(z, U(z))\,\mathrm{d}z.
$$

Consequently,

$$
\begin{aligned}
&\int_0^{\gamma(s)} f(z, U(z))\,\mathrm{d}z - \int_0^{\gamma_n(s)} f(z, U(z))\,\mathrm{d}z \\
&\qquad = \int_0^{\gamma_n(s)} f(z, \bar{Y}_n(z))\,\mathrm{d}z - \int_0^{\gamma_n(s)} f(z, U(z))\,\mathrm{d}z. \qquad (4.8)
\end{aligned}
$$

Since $f(z, U(z))$ is bounded below away from 0 (see Remark 2.1), there exists a $c > 0$ such that

$$
\bigg|\int_0^{\gamma(s)} f(z, U(z))\,\mathrm{d}z - \int_0^{\gamma_n(s)} f(z, U(z))\,\mathrm{d}z\bigg| \geq c|\gamma(s) - \gamma_n(s)|. \qquad (4.9)
$$

On the other hand, from (3.13), we obtain, for any $s \leq \Lambda_n(t)$ (equivalently, $\gamma_n(s) \leq t$),

$$
\begin{aligned}
&\bigg|\int_0^{\gamma_n(s)} f(z, \bar{Y}_n(z))\,\mathrm{d}z - \int_0^{\gamma_n(s)} f(z, U(z))\,\mathrm{d}z\bigg| \\
&\qquad \leq L \sup_{0\leq z\leq\gamma_n(s)} |\bar{Y}_n(z) - U(z)| \int_0^{\gamma_n(s)} \exp\{-g(-bu)\}\,\mathrm{d}u \\
&\qquad \leq L \sup_{0\leq s\leq t} |\bar{Y}_n(s) - U(s)| \int_0^t \exp\{-g(-bu)\}\,\mathrm{d}u. \qquad (4.10)
\end{aligned}
$$

Combining (4.8)–(4.10) we have

$$
\mathbf{1}_{[0,\Lambda_n(t)]}(s)|\gamma_n(s) - \gamma(s)| \leq \frac{L}{c} \sup_{0\leq s\leq t} |\bar{Y}_n(s) - U(s)| \int_0^t \exp\{-g(-bu)\}\,\mathrm{d}u.
$$

Using (2.8) we now obtain

$$
\mathbf{1}_{[0,\Lambda_n(t)]}(s)|\gamma_n(s) - \gamma(s)| \to 0
$$

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in probability, as $n \to \infty$. An application of the dominated convergence theorem now shows that

$$\lim_{n\to\infty} \mathbb{E}\left\{\int_0^\infty \int_0^\infty \mathbf{1}_{[0,\Lambda_n(t)]}(s)[r \wedge (t-\gamma_n(s)) - r \wedge (t-\gamma(s))]^2 \frac{\theta(\beta-1)n^\beta}{(n\theta r+1)^\beta}\,\mathrm{d}r\,\mathrm{d}s\right\} = 0.$$

Thus, the first term on the right-hand side of (4.7) converges to 0 as $n \to \infty$.

Next, we consider the second term on the right-hand side of (4.7). By the dominated convergence theorem, we have by using the fact that $\beta \in (2, 3)$,

$$\lim_{n\to\infty} \int_0^\infty (r \wedge t)^2 \theta(\beta-1)n^\beta (n\theta r+1)^{-\beta}\,\mathrm{d}r = \int_0^\infty (r\wedge t)^2 \theta^{1-\beta}(\beta-1)r^{-\beta}\,\mathrm{d}r < \infty. \tag{4.11}$$

Note that

$$0 \le \min\{\Lambda(t), \Lambda_n(t)\} \le \max\{\Lambda(t), \Lambda_n(t)\} \le \int_0^t \exp\{-g(-bs)\}\,\mathrm{d}s.$$

Consequently,

$$|\Lambda_n(t) - \Lambda(t)| \le 2\int_0^t \exp\{-g(-bs)\}\,\mathrm{d}s.$$

Also, from (3.13) we have

$$|\Lambda_n(t) - \Lambda(t)| \le L \sup_{0\le s\le t} |\bar{Y}_n(s) - U(s)| \int_0^t \exp\{-g(-bs)\}\,\mathrm{d}s.$$

Thus, (2.8) and the dominated convergence theorem yield

$$\lim_{n\to\infty} \mathbb{E}\{|\Lambda_n(t) - \Lambda(t)|\} = 0. \tag{4.12}$$

Combining (4.11) and (4.12) it follows that the second term on the right-hand side of (4.7) converges to 0 as $n \to \infty$.

Combining the above observations it follows that, for each $i = 1, \dots, d$ and $t \ge 0$,

$$\lim_{n\to\infty} \mathbb{E}\{R_{n,i}(t) - \tilde{R}_{n.i}(t)\}^2 = 0. \tag{4.13}$$

Note that

$$\tilde{R}_{n.i}(t) = \int_{[0,\infty)\times[0,\infty)} n^{(\beta-\alpha-1)/2}\,\mathbf{1}_{B(t)}(s,r)(r\wedge(t-\gamma(s)))\tilde{\xi}_{n,i}(\mathrm{d}s,\mathrm{d}r).$$

For each fixed $i = 1, \dots, d$, we will now show the weak convergence of the finite-dimensional distribution of $\tilde{R}_{n,i}$. For any $0 < t_1 < \cdots < t_k < \infty$, denote $f^n(s,r) = (f_1^n(s,r), \dots, f_k^n(s,r))^\top$ where $f_j^n(s,r) = n^{(\beta-\alpha-1)/2}\,\mathbf{1}_{B(t_j)}(s,r)(r \wedge (t_j - \gamma(s)))$, $j = 1, \dots, k$. Then

$$\tilde{R}_{n,i}(t_j) = \int_{[0,\infty)\times[0,\infty)} f_j^n(s,r)\tilde{\xi}_{n,i}(\mathrm{d}s,\mathrm{d}r).$$

One can show by a change of variables that

$$\begin{aligned}
&\lim_{n\to\infty}\int_{[0,\infty)\times[0,\infty)} f_j^n(s,r) f_l^n(s,r)\eta_n(\mathrm{d}s,\mathrm{d}r)\\
&\quad = \lim_{n\to\infty}\int_0^{t_j\wedge t_l}\int_0^{\infty} f(s,U(s))[r\wedge(t_j-s)][r\wedge(t_l-s)]n^{\beta}(\beta-1)\\
&\qquad\qquad\qquad\qquad \times\theta(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\\
&\quad = \int_0^{t_j\wedge t_l}\int_0^{\infty} f(s,U(s))[r\wedge(t_j-s)][r\wedge(t_l-s)](\beta-1)\theta^{1-\beta}r^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\\
&\quad = \mathbb{E}\{R_i(t_j)R_i(t_l)\}.
\end{aligned}$$

Since $|f^n|\le n^{(\beta-\alpha-1)/2}t_k$ and $\lim_{n\to\infty} n^{(\beta-\alpha-1)/2}=0$, we deduce that $\mathbf{1}_{\{|f^n|>\varepsilon\}}=0$ for large enough n, and, hence, for each $\varepsilon>0$ and j, if n is large enough,

$$\int_{[0,\infty)\times[0,\infty)} \mathbf{1}_{\{|f^n|>\varepsilon\}}\,|f_j^n(s,r)|^2\eta_n(\mathrm{d}s,\mathrm{d}r)=0.$$

From Theorem 6.1 of [6] it now follows that

$$(\tilde{R}_{n,i}(t_1),\dots,\tilde{R}_{n,i}(t_k))^{\top}\Longrightarrow(R_i(t_1),\dots,R_i(t_k))^{\top}$$

as n increases to ∞, for each $i=1,\dots,d$. Since $\tilde{\boldsymbol{R}}_n$ has independent components, it follows that the finite-dimensional distributions of $\tilde{\boldsymbol{R}}_n$ converge to those of $\boldsymbol{R}$. Using (4.13), we then obtain that the finite-dimensional distributions of $\boldsymbol{R}_n$ converge to those of $\boldsymbol{R}$.

Thus, in order to prove the lemma it suffices to show that $\{\boldsymbol{R}_n\}$ is tight in $D_{\mathbb{R}^d}[0,\infty)$, for which, it suffices to prove the following estimate: for each fixed $T>0$ there exists a constant $C_T>0$ such that for $0\le h\le 1$ and $0\le t\le T$

$$\mathbb{E}\{|\boldsymbol{R}_n(t+h)-\boldsymbol{R}_n(t)|^4\}\le C_T h^2.$$

Recall the definition of $\mathcal{U}_{n,i}^{(2)}$ in (3.25). Then

$$R_{n,i}(t+h)-R_{n,i}(t)=n^{(\alpha+\beta-3)/2}\mathcal{U}_{n,i}^{(2)}(\Lambda_n(t+h))=n^{(\alpha+\beta-3)/2}(A_{n,i}(t+h)-A_{n,i}(t)).$$

From (3.26), (3.27), and (3.28), we now have

$$\begin{aligned}
\mathbb{E}\{|\boldsymbol{R}_n(t+h)-\boldsymbol{R}_n(t)|^4\} &= n^{4(\alpha+\beta-3)/2}\mathbb{E}\{|\boldsymbol{A}_n(t+h)-\boldsymbol{A}_n(t)|^4\}\\
&\le Cn^{2\alpha+2\beta-6}(n^{5-\beta-3\alpha}h^{6-\beta}+n^{6-2\beta-2\alpha}h^{2(4-\beta)})\\
&= C(n^{-(\alpha-\beta+1)}h^{6-\beta}+h^{2(4-\beta)})\\
&\le Ch^2,
\end{aligned}$$

where the last inequality follows from $\alpha>\beta-1$ and $2<\beta<3$. This proves the desired tightness and the result follows.

Lemma 4.2. *Let C_n be as given in (4.3). As $n\to\infty$, $\sup_{0\le s\le t}|C_n(s)|\to 0$, in probability, for every $t\ge 0$.*

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Proof. From Assumption 1.1, we have

$$\begin{aligned}|f_{yy}(t,y)| &= |\exp\{-g(y-bt)\}[g'(y-bt)]^2 - \exp\{-g(y-bt)\}g''(y-bt)|\\ &\le (L^2+L)\exp\{-g(y-bt)\}\\ &\le (L^2+L)\exp\{-g(-bt)\}\\ &=: c(t)\end{aligned}$$

for all $y \in [0,\infty)$. Consequently, $y \mapsto f_y(t,y)$ is a Lipschitz function on $\mathbb{R}_+$, uniformly in t in compact intervals. Therefore,

$$|C_n(t)| \le a\int_0^t c(s)(n^{(\alpha+\beta-3)/4}|\bar{Y}_n(s)-U(s)|)^2\,\mathrm{d}s.$$

The result now follows by noting that $(\alpha+\beta-3)/4 < \beta-2$ (see (1.9)) and using (2.8).

Lemma 4.3. *Let D_n be as given in (4.4). As $n\to\infty$, $\sup_{0\le s\le t}|D_n(s)| \to 0$, in probability, for every $t \ge 0$.*

Proof. Note that

$$\begin{aligned}D_n(t) &= n^{(\alpha-\beta+1)/2}\bigg(\int_0^t \frac{a\theta^{2-\beta}f(s,U(s))}{(t-s)^{\beta-2}}\,\mathrm{d}s - n^{\beta-2}\int_0^t \frac{af(s,U(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s\bigg)\\ &\quad + n^{(\alpha-\beta+1)/2}\bigg(n^{\beta-2}\int_0^t \frac{af(s,U(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s\\ &\qquad\qquad - n^{\beta-2}\int_0^t \frac{af(s,\bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s\bigg).\end{aligned} \tag{4.14}$$

For the first term, note that

$$\begin{aligned}0 &< \int_0^t \frac{a\theta^{2-\beta}f(s,U(s))}{(t-s)^{\beta-2}}\,\mathrm{d}s - n^{\beta-2}\int_0^t \frac{af(s,U(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s\\ &= \int_0^t a\theta^{2-\beta}f(s,U(s))\bigg(\frac{1}{(t-s)^{\beta-2}} - \frac{1}{(t-s+1/n\theta)^{\beta-2}}\bigg)\,\mathrm{d}s\\ &\le \frac{a\theta^{2-\beta}}{3-\beta}\sup_{0\le s\le t}\{\exp\{-g(-bs)\}\}\bigg(\frac{1}{n\theta}\bigg)^{3-\beta}.\end{aligned} \tag{4.15}$$

For the second term, from the Lipschitz property of f (see (3.13)), we have

$$\begin{aligned}&\bigg|n^{\beta-2}\int_0^t \frac{af(s,U(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s - n^{\beta-2}\int_0^t \frac{af(s,\bar{Y}_n(s))}{(n\theta(t-s)+1)^{\beta-2}}\,\mathrm{d}s\bigg|\\ &\quad\le \int_0^t \frac{a\theta^{2-\beta}L\mathrm{e}^{-g(-bs)}\,|\,\bar{Y}_n(s)-U(s)|}{(t-s+1/n\theta)^{\beta-2}}\,\mathrm{d}s\\ &\quad\le \frac{a\theta^{2-\beta}Lt^{3-\beta}}{3-\beta}\sup_{0\le s\le t}\{\exp\{-g(-bs)\}\}\sup_{0\le s\le t}\{|\bar{Y}_n(s)-U(s)|\}.\end{aligned} \tag{4.16}$$

Combining (4.14)–(4.16), we have

$$\sup_{0\le s\le t}|D_n(s)| \le C[n^{(\alpha+\beta-5)/2} + n^{(\alpha-\beta+1)/2}|U_n(s)-U(s)|].$$

From (1.9) we see that $(\alpha+\beta-5)/2 < 0$ and $(\alpha-\beta+1)/2 < \beta-2$. The result follows using (2.8).

5. The moment stabilization property

In this section, we will prove Theorem 2.3. Let $\boldsymbol{Z}$ be as in Theorem 2.2 and let $\boldsymbol{R} = (R_1, \dots, R_d)^\top$ be the Gaussian process introduced in (4.6). Then $\bar{Z} = (1/d)\sum_{i=1}^d Z_i$ satisfies

$$\bar{Z}(t) = \bar{R}(t) + a\int_0^t f_y(s, U(s))\bar{Z}(s)\,\mathrm{d}s, \tag{5.1}$$

where $\bar{R} = (1/d)\sum_{i=1}^d R_i$. Note that $\bar{R}$ is a zero mean Gaussian process. We begin by computing the covariance functions of R_i, $i = 1, \dots, d$, and $\bar{R}$. The proof is omitted due to space constraints.

Lemma 5.1. *The covariance functions of the Gaussian processes R_i, $i = 1, \dots, d$, and $\bar{R}$ are given respectively by*

$$\begin{aligned}\operatorname{cov}(R_i(s), R_i(t)) &= \mathbb{E}\{R_i(s)R_i(t)\}\\ &= \theta^{1-\beta}\int_0^s\int_0^t\int_0^{u\wedge v}\exp\{-g(U(z)-bz)\}(u\vee v - z)^{1-\beta}\,\mathrm{d}z\,\mathrm{d}u\,\mathrm{d}v,\end{aligned} \tag{5.2}$$

and $\operatorname{cov}(\bar{R}(s), \bar{R}(t)) = \operatorname{cov}(R_i(s), R_i(t))/d$, *for any* $s, t \ge 0$.

In the next lemma, we provide a bound on the second moments of the increment of the Gaussian processes R_i, $i = 1, \dots, d$, and $\bar{R}$.

Lemma 5.2. *For any $s, t \ge 0$, the following bound holds:*

$$\mathbb{E}\{|R_i(t) - R_i(s)|^2\} \le \frac{2K_1\theta^{1-\beta}}{(\beta-2)(3-\beta)(4-\beta)}(t-s)^{4-\beta}, \tag{5.3}$$

where $K_1 := \sup_{s\ge 0}\{\exp\{-g(U(s) - bs)\}\}$. The bound (5.3) also holds with R_i replaced by $\bar{R}$ when its right-hand side is divided by d.

Consequently, the Gaussian processes $R_1, \dots, R_d, \bar{R}$ have versions that are Hölder continuous of any order $\rho \in (0, (4-\beta)/2)$ on $[0, T]$ for all $T > 0$.

Proof. Fix $0 \le s \le t < \infty$. From Lemma 5.1, for each $i = 1, \dots, d$,

$$\begin{aligned}\mathbb{E}\{|R_i(t) - R_i(s)|^2\} &= \theta^{1-\beta}\int_s^t\int_s^t\int_0^{u\wedge v}\exp\{-g(U(z)-bz)\}(u\vee v - z)^{1-\beta}\,\mathrm{d}z\,\mathrm{d}u\,\mathrm{d}v\\ &\le \frac{K_1\theta^{1-\beta}}{\beta-2}\int_s^t\int_s^t[|u-v|^{2-\beta} - (u\vee v)^{2-\beta}]\,\mathrm{d}u\,\mathrm{d}v\\ &\le \frac{2K_1\theta^{1-\beta}}{\beta-2}\int_s^t\int_v^t(u-v)^{2-\beta}\,\mathrm{d}u\,\mathrm{d}v\\ &= \frac{2K_1\theta^{1-\beta}}{(\beta-2)(3-\beta)(4-\beta)}(t-s)^{4-\beta}.\end{aligned} \tag{5.4}$$

This completes the proof of (5.3). The result for $\bar{R}$ is now immediate. The last statement in the lemma now follows from Kolmogorov's continuity criterion.

The proof of Theorem 2.3 relies on an explicit representation for the solution of (5.1). For that we begin with an indefinite integral of a deterministic function with respect to the Gaussian process $\bar{R}$.

Denote by $\mathcal{E}$ the linear span of indicator functions of the form $\mathbf{1}_{(s,t]}\colon \mathbb{R}_+ \to \mathbb{R}$, $0 \le s \le t < \infty$. Consider the inner product on $\mathcal{E}$ given by

$$\langle \mathbf{1}_{(0,s]}, \mathbf{1}_{(0,t]} \rangle_{\mathcal{H}_{\bar{R}}} = \operatorname{cov}(\bar{R}(s), \bar{R}(t)) = \int_0^s \int_0^t \rho(u, v)\, \mathrm{d}u\, \mathrm{d}v,$$

where, by (5.2),

$$\rho(u, v) = \begin{cases} \dfrac{\theta^{1-\beta}}{d} \displaystyle\int_0^{u \wedge v} \exp\{-g(U(z) - bz)\}(u \vee v - z)^{1-\beta}\, \mathrm{d}z & \text{if } u \neq v, \\ 0 & \text{if } u = v. \end{cases}$$

We denote by $\mathcal{H}_{\bar{R}}$ the Hilbert space obtained as the closure of $\mathcal{E}$ with respect to this inner product. Define $\bar{\boldsymbol{R}}\colon \mathcal{E} \to L^2(\Omega, \mathcal{F}, \mathbb{P})$ as

$$\bar{\boldsymbol{R}}(\mathbf{1}_{(0,t]}) = \bar{R}(t), \qquad 0 \le t < \infty,$$

where the definition is extended to all $\mathcal{E}$ by linearity. Clearly, $\mathbb{E}\{|\bar{\boldsymbol{R}}(\phi)|^2\} = \langle \phi, \phi \rangle_{\mathcal{H}_{\bar{R}}}$ for all $\phi \in \mathcal{E}$. We can now extend the definition of $\bar{\boldsymbol{R}}$ to all $\mathcal{H}_{\bar{R}}$ by isometry. Occasionally, we will use the notation

$$\bar{\boldsymbol{R}}(\phi) = \int_0^\infty \phi(t)\, \mathrm{d}\bar{R}(t), \ \ \phi \in \mathcal{H}_{\bar{R}}.$$

For any $\phi,\ \tilde{\phi} \in \mathcal{E}$, it holds that

$$\langle \phi, \tilde{\phi} \rangle_{\mathcal{H}_{\bar{R}}} = \int_0^\infty \int_0^\infty \phi(u)\tilde{\phi}(v)\rho(u, v)\, \mathrm{d}u\, \mathrm{d}v. \tag{5.5}$$

It can be shown that $\mathcal{H}_{\bar{R}}$ contains all measurable functions ϕ on $\mathbb{R}_+$ satisfying

$$\int_0^\infty \int_0^\infty |\phi(u)||\phi(v)|\rho(u, v)\, \mathrm{d}u\, \mathrm{d}v < \infty, \tag{5.6}$$

and that equality (5.5) holds for $\phi,\ \tilde{\phi}$ that satisfy (5.6).

This type of isometry is considered in [9] (see Chapter 5) and [1] with respect to fractional Brownian motion and general Gaussian processes respectively.

Remark 5.1. If $\phi\colon [0, \infty) \to \mathbb{R}$ is continuous, then, for any $t > 0$, the function ϕ_t defined by $\phi_t(\cdot) = \mathbf{1}_{[0,t]}(\cdot)\phi(\cdot)$ satisfies (5.6). Consequently, ϕ_t is in $\mathcal{H}_{\bar{R}}$ and we write, formally,

$$\bar{\boldsymbol{R}}(\phi)(t) := \bar{\boldsymbol{R}}(\phi_t) = \int_0^t \phi(s)\, \mathrm{d}\bar{R}(s). \tag{5.7}$$

Remark 5.2. If $\phi(\cdot)$ is Hölder continuous of order $\rho_1 > 1 - (4 - \beta)/2$ on $[0, t]$, for every $t > 0$, as a result of Young's integration theory [13], the pathwise Riemann–Stieltjes integral $\int_0^t \phi(s)\, \mathrm{d}\bar{R}(s)$ exists, since $\bar{R}$ is Hölder continuous of any order $\rho \in (0, (4 - \beta)/2)$. Zähle [14] showed (see Proposition 4.4.1 therein) that $\bar{\boldsymbol{R}}(\phi)(t)$ is Hölder continuous of the same order as $\bar{R}$ on $[0, T]$, for every $T > 0$. The indefinite integral $\bar{\boldsymbol{R}}(\phi)(\cdot)$ on the right-hand side of (5.7) coincides with the pathwise Riemann–Stieltjes integral.

We now proceed to the proof of Theorem 2.3.

Proof of Theorem 2.3. Define

$$\phi(t) := \exp\biggl\{-a\int_0^t f_y(z, U(z))\,\mathrm{d}z\biggr\}, \qquad \tilde{\phi}(t) = \exp\biggl\{a\int_0^t f_y(z, U(z))\,\mathrm{d}z\biggr\}.$$

Then, the derivatives of ϕ and $\tilde{\phi}$ are

$$\phi'(t) = -af_y(t, U(t))\phi(t), \qquad \tilde{\phi}'(t) = af_y(t, U(t))\tilde{\phi}(t).$$

Remark 2.1 implies that ϕ' and $\tilde{\phi}'$ are bounded on any compact interval and, hence, ϕ and $\tilde{\phi}$ are locally Lipschitz continuous. From Remark 5.2, the indefinite integral

$$\bar{\boldsymbol{R}}(\phi)(t) = \int_0^t \phi(s)\,\mathrm{d}\bar{R}(s) = \int_0^t \exp\biggl\{-a\int_0^s f_y(z, U(z))\,\mathrm{d}z\biggr\}\,\mathrm{d}\bar{R}(s)$$

is well defined as a Riemann–Stieltjes integral, and, for every $T > 0$, $\bar{\boldsymbol{R}}(\phi)(t)$ is Hölder continuous on $[0, T]$ of any order $\rho \in (0, (4-\beta)/2)$.

It follows from Theorems 3.1 and 4.4.2 of [14] that

$$\begin{aligned}\tilde{\phi}(t)\bar{\boldsymbol{R}}(\phi)(t) &= \int_0^t \tilde{\phi}(s)\phi(s)\,\mathrm{d}\bar{R}(s) + a\int_0^t f_y(s, U(s))\tilde{\phi}(s)\bar{\boldsymbol{R}}(\phi)(s)\,\mathrm{d}s \\ &= \bar{R}(t) + a\int_0^t f_y(s, U(s))\tilde{\phi}(s)\bar{\boldsymbol{R}}(\phi)(s)\,\mathrm{d}s,\end{aligned}$$

which implies that $\tilde{\phi}(t)\bar{\boldsymbol{R}}(\phi)(t)$ solves (5.1). Thus, the solution $\bar{Z}$ to (5.1) can be written explicitly as

$$\begin{aligned}\bar{Z}(t) &= \tilde{\phi}(t)\bar{\boldsymbol{R}}(\phi)(t) \\ &= \exp\biggl\{a\int_0^t f_y(z, U(z))\,\mathrm{d}z\biggr\}\int_0^t \exp\biggl\{-a\int_0^s f_y(z, U(z))\,\mathrm{d}z\biggr\}\,\mathrm{d}\bar{R}(s).\end{aligned}$$

By the isometry of the mapping $\bar{\boldsymbol{R}}$, we have, on letting $\phi_t(u) = \phi(u)\,\mathbf{1}_{[0,t]}(u)$,

$$\begin{aligned}\mathbb{E}\{|\bar{Z}(t)|^2\} &= |\tilde{\phi}(t)|^2\int_0^\infty\int_0^\infty \phi_t(u)\phi_t(v)\rho(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &= |\tilde{\phi}(t)|^2\int_0^t\int_0^t \phi(u)\phi(v)\rho(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &= \int_0^t\int_0^t \exp\biggl\{a\int_u^t f_y(z, U(z))\,\mathrm{d}z\biggr\}\exp\biggl\{a\int_v^t f_y(z, U(z))\,\mathrm{d}z\biggr\}\rho(u, v)\,\mathrm{d}u\,\mathrm{d}v.\end{aligned}$$

Recall the definition of μ from the statement of Theorem 2.3. From Remark 2.1, $\mu \in (0, \infty)$. Then, by a calculation similar to (5.4), we have, for all $t \geq 0$,

$$\begin{aligned}\theta^{\beta-1}\mathbb{E}\{|\bar{Z}(t)|^2\} &\leq \int_0^t\int_0^t \mathrm{e}^{-a\mu(t-u)}\mathrm{e}^{-a\mu(t-v)}\rho(u, v)\,\mathrm{d}u\,\mathrm{d}v \\ &\leq \frac{2}{d(\beta-2)}\int_0^t\int_v^t \mathrm{e}^{-a\mu(t-v)}(u-v)^{2-\beta}\,\mathrm{d}u\,\mathrm{d}v \\ &\leq \frac{2}{d(\beta-2)(3-\beta)(a\mu)^{4-\beta}}\Gamma(4-\beta).\end{aligned}$$

The result follows.

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6. Fractional Ornstein–Uhlenbeck process

We now proceed to the proof of Theorem 2.4. Throughout this section we take $b = a$. From Proposition 2.1 it follows that

$$U(t) = bt, \qquad f(t, U(t)) = 1, \qquad f_y(t, U(t)) = -g'(0)$$

for all $t \geq 0$. For notational simplicity, we will only present the proof for the $\theta = 1$ case.

In this special case, the SDE (5.1) can be written as

$$\bar{Z}(t) = \bar{R}(t) - a \int_0^t g'(0)\bar{Z}(s)\,\mathrm{d}s = \bar{R}(t) - \kappa \int_0^t \bar{Z}(s)\,\mathrm{d}s, \tag{6.1}$$

where $\kappa = a\mu = ag'(0) > 0$, and

$$\bar{R}(t) = \frac{1}{d}\sum_{i=1}^{d}\int_0^t \int_0^\infty r \wedge (t-s)\Sigma_i(\mathrm{d}s, \mathrm{d}r)$$

for any $t \geq 0$.

From Lemma 5.1, it follows that the covariance of $\bar{R}$ is given by

$$\begin{aligned}\operatorname{cov}(\bar{R}(s), \bar{R}(t)) = \mathbb{E}\{\bar{R}(s)\bar{R}(t)\} &= \frac{1}{d}\int_0^s \int_0^t \int_0^{u\wedge v} (u \vee v - z)^{1-\beta}\,\mathrm{d}z\,\mathrm{d}u\,\mathrm{d}v \\ &= \frac{1}{d(\beta-2)}\int_0^s \int_0^t (|u-v|^{2-\beta} - (u\vee v)^{2-\beta})\,\mathrm{d}u\,\mathrm{d}v,\end{aligned}$$

and, from Lemma 5.2, we recall that the sample paths of the process $\bar{R}$ are Hölder continuous on $[0, T]$ of order ρ, for any $\rho \in (0, (4-\beta)/2)$.

Recalling the definition of the indefinite integrals with respect to the Gaussian process $\bar{R}$, the solution of the SDE in (6.1) can be explicitly written as

$$\bar{Z}(t) = \mathrm{e}^{-\kappa t}\int_0^t \mathrm{e}^{\kappa s}\,\mathrm{d}\bar{R}(s).$$

We now consider the asymptotic behavior of the process $\bar{Z}(t)$ as $t \to \infty$. For $T,\ t \geq 0$, let $\bar{R}_T(t) = \bar{R}(T+t) - \bar{R}(T)$. From (6.1), we can write

$$\bar{Z}(T+t) = \bar{Z}(T) + \bar{R}_T(t) - \kappa\int_0^t \bar{Z}(T+s)\,\mathrm{d}s.$$

Recall the parameters σ_0^2 and σ (also recall that $\theta = 1$). The proof of the following lemma is omitted due to space constraints.

Lemma 6.1. *We have*

(a) $\lim_{T\to\infty} \mathbb{E}\{|\bar{Z}(T)|^2\} = \sigma_0^2$.

(b) *For any* $t \geq s \geq 0$,

$$\lim_{T\to\infty} \mathbb{E}\{\bar{R}_T(t)\bar{R}_T(s)\} = \frac{\sigma^2}{2}[t^{4-\beta} + s^{4-\beta} - (t-s)^{4-\beta}].$$

(c) *For any $t \geq 0$,*

$$\lim_{T\to\infty} \operatorname{cov}(\bar{Z}(T), \bar{R}_T(t)) = \lim_{T\to\infty} \mathbb{E}\{\bar{Z}(T)\bar{R}_T(t)\}$$
$$= \frac{1}{d\,(\beta-2)} \int_0^t \int_0^\infty \mathrm{e}^{-\kappa v}(u+v)^{2-\beta}\,\mathrm{d}v\,\mathrm{d}u.$$

Proof of Theorem 2.4. Define $\varphi\colon C_{\mathbb{R}}[0,\infty) \to C_{\mathbb{R}}[0,\infty)$ by

$$[\varphi(x)](t) = x(t) - \kappa \int_0^t [\varphi(x)](s)\,\mathrm{d}s.$$

Then, φ is a continuous mapping from $C_{\mathbb{R}}[0,\infty)$ to $C_{\mathbb{R}}[0,\infty)$.

For any $t,\ T \geq 0$, denote $\bar{\mathcal{R}}_T(t) = \bar{Z}(T) + \bar{R}_T(t)$. Then $\bar{\mathcal{R}}_T$ is a Gaussian process with continuous trajectories and $\bar{Z}(T+\cdot) = \varphi(\bar{\mathcal{R}}_T)(\cdot)$. Therefore, in order to prove that $\bar{Z}(T+\cdot)$ converges in distribution in $C_{\mathbb{R}}[0,\infty)$ to Z_∞, it suffices to show the convergence of $\bar{\mathcal{R}}_T$ to $Z_\infty(0) + \sigma B_H(\cdot)$, where $Z_\infty(0)$, B_H, and σ are as defined earlier. From Lemmas 6.1, it follows that the finite-dimensional distributions of $\bar{\mathcal{R}}_T$ converge to those of $Z_\infty(0) + \sigma B_H(\cdot)$. It thus suffices to verify that $\{\bar{\mathcal{R}}_T(\cdot)\}_{T>0}$ is tight in $C_{\mathbb{R}}[0,\infty)$. By the Cauchy–Schwartz inequality and (5.3) for $\bar{R}$, it follows that, for any $h \geq 0$, $t \geq h$, and $T \geq 0$,

$$\begin{aligned}
&\mathbb{E}\{|\bar{\mathcal{R}}_T(t+h) - \bar{\mathcal{R}}_T(t)||\bar{\mathcal{R}}_T(t) - \bar{\mathcal{R}}_T(t-h)|\}\\
&\quad\leq (\mathbb{E}|\bar{R}(T+t+h) - \bar{R}(T+t)|^2)^{1/2}(\mathbb{E}|\bar{R}(T+t) - \bar{R}(T+t-h)|^2)^{1/2}\\
&\quad\leq \frac{2h^{4-\beta}}{d\,(\beta-2)(3-\beta)(4-\beta)}.
\end{aligned}$$

Note that $4-\beta > 1$, since $\beta < 3$. The desired tightness now follows from standard results (cf. Theorems 3.8.6 and 3.8.8 of [2]). This proves the convergence of $\bar{Z}(T+\cdot)$ to $Z_\infty(\cdot)$. The stationarity of Z_∞ is now immediate. The result follows.

7. Conclusions

An infinite source Poisson arrival model with heavy-tailed workload input distributions has been extensively used for modeling data packet traffic in communication networks. In this paper we introduce, in the context of such a model, a natural family of admission control policies that keep the associated scaled cumulative workload asymptotically close to a prespecified linear trajectory, uniformly over time. A law of large numbers for the scaled workload is proved and fluctuations studied by establishing a functional central limit theorem for suitable scaled and centered workload processes. The moment stabilization property of the control policy is demonstrated by establishing that the asymptotic second moment of the scaled and centered workload process is uniformly bounded in time. The slope of the linear trajectory represents the system processing rate and, thus, such control policies yield, uniform in time, reliability bounds on probabilities of processor underutilization and overload. Finally, a stationary Ornstein–Uhlenbeck process driven by a fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$ is obtained as the large time limit of the asymptotic scaled and centered workload process.

Appendix A. Auxiliary results

Recall that $2 < \beta < 3$ and $\alpha > \beta - 1$. Also, recall the notation $\vartheta_n^{h,t}(r,s)$ and $\tilde{\vartheta}_n^{h,t}(r,s)$ introduced in the proof of Theorem 2.1. The following lemma provides a key estimate for the proof of the theorem.

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Lemma A.1. *There exists a constant $C > 0$ depending only on β and θ, such that, for any $0 \le h \le 1$ and $h \le t < \infty$, we have the following estimates:*

$$\frac{1}{n^{4(\alpha-1)}} \int_0^{t+h} \int_0^{\infty} [\tilde{\vartheta}_n^{h,t}(r,s)]^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \le Cn^{5-\beta-3\alpha}h^{6-\beta} \tag{A.1}$$

and

$$\frac{1}{n^{4(\alpha-1)}} \left(\int_0^{t+h} \int_0^{\infty} [\tilde{\vartheta}_n^{h,t}(r,s)]^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\right)^2 \le Cn^{6-2\beta-2\alpha}h^{2(4-\beta)}. \tag{A.2}$$

Proof. We first prove (A.1). We can write the left-hand side of (A.1) as

$$\begin{aligned} &\frac{1}{n^{4(\alpha-1)}} \int_t^{t+h} \int_0^{\infty} [r \wedge (t+h-s)]^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\ &+ \frac{1}{n^{4(\alpha-1)}} \int_0^{t} \int_0^{\infty} [r \wedge (t+h-s) - r\wedge(t-s)]^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s. \end{aligned} \tag{A.3}$$

We now bound the two terms in (A.3) separately. In the following, we use $C > 0$ to denote a generic constant depending only on β and θ; the value of C may change from one line to the next. Using $(n\theta r+1)^{-\beta} < (n\theta r)^{-\beta}$, we have, on splitting the inner integral in the first term in (A.3) as $[0, t+h-s] \cup (t+h-s, \infty)$,

$$\begin{aligned} &\frac{1}{n^{4(\alpha-1)}} \int_t^{t+h} \int_0^{\infty} [r \wedge (t+h-s)]^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\ &\quad \le \frac{n^{\alpha-\beta+1}}{n^{4(\alpha-1)}} \int_t^{t+h} \int_0^{t+h-s} r^{4-\beta}\,\mathrm{d}r\,\mathrm{d}s + \frac{n^{\alpha-\beta+1}}{n^{4(\alpha-1)}} \int_t^{t+h} \int_{t+h-s}^{\infty} (t+h-s)^4 r^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\ &\quad = Cn^{5-\beta-3\alpha} \int_t^{t+h} (t+h-s)^{5-\beta}\,\mathrm{d}s \le Cn^{5-\beta-3\alpha}h^{6-\beta}. \end{aligned} \tag{A.4}$$

For the second term in (A.3), we have, on splitting the inner integral as $[0, t-s] \cup (t-s, t+h-s] \cup (t+h-s, \infty)$, noting that the contribution from the first summand is zero and the changing of variables, that

$$\begin{aligned} &\frac{1}{n^{4(\alpha-1)}} \int_0^{t} \int_0^{\infty} [r \wedge (t+h-s) - r\wedge(t-s)]^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\ &\quad = \frac{1}{n^{4(\alpha-1)}} \int_0^{t} \int_s^{h+s} (r-s)^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\ &\qquad + \frac{1}{n^{4(\alpha-1)}} \int_0^{t} \int_{t+h-s}^{\infty} h^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s. \end{aligned} \tag{A.5}$$

For the first term on the right-hand side of (A.5), we change the order of integration and obtain

$$\begin{aligned}
&\frac{1}{n^{4(\alpha-1)}}\int_0^t\int_s^{h+s}(r-s)^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\
&\quad\le \frac{1}{n^{4(\alpha-1)}}\int_0^h\int_0^r (r-s)^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}s\,\mathrm{d}r \\
&\qquad+\frac{1}{n^{4(\alpha-1)}}\int_h^{t+h}\int_{r-h}^r (r-s)^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}s\,\mathrm{d}r \\
&\quad\le \frac{Cn^{\alpha-\beta+1}}{n^{4(\alpha-1)}}\int_0^h r^{5-\beta}\,\mathrm{d}r+\frac{Cn^{\alpha-\beta+1}}{n^{4(\alpha-1)}}\int_h^{t+h} h^5\Big(r+\frac{1}{n\theta}\Big)^{-\beta}\,\mathrm{d}r \\
&\quad\le Cn^{5-\beta-3\alpha}h^{6-\beta}, \qquad\qquad (\text{A.6})
\end{aligned}$$

since $(h+1/n\theta)^{1-\beta}-(t+h+1/n\theta)^{1-\beta}<(h+1/n\theta)^{1-\beta}<h^{1-\beta}$ for all $n\in\mathbb{N}$ and $\theta\in(0,\infty)$.

For the second term on the right-hand side of (A.5), we have

$$\begin{aligned}
\frac{1}{n^{4(\alpha-1)}}\int_0^t\int_{t+h-s}^{\infty} h^4 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s &\le \frac{Cn^{\alpha-\beta+1}}{n^{4(\alpha-1)}}\int_0^t\int_{t+h-s}^{\infty} h^4 r^{-\beta}\,\mathrm{d}r\,\mathrm{d}s \\
&\le Cn^{5-\beta-3\alpha}h^{6-\beta}. \qquad (\text{A.7})
\end{aligned}$$

Combining (A.3)–(A.7), the bound (A.1) follows.

Next, we show (A.2). The left-hand side of (A.2) is bounded by

$$\begin{aligned}
&\frac{2}{n^{4(\alpha-1)}}\bigg(\int_t^{t+h}\int_0^\infty [r\wedge(t+h-s)]^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&+\frac{2}{n^{4(\alpha-1)}}\bigg(\int_0^t\int_0^\infty [r\wedge(t+h-s)-r\wedge(t-s)]^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2. \qquad (\text{A.8})
\end{aligned}$$

We bound the first term in (A.8) by splitting the inner integral as $[0,t+h-s]\cup(t+h-s,\infty)$, as

$$\begin{aligned}
&\frac{2}{n^{4(\alpha-1)}}\bigg(\int_t^{t+h}\int_0^\infty [r\wedge(t+h-s)]^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&\quad\le \frac{Cn^{2(\alpha-\beta+1)}}{n^{4(\alpha-1)}}\bigg(\int_t^{t+h}\int_0^{t+h-s} r^{2-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&\qquad+\frac{Cn^{2(\alpha-\beta+1)}}{n^{4(\alpha-1)}}\bigg(\int_t^{t+h}\int_{t+h-s}^{\infty}(t+h-s)^2 r^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&\quad\le Cn^{6-2\beta-2\alpha}\bigg(\int_t^{t+h}(t+h-s)^{3-\beta}\,\mathrm{d}s\bigg)^2\le Cn^{6-2\beta-2\alpha}h^{2(4-\beta)}. \qquad (\text{A.9})
\end{aligned}$$

For the second term in (A.8), by a change of variables, we obtain, on splitting the inner integral as $[0, t-s] \cup (t-s, t+h-s] \cup (t+h-s, \infty)$,

$$\begin{aligned}
&\frac{2}{n^{4(\alpha-1)}}\bigg(\int_0^t \int_0^\infty [r \wedge (t+h-s) - r \wedge (t-s)]^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&\quad \le \frac{4}{n^{4(\alpha-1)}}\bigg(\int_0^t \int_s^{h+s} (r-s)^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&\qquad + \frac{4}{n^{4(\alpha-1)}}\bigg(\int_0^t \int_{t+h-s}^\infty h^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2. \qquad \text{(A.10)}
\end{aligned}$$

By changing the order of integration, the first term on the right-hand side of (A.10) is bounded as

$$\begin{aligned}
&\frac{4}{n^{4(\alpha-1)}}\bigg(\int_0^t \int_s^{h+s} (r-s)^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 \\
&\quad \le \frac{8}{n^{4(\alpha-1)}}\bigg(\int_0^h \int_0^r (r-s)^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}s\,\mathrm{d}r\bigg)^2 \\
&\qquad + \frac{8}{n^{4(\alpha-1)}}\bigg(\int_h^{t+h} \int_{r-h}^r (r-s)^2 n^{\alpha+1}(n\theta r+1)^{-\beta}\,\mathrm{d}s\,\mathrm{d}r\bigg)^2 \\
&\quad \le C n^{6-2\alpha-2\beta} h^{2(4-\beta)}. \qquad \text{(A.11)}
\end{aligned}$$

The second term on the right-hand side of (A.10) can be bounded as

$$\begin{aligned}
\frac{4n^{2(\alpha-\beta+1)}}{n^{4(\alpha-1)}}\bigg(\int_0^t \int_{t+h-s}^\infty h^2 r^{-\beta}\,\mathrm{d}r\,\mathrm{d}s\bigg)^2 &\le C n^{6-2\alpha-2\beta}\bigg(\int_0^t h^2 (t+h-s)^{1-\beta}\,\mathrm{d}s\bigg)^2 \\
&\le C n^{6-2\alpha-2\beta} h^{2(4-\beta)}. \qquad \text{(A.12)}
\end{aligned}$$

The bound in (A.2) now follows from (A.8)–(A.12).

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