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Martin Orr and Alexei N. Skorobogatov

Compositio Math. **154** (2018), 1571–1592.

[doi:10.1112/S0010437X18007169](https://doi.org/10.1112/S0010437X18007169)



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ABSTRACT

We study abelian varieties and K3 surfaces with complex multiplication defined over number fields of fixed degree. We show that these varieties fall into finitely many isomorphism classes over an algebraic closure of the field of rational numbers. As an application we confirm finiteness conjectures of Shafarevich and Coleman in the CM case. In addition we prove the uniform boundedness of the Galois invariant subgroup of the geometric Brauer group for forms of a smooth projective variety satisfying the integral Mumford–Tate conjecture. When applied to K3 surfaces, this affirms a conjecture of Várilly-Alvarado in the CM case.

1. Introduction

In a recent paper Tsimerman [Tsi18], building on the work of Andreatta *et al.* [AGHM18] and of Yuan and Zhang [YZ18], obtained a lower bound for the size of Galois orbits of CM points, reproduced below as Theorem 2.1. We view this inequality as an upper bound for the discriminant of the centre of the endomorphism ring of a principally polarised CM abelian variety in terms of the degree of a field of definition. With a little extra work, using the classical Masser–Wüstholz bound for the minimal degree of an isogeny between abelian varieties, and Zarhin’s quaternion trick, we deduce the following result (see Theorem 2.5).

THEOREM A. *There are only finitely many $\overline{\mathbb{Q}}$ -isomorphism classes of abelian varieties of CM type of given dimension which can be defined over number fields of given degree.*

Robert Coleman conjectured that for a given number field k only finitely many rings, up to isomorphism, can be realised as the ring of $\overline{\mathbb{Q}}$ -endomorphisms of an abelian variety defined over k (see Remark 4 at the end of [Sha96] and conjecture C(e, g) in [BFGR06, p. 384]). For abelian varieties of CM type this conjecture follows from Theorem A.

COROLLARY A.1. *Let g and d be positive integers. There are only finitely many rings R , up to isomorphism, for which there exists an abelian variety A of CM type over a number field of degree d such that $\dim(A) = g$ and $\text{End}(A \times_k \overline{\mathbb{Q}}) \cong R$.*

Under the generalised Riemann hypothesis this had been proved by Greenberg, see [BFGR06, p. 384].

Using Theorem A we show that any Shimura variety of abelian type has only finitely many CM points defined over number fields of bounded degree, see Proposition 3.1. By the work of

Received 9 June 2017, accepted in final form 13 November 2017, published online 18 July 2018.

2010 Mathematics Subject Classification 11G15, 14J28, 14F22 (primary).

Keywords: abelian varieties, K3 surfaces, complex multiplication, Brauer group.

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Rizov and Madapusi Pera, a double cover of the moduli space of K3 surfaces with a polarisation of fixed degree is an open subset of a Shimura variety of abelian type. Hence we get a finiteness theorem for K3 surfaces of CM type with a polarisation of fixed degree defined over number fields of bounded degree.

Using an idea suggested to us by François Charles we develop a lattice-theoretic device that plays the role of Zarhin's trick for K3 surfaces, related to the method of [Cha16]. The analogy with Zarhin's trick becomes apparent when stated in terms of Shimura varieties, see §4.1. This allows us to remove the dependence on the degree of polarisation and prove the following main result of this paper (Theorem 4.1).

THEOREM B. *There are only finitely many $\overline{\mathbb{Q}}$ -isomorphism classes of K3 surfaces of CM type which can be defined over number fields of given degree.*

Recall that a K3 surface has CM type if its Mumford–Tate group is commutative. K3 surfaces of CM type were introduced by Piatetski-Shapiro and Shafarevich in [PS73]. They proved that such a surface is always defined over a number field [PS73, Theorem 4]. Examples of K3 surfaces of CM type are diagonal quartic surfaces in \mathbb{P}^3 and, more generally, arbitrary K3 surfaces of Picard rank 20, as well as the Kummer surfaces attached to simple abelian surfaces of CM type. In the latter case the Picard rank is 18. Taelman proved that there exist K3 surfaces of CM type over \mathbb{C} with arbitrary even Picard rank from 2 to 20 [Tae16].

Let us understand by a lattice a free abelian group of finite rank with an integral symmetric bilinear form. Shafarevich conjectured that only finitely many lattices, up to isomorphism, can be realised as the Néron–Severi lattice of a K3 surface defined over a number field of fixed degree [Sha96]. Equivalently, the discriminants of the Néron–Severi lattices of such K3 surfaces are bounded. Theorem B confirms this conjecture in the case of K3 surfaces of CM type.

COROLLARY B.1. *Only finitely many lattices, up to isomorphism, can be realised as the Néron–Severi lattice $\text{NS}(\overline{X})$ where X is a K3 surface of CM type defined over a number field of given degree.*

In [Sha96] Shafarevich proved this conjecture for K3 surfaces of Picard rank 20. In fact, he proved that K3 surfaces of Picard rank 20 that can be defined over a number field of fixed degree fall into finitely many isomorphism classes over $\overline{\mathbb{Q}}$. This paper was born from our reflections on this theorem of Shafarevich.

In contrast to the above results, the following statement does not use the assumption of complex multiplication. The main ingredients of its proof are the results of Cadoret and Moonen on the Mumford–Tate conjecture [CM15, Theorems A and B], which build on the previous work of many authors, including Serre, Wintenberger, Larsen, Pink, Cadoret and Kret, and the proof of the Mumford–Tate conjecture for K3 surfaces by Tankeev and André.

THEOREM C. *Let k be a field finitely generated over \mathbb{Q} . Let X over k be either an abelian variety satisfying the Mumford–Tate conjecture or a K3 surface. For each positive integer n , there exists a constant $C = C_{n,X}$ such that for every (\overline{k}/L) -form Y of X defined over a field extension L of degree $[L : k] \leq n$, we have $|\text{Br}(\overline{Y})^{\text{Gal}(\overline{k}/L)}| < C$.*

Here and elsewhere in this paper we write \overline{k} for an algebraic closure of k and $\overline{Y} = Y \times_L \overline{k}$. A (\overline{k}/L) -form of X is a variety Y over a field L , where $k \subset L \subset \overline{k}$, such that $\overline{Y} \cong \overline{X}$.

Although the finiteness of $\mathrm{Br}(\bar{Y})^{\mathrm{Gal}(\bar{k}/L)}$ in Theorem C follows from the main results of [SZ08], they are not used in the present proof, so Theorem C also gives a new proof of [SZ08, Theorem 1.2].

Theorem C is the combination of Corollaries 5.2 and 5.3 of a more general Theorem 5.1. For an analogue of Theorem C concerning torsion of abelian varieties which are (\bar{k}/L) -forms of a given abelian variety, see the remark at the end of the paper.

When Y is a (\bar{k}/L) -form of a K3 surface X , Theorem C together with a classical observation of Minkowski [Min87] that there are only finitely many isomorphism classes of finite subgroups in $\mathrm{GL}_n(\mathbb{Z})$ gives the boundedness of $|\mathrm{Br}(Y)/\mathrm{Br}_0(Y)|$, where $\mathrm{Br}_0(Y)$ is the image of the canonical map $\mathrm{Br}(k) \rightarrow \mathrm{Br}(Y)$ (see [VV17, Lemma 6.4]). For example, this can be applied to the diagonal quartic surface $X \subset \mathbb{P}_{\mathbb{Q}}^3$ given by $x^4 + y^4 + z^4 + w^4 = 0$. Any surface $Y \subset \mathbb{P}_L^3$ given by

$$ax^4 + by^4 + cz^4 + dw^4 = 0,$$

where $a, b, c, d \in L^*$, is a $(\bar{\mathbb{Q}}/L)$ -form of X . In the particular case $L = \mathbb{Q}$ a bound for $|\mathrm{Br}(Y)/\mathrm{Br}_0(Y)|$ was obtained with an explicit constant in [ISZ11, Corollary 4.6], see also [Ier10, IS15, New16] for related work.

Recently, Tony Várilly-Alvarado conjectured that for any positive integer n and any isomorphism class of a primitive sublattice Λ of the K3 lattice Λ_{K3} , there is a constant $c(n, \Lambda)$ such that for any K3 surface X defined over a number field of degree n with $\mathrm{NS}(\bar{X}) \cong \Lambda$ we have $|\mathrm{Br}(X)/\mathrm{Br}_0(X)| < c(n, \Lambda)$; see [Vár17, Conjecture 4.6] and [VV17, Question 1.1]. Combining Theorems B and C we confirm this conjecture in the case of K3 surfaces of CM type, even without fixing the isomorphism type of the Néron–Severi lattice.

COROLLARY C.1. *For any positive integer n there is a constant $C = C_n$ such that $|\mathrm{Br}(X)/\mathrm{Br}_0(X)| < C$ and $|\mathrm{Br}(\bar{X})^{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}| < C$ for any K3 surface X of CM type defined over a number field k of degree n .*

Combining Theorems A and C with the fact that the Mumford–Tate conjecture holds for abelian varieties of CM type, we obtain the following.

COROLLARY C.2. *For any positive integers n and g there is a constant $C = C_{n,g}$ such that $|\mathrm{Br}(\bar{X})^{\mathrm{Gal}(\bar{\mathbb{Q}}/k)}| < C$ for any form X of an abelian variety of dimension g of CM type, where X is defined over a number field k of degree n .*

It would be interesting to find explicit expressions for the constants in Corollaries C.1 and C.2.

Theorem A is proved in §2, see Theorem 2.5. In §3 we recall the background on Shimura varieties used to prove Theorem B in §4, see Theorem 4.1. Finally, in §5 we discuss the Brauer groups of (\bar{k}/L) -forms of projective varieties in relation with the Mumford–Tate conjecture and prove Theorem 5.1 and its corollaries, including Theorem C.

2. Finiteness theorem for abelian varieties of CM type

In this section, we prove a finiteness theorem for abelian varieties of CM type. The proof relies on the following recent theorem of Tsimerman.

THEOREM 2.1 [Tsi18, Theorem 5.1]. *Let g be a positive integer. There exist constants $b_g, C_g > 0$ such that, for every principally polarised abelian variety A of dimension g defined over a number field k , if \bar{A} is of CM type, then*

$$|\text{disc}(R)| < C_g [k : \mathbb{Q}]^{b_g},$$

where R is the centre of $\text{End}(\bar{A})$.

The exact form of the bound in Theorem 2.1 is not important for our application: it only matters that when we fix k and g , we get a uniform bound for $|\text{disc}(R)|$.

It is straightforward to deduce a finiteness theorem for *absolutely simple* principally polarised abelian varieties of CM type from Theorem 2.1 and classical results.

COROLLARY 2.2. *For all positive integers g and n , there are only finitely many $\bar{\mathbb{Q}}$ -isomorphism classes of absolutely simple principally polarised abelian varieties of dimension g of CM type which can be defined over number fields of degree n .*

Proof. If A is an absolutely simple principally polarised abelian variety of dimension g of CM type, then the endomorphism ring $R = \text{End}(\bar{A})$ is an order in a CM field of degree $2g$. By Theorem 2.1, the discriminant of this ring is bounded in terms of g and n .

It follows from the Hermite–Minkowski theorem that there are only finitely many orders R in number fields with a given value of $\text{disc}(R)$ [Wei67, ch. V, Proposition 7, Corollary 2]. By the classical theory of abelian varieties of CM type, an absolutely simple abelian variety of CM type with endomorphism ring R is determined (up to $\bar{\mathbb{Q}}$ -isomorphism) by a class in the ideal class group of R and a CM type for $R \otimes_{\mathbb{Z}} \mathbb{Q}$ [Mil, 3.11]. Hence there are only finitely many $\bar{\mathbb{Q}}$ -isomorphism classes of absolutely simple abelian varieties of CM type with a given endomorphism ring. \square

To generalise Corollary 2.2 to non-simple abelian varieties one needs to do a little more work because the endomorphism ring may not be commutative so its centre R does not contain enough information to determine the abelian variety up to finite ambiguity. Our proof of this generalisation is based on Shafarevich’s proof for the case of abelian surfaces isogenous to the square of a CM elliptic curve [Sha96].

There is an alternative approach which also proves Proposition 2.4 from Theorem 2.1, using Pila and Tsimerman’s height bound for the preimages of CM points in a fundamental set of \mathcal{H}_g [PT13, Theorem 3.1].¹ The final step, using Zarhin’s trick to deduce Theorem 2.5 from Proposition 2.4, has an interpretation in terms of Shimura varieties which we describe below in § 4.1. However we preferred to give a proof here which does not require the machinery of Shimura varieties.

PROPOSITION 2.3. *Let g and n be positive integers. Consider abelian varieties of dimension g of CM type which are defined over a number field of degree n and have a principal polarisation defined over the same number field. For each g and n , such abelian varieties fall into only finitely many isogeny classes over $\bar{\mathbb{Q}}$.*

¹ This alternative approach was carried out at [PT14, Lemma 7.4].

Proof. Let A be a principally polarised abelian variety of dimension g of CM type defined over a number field of degree n . Then \bar{A} is isogenous to

$$\prod_{i=1}^t A_i^{n_i} \quad (2.1)$$

for some pairwise non-isogenous simple abelian varieties A_1, \dots, A_t over $\bar{\mathbb{Q}}$ of CM type, where n_1, \dots, n_t are positive integers. Then $F_i = \text{End}(A_i) \otimes \mathbb{Q}$ is a CM field. We have

$$\text{End}(\bar{A}) \otimes \mathbb{Q} = \prod_{i=1}^t M_{n_i}(F_i).$$

Let R be the centre of $\text{End}(\bar{A})$. Then R is an order in $\prod_{i=1}^t F_i$. It follows that

$$|\text{disc}(R)| \geq \prod_{i=1}^t |\text{disc}(F_i)|.$$

By Theorem 2.1, $|\text{disc}(R)|$ is bounded by a value depending only on g and n . (This uses the hypothesis that A has a principal polarisation defined over k .) Hence the discriminants $|\text{disc}(F_i)|$ are also bounded. Therefore the Hermite–Minkowski theorem [Wei67, ch. V, Proposition 7, Corollary 2] implies that there are finitely many possible choices for the F_i .

For each possible field F_i there is a bijection between isogeny classes of simple CM abelian varieties over $\bar{\mathbb{Q}}$ with endomorphism algebra F_i and primitive CM types for F_i [Mil, Proposition 3.13]. Each field has finitely many primitive CM types, so we conclude that the simple CM abelian varieties A_i which can appear in (2.1) lie in finitely many $\bar{\mathbb{Q}}$ -isogeny classes. Because the integers n_i and t in (2.1) are bounded by g , this proves that A itself must lie in one of finitely many $\bar{\mathbb{Q}}$ -isogeny classes. \square

PROPOSITION 2.4. *For all positive integers g and n there are only finitely many $\bar{\mathbb{Q}}$ -isomorphism classes of principally polarised abelian varieties of dimension g of CM type defined over number fields of degree n .*

Proof. Let (A, λ) be a principally polarised abelian variety of CM type of dimension g defined over a number field k of degree n .

By Proposition 2.3 there is a finite set \mathcal{S} of principally polarised CM abelian varieties of dimension g defined over number fields of degree n , which contains one representative from each $\bar{\mathbb{Q}}$ -isogeny class of such abelian varieties. Let B be an abelian variety in \mathcal{S} which is $\bar{\mathbb{Q}}$ -isogenous to A . Let K be a common field of definition of A and B . We can choose K such that $[K : \mathbb{Q}] \leq n^2$. The main theorem of [MW93] tells us that there are constants c and κ depending only on g such that there exists an isogeny $f : \bar{B} \rightarrow \bar{A}$ of degree at most

$$c \max(1, h_F(B), [K : \mathbb{Q}], \delta(A), \delta(B))^\kappa.$$

Here h_F denotes the Faltings height and δ denotes the minimum degree of a polarisation of an abelian variety. The value of $h_F(B)$ is bounded because B comes from the finite set \mathcal{S} . Since A and B are principally polarised, we have $\delta(A) = \delta(B) = 1$. We conclude that there is a bound for the degree of f depending only on g and n .

The kernel of f is a subgroup of $B(\bar{\mathbb{Q}})$ of order $\deg(f)$. Since $\deg(f)$ is bounded, there are finitely many possible subgroups. If we know B and $\ker(f)$, then the $\bar{\mathbb{Q}}$ -isomorphism class of A is

determined because $\bar{A} \cong \bar{B}/\ker(f)$. Thus we conclude that there are only finitely many possible $\bar{\mathbb{Q}}$ -isomorphism classes for the abelian variety A .

By [Mil86, Theorem 18.1], each of these abelian varieties has finitely many principal polarisations, up to isomorphisms of polarised abelian varieties. □

THEOREM 2.5. *For all positive integers g and n there are only finitely many $\bar{\mathbb{Q}}$ -isomorphism classes of abelian varieties of dimension g of CM type defined over number fields of degree n .*

Proof. Let A be an abelian variety of CM type of dimension g defined over a number field k of degree n . According to [Zar85, § 5.3], $(A \times A^\vee)^4$ has a principal polarisation over k . Hence by Proposition 2.4, there are only finitely many possible isomorphism classes for $(\bar{A} \times \bar{A}^\vee)^4$. By [Mil86, Theorem 18.7], $(\bar{A} \times \bar{A}^\vee)^4$ has finitely many direct factors up to isomorphism, which proves that there are finitely many possibilities for the $\bar{\mathbb{Q}}$ -isomorphism class of A . □

3. Shimura varieties

The purposes of this section are both to assist the reader who is not familiar with the theory of Shimura varieties and to set out the notation and terminology.

3.1 Definition of Shimura variety components

For the purposes of this paper, we do not need to worry about multiple connected components of a Shimura variety or about their exact field of definition. We shall therefore omit the complexities of the adelic definition of Shimura varieties. We simply define a ‘Shimura variety component’, which is a geometrically connected component of the canonical model of a Shimura variety. One can describe the complex points of a Shimura variety component as follows.

A *Shimura datum* is a pair (\mathbf{G}, X) where \mathbf{G} is a connected reductive \mathbb{Q} -algebraic group and X is a $\mathbf{G}(\mathbb{R})$ -conjugacy class in $\text{Hom}(\mathbb{S}, \mathbf{G}_{\mathbb{R}})$ satisfying [Del79, Axioms 2.1.1.1–2.1.1.3]. Here \mathbb{S} denotes the Deligne torus $\text{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_m$. These axioms imply that X is a finite disjoint union of Hermitian symmetric domains [Del79, Corollaire 1.1.17].

Since we only wish to define connected components of Shimura varieties, we choose a connected component $X^+ \subset X$. Let $\mathbf{G}(\mathbb{Q})_+$ denote the stabiliser of X^+ in $\mathbf{G}(\mathbb{Q})$. In order to define congruence subgroups of $\mathbf{G}(\mathbb{Q})_+$, pick a representation of \mathbb{Q} -algebraic groups $\rho: \mathbf{G} \rightarrow \mathbf{GL}(V_{\mathbb{Q}})$ and a lattice $V_{\mathbb{Z}} \subset V_{\mathbb{Q}}$. For each positive integer N , let

$$\Gamma(V_{\mathbb{Z}}, N) = \ker(\text{GL}(V_{\mathbb{Z}}) \rightarrow \text{GL}(V_{\mathbb{Z}}/NV_{\mathbb{Z}})).$$

A *congruence subgroup* of $\mathbf{G}(\mathbb{Q})_+$ is defined to be a subgroup of $\mathbf{G}(\mathbb{Q})_+$ which contains the intersection of $\mathbf{G}(\mathbb{Q})_+$ with $\rho^{-1}(\Gamma(V_{\mathbb{Z}}, N))$ as a finite index subgroup for some N . By [Mil05, Proposition 4.1], this is equivalent to defining a congruence subgroup of $\mathbf{G}(\mathbb{Q})_+$ to be the intersection of $\mathbf{G}(\mathbb{Q})_+$ with a compact open subgroup of $\mathbf{G}(\mathbb{A}^f)$. Hence the definition of congruence subgroups is independent of the choice of ρ and $V_{\mathbb{Z}}$.

If Γ is a congruence subgroup of $\mathbf{G}(\mathbb{Q})_+$, then the quotient space $S_{\mathbb{C}} = \Gamma \backslash X^+$ has a canonical structure as a quasi-projective variety over \mathbb{C} , by [BB66]. This variety $S_{\mathbb{C}}$ is a connected component of a Shimura variety $M_{\mathbb{C}}$.

According to Deligne’s theory of canonical models ([Del79], completed in [Mil83] and [Bor83/84]), the Shimura variety $M_{\mathbb{C}}$ has a canonical model over a number field. Hence the connected component $S_{\mathbb{C}}$ also has a model over a number field. The field of definition of the canonical model of the disconnected Shimura variety is the reflex field E of (\mathbf{G}, X) . The field

of definition for the model of $S_{\mathbb{C}}$ is an extension of the reflex field determined by the action of $\text{Gal}(\overline{E}/E)$ on components of $M_{\mathbb{C}}$.

We use the phrase *Shimura variety component* to mean a variety over a number field whose extension to \mathbb{C} is of the form $\Gamma \backslash X^+$ and whose structure over a number field comes from the theory of canonical models, as described above.

A *morphism of Shimura data* $f: (\mathbf{G}, X) \rightarrow (\mathbf{H}, Y)$ is a homomorphism of algebraic groups $f: \mathbf{G} \rightarrow \mathbf{H}$ such that composition with f induces a map $f_*: X \rightarrow Y$. If $f_*(X^+) \subset Y^+$ and we have congruence subgroups $\Gamma \subset \mathbf{G}(\mathbb{Q})_+$ and $\Gamma_{\mathbf{H}} \subset \mathbf{H}(\mathbb{Q})_+$ such that $f(\Gamma) \subset \Gamma_{\mathbf{H}}$, then f induces a morphism of algebraic varieties

$$[f]: \Gamma \backslash X^+ \rightarrow \Gamma_{\mathbf{H}} \backslash Y^+.$$

The morphism $[f]$ is defined over the compositum of the natural fields of definition of the Shimura variety components $\Gamma \backslash X^+$ and $\Gamma_{\mathbf{H}} \backslash Y^+$.

3.2 Shimura varieties of Hodge and of abelian type

A fundamental example of a Shimura variety component is \mathcal{A}_g , the coarse moduli space of principally polarised abelian varieties of dimension g . This arises from the Shimura datum $(\mathbf{GSp}_{2g}, \mathcal{H}_g^{\pm})$, where \mathcal{H}_g^{\pm} denotes a certain conjugacy class in $\text{Hom}(\mathbb{S}, \mathbf{GSp}_{2g, \mathbb{R}})$. Using period matrices, there is a natural identification between \mathcal{H}_g^{\pm} and the union of the upper and lower Siegel half-spaces. The complex points of \mathcal{A}_g are obtained as the quotient of a connected component $\mathcal{H}_g \subset \mathcal{H}_g^{\pm}$ by the congruence subgroup $\text{Sp}_{2g}(\mathbb{Z})$. The canonical model of \mathcal{A}_g is defined over \mathbb{Q} , and the model of \mathcal{A}_g over \mathbb{Q} which comes from the theory of Shimura varieties is the same as the model over \mathbb{Q} which comes from the interpretation as a moduli space (Deligne's definition of canonical models of Shimura varieties was motivated by this case).

A Shimura datum (\mathbf{G}, X) is said to be *of Hodge type* if there exists a morphism of Shimura data

$$i: (\mathbf{G}, X) \rightarrow (\mathbf{GSp}_{2g}, \mathcal{H}_g^{\pm})$$

such that the underlying homomorphism of algebraic groups $\mathbf{G} \rightarrow \mathbf{GSp}_{2g}$ is injective. If $\Gamma \subset \mathbf{G}(\mathbb{Q})_+$ is a congruence subgroup such that $i(\Gamma) \subset \text{Sp}_{2g}(\mathbb{Z})$, then the induced morphism of Shimura variety components

$$[i]: \Gamma \backslash X^+ \rightarrow \mathcal{A}_g$$

is finite by [Pin90, Proposition 3.8(a)]. Shimura varieties of Hodge type can be described as moduli spaces of abelian varieties with prescribed Hodge classes (for example, polarisations and endomorphisms) and a level structure.

A Shimura datum (\mathbf{H}, Y) is said to be *of abelian type* if there exists a Shimura datum (\mathbf{G}, X) of Hodge type and a morphism of Shimura data

$$p: (\mathbf{G}, X) \rightarrow (\mathbf{H}, Y)$$

such that the underlying homomorphism of algebraic groups is surjective and has kernel contained in the centre of \mathbf{G} . If $\Gamma \subset \mathbf{G}(\mathbb{Q})_+$ and $\Gamma_{\mathbf{H}} \subset \mathbf{H}(\mathbb{Q})_+$ are congruence subgroups such that $p(\Gamma) \subset \Gamma_{\mathbf{H}}$, then the resulting morphism of Shimura varieties

$$[p]: \Gamma \backslash X^+ \rightarrow \Gamma_{\mathbf{H}} \backslash Y^+$$

is finite and surjective by [Pin05, Facts 2.6].

3.3 Orthogonal Shimura varieties

Moduli spaces of K3 surfaces are closely related to Shimura varieties of abelian type associated with orthogonal groups.

Let Λ be a *lattice*, that is, a finitely generated free \mathbb{Z} -module equipped with a non-degenerate symmetric bilinear form $\psi: \Lambda \times \Lambda \rightarrow \mathbb{Z}$. For any ring R , we shall write $\Lambda_R = \Lambda \otimes_{\mathbb{Z}} R$. The *orthogonal group* $O(\Lambda)$ is the group of automorphisms of Λ which preserve the bilinear form ψ . We write $O(\Lambda)_{\mathbb{Q}}$ for the \mathbb{Q} -algebraic group whose functor of points is given by

$$O(\Lambda)_{\mathbb{Q}}(R) = O(\Lambda_R).$$

Let $SO(\Lambda) \subset O(\Lambda)$ be the subgroup of automorphisms with determinant $+1$. The \mathbb{Q} -algebraic group $SO(\Lambda)_{\mathbb{Q}}$ is defined in the obvious fashion. Observe that $SO(\Lambda)_{\mathbb{Q}}$ is geometrically connected and absolutely almost simple as an algebraic group.

Each homomorphism $h: \mathbb{S} \rightarrow SO(\Lambda)_{\mathbb{R}}$ induces a \mathbb{Z} -Hodge structure Λ_h with underlying \mathbb{Z} -module Λ . Therefore it makes sense to define Ω_{Λ} to be the set of $h \in \text{Hom}(\mathbb{S}, SO(\Lambda)_{\mathbb{R}})$ which satisfy:

- (1) $\dim \Lambda_h^{-1,1} = \dim \Lambda_h^{1,-1} = 1$ and $\dim \Lambda_h^{0,0} = \text{rk } \Lambda - 2$;
- (2) for every non-zero $v \in \Lambda_h^{1,-1}$ we have $\psi(v, v) = 0$ and $\psi(v, \bar{v}) > 0$;
- (3) $\psi(\Lambda_h^{0,0}, \Lambda_h^{1,-1}) = 0$.

By [Huy16, Proposition 6.1.2], the map $h \mapsto \Lambda_h^{-1,1}$ is a bijection between Ω_{Λ} and

$$\{[v] \in \mathbb{P}(\Lambda_{\mathbb{C}}) : \psi(v, v) = 0, \psi(v, \bar{v}) > 0\}.$$

Note that we have shifted the labelling of the Hodge structures to have type $\{(1, -1), (0, 0), (-1, 1)\}$ instead of type $\{(2, 0), (1, 1), (0, 2)\}$ as in [Huy16, Proposition 6.1.2]. This is necessary to ensure that the associated homomorphisms $\mathbb{S} \rightarrow GL(\Lambda)_{\mathbb{R}}$ factor through $SO(\Lambda)_{\mathbb{R}}$.

If Λ has signature $(2, n)$, then $(SO(\Lambda)_{\mathbb{Q}}, \Omega_{\Lambda})$ forms a Shimura datum. One can use the Kuga–Satake construction to show that this is a Shimura datum of abelian type: $(SO(\Lambda)_{\mathbb{Q}}, \Omega_{\Lambda})$ can be covered by a Shimura datum associated with the group $GSpin(\Lambda)_{\mathbb{Q}}$, and this embeds into the Shimura datum $(GSp_{2g}, \mathcal{H}_g^{\pm})$ where $g = 2^n = 2^{\text{rk } \Lambda - 2}$ (for more details, see [Riz10, § 5.5]).

For future use we note that if Λ has signature $(2, n)$, then Ω_{Λ} has two connected components interchanged by complex conjugation.

3.4 Finiteness theorem for CM points

We recall the definition of CM points in a Shimura variety component. Let (\mathbf{G}, X) be a Shimura datum and let S be a Shimura variety component whose \mathbb{C} -points are $\Gamma \backslash X^+$. A point $s \in S(\mathbb{C})$ is said to be a *CM point* if it is the image (under $X^+ \rightarrow \Gamma \backslash X^+$) of a homomorphism $h \in X^+$ which factors through $\mathbf{T}_{\mathbb{R}}$ for some \mathbb{Q} -torus $\mathbf{T} \subset \mathbf{G}$. This terminology is used because CM points in \mathcal{A}_g are precisely the points which correspond to abelian varieties of CM type. It is part of the definition of canonical models of Shimura varieties that CM points are defined over number fields.

PROPOSITION 3.1. *Let S be a Shimura variety component of abelian type. For every positive integer n the set of CM points in S defined over number fields of degree n is finite.*

Proof. Proposition 2.4 is precisely the statement of this proposition for \mathcal{A}_g . We will use this to prove the proposition for other cases.

Consider a Shimura variety component S of abelian type. Let (\mathbf{H}, Y) be the associated Shimura datum. By the definition of Shimura data of abelian type, there exists a Shimura datum (\mathbf{G}, X) of Hodge type and a morphism of Shimura data

$$p: (\mathbf{G}, X) \rightarrow (\mathbf{H}, Y)$$

such that the underlying homomorphism of algebraic groups is surjective and has kernel contained in the centre of \mathbf{G} . Because (\mathbf{G}, X) is of Hodge type, there is an injective morphism of Shimura data

$$i: (\mathbf{G}, X) \rightarrow (\mathbf{GSp}_{2g}, \mathcal{H}_g^\pm)$$

for some positive integer g .

From the definition of Shimura variety components, $S(\mathbb{C}) = \Gamma_{\mathbf{H}} \backslash Y^+$ for some connected component $Y^+ \subset Y$ and some congruence subgroup $\Gamma_{\mathbf{H}} \subset \mathbf{H}(\mathbb{Q})_+$. Let X^+ be a connected component of X which maps onto Y^+ .

By [Mar91, Lemma I.3.1.1(ii)] there exist congruence subgroups $\Gamma_1, \Gamma_2 \subset \mathbf{G}(\mathbb{Q})_+$ such that

$$i(\Gamma_1) \subset \mathrm{Sp}_{2g}(\mathbb{Z}) \quad \text{and} \quad p(\Gamma_2) \subset \Gamma_{\mathbf{H}}.$$

Let $\Gamma = \Gamma_1 \cap \Gamma_2$. Let S' be the Shimura variety component whose \mathbb{C} -points are $\Gamma \backslash X^+$. As discussed in §3.2, p induces a finite surjective morphism $[p]: S' \rightarrow S$. Let d be the degree of $[p]$. Because the kernel of p is contained in the centre of \mathbf{G} , a point $s \in S'(\mathbb{C})$ is a CM point if and only if $[p](s)$ is a CM point of S . Hence every CM point of S defined over a number field of degree n has a preimage in S' which is a CM point defined over a number field of degree dn . Thus it suffices to show that S' has finitely many CM points defined over number fields of degree dn .

As discussed in §3.2, i induces a finite morphism $[i]: S' \rightarrow \mathcal{A}_g$. Any morphism of Shimura variety components induced by a morphism of Shimura data maps CM points to CM points. Because the proposition holds for \mathcal{A}_g and because $[i]$ is finite, S' has finitely many CM points defined over number fields of degree dn . This completes the proof of the proposition. \square

4. Finiteness theorem for K3 surfaces of CM type

In this section, we prove our finiteness theorem for K3 surfaces of CM type defined over number fields.

THEOREM 4.1. *For each positive integer n there are only finitely many $\overline{\mathbb{Q}}$ -isomorphism classes of K3 surfaces of CM type defined over number fields of degree n .*

We prove Theorem 4.1 by using orthogonal Shimura varieties. Before discussing the proof further, we recall the definition of a polarisation of a K3 surface. Let X be a K3 surface over a perfect field k . A *polarisation* of X is a k -point of the relative Picard scheme $\mathrm{Pic}_{X/k}$ (equivalently, by [Huy16, Proposition 1.2.4], an element of $\mathrm{NS}(\overline{X})^{\mathrm{Gal}(\overline{k}/k)}$) which over \overline{k} is the class of a primitive ample line bundle on \overline{X} .

For each positive integer d there is a coarse moduli space over \mathbb{Q} of polarised K3 surfaces of degree $2d$, which is a quasi-projective variety M_{2d} defined over \mathbb{Q} [Huy16, ch. 5]. There is a degree-2 covering $\tilde{M}_{2d} \rightarrow M_{2d}$ such that \tilde{M}_{2d} is a Zariski open subset of an orthogonal Shimura variety component S_{2d} . The K3 surfaces of CM type are precisely those which correspond to CM points in S_{2d} . The Shimura variety component S_{2d} is of abelian type, and therefore Proposition 3.1 tells

us that each S_{2d} contains finitely many CM points defined over number fields of degree n . This proves Theorem 4.1 if we restrict to K3 surfaces with a polarisation of degree $2d$.

However Theorem 4.1 does not require an *a priori* restriction on the degree of polarisation of the K3 surfaces involved. Indeed, Theorem 4.1 implies that there is a bound $d(n)$ such that every CM K3 surface defined over a number field of degree n possesses a polarisation (over $\overline{\mathbb{Q}}$) of degree at most $d(n)$.

In order to remove the dependence on the degree of the polarisation, we use Nikulin's results on lattices to construct a Shimura variety component $S_{\#}$ associated with an orthogonal group of greater rank, such that there is a finite map $S_{2d} \rightarrow S_{\#}$ for every d . The idea of constructing such an $S_{\#}$ was suggested to the authors by François Charles, who used a similar method in [Cha16]. Our construction of $S_{\#}$ differs from the construction in [Cha16] because we require S_{2d} to map into $S_{\#}$ for every positive integer d , while [Cha16] requires this only for an infinite set of values of d . On the other hand, the Shimura variety constructed in [Cha16] has an interpretation as a moduli space of irreducible holomorphic symplectic varieties whereas our $S_{\#}$ does not appear to have a natural interpretation as a moduli space of geometric objects.

The Shimura variety component $S_{\#}$ is again of abelian type, and hence there are finitely many CM points in $S_{\#}$ defined over number fields of given degree. This is not sufficient to prove Theorem 4.1, because a single point in $S_{\#}$ might be in the image of S_{2d} for infinitely many values of d . We shall use some calculations with Hodge structures to show that whenever different K3 surfaces correspond to the same point in $S_{\#}$, they must have isometric transcendental lattices. Finally a result of Bridgeland and Maciocia [BM01] allows us to conclude that each point in $S_{\#}$ can only come from finitely many K3 surfaces, completing the proof of Theorem 4.1.

4.1 Comparison with the case of abelian varieties

The structure of the proof of Theorem 4.1, for K3 surfaces, can be compared with the proof of Theorem 2.5, for abelian varieties. In both cases, we can use Proposition 3.1 to easily deduce that there are finitely many $\overline{\mathbb{Q}}$ -isomorphism classes of the appropriate object equipped with a polarisation of given degree.

In the abelian varieties case, in order to get a finiteness statement without restricting the degree of a polarisation, we used Zarhin's trick (Theorem 2.5). This can be described in terms of Shimura varieties as follows.

Define a *polarisation type* to be a sequence of g positive integers (d_1, \dots, d_g) such that d_i divides d_{i+1} for each i . For any polarised abelian variety (A, λ) , the elementary divisors of the associated symplectic form on $H_1(A(\mathbb{C}), \mathbb{Z})$ form a polarisation type.

For each polarisation type $D = (d_1, \dots, d_g)$, let $\mathcal{A}_{g,D}$ denote the moduli space of abelian varieties of dimension g with a polarisation of type D . This is a Shimura variety component. Zarhin's trick [Zar85, § 5.3] can be interpreted as constructing a morphism of Shimura variety components $f_{g,D}: \mathcal{A}_{g,D} \rightarrow \mathcal{A}_{8g}$. (Note that Zarhin's trick involves the choice of an integer quaternion of norm $\prod_{i=1}^g d_i$. We can make this choice once for each D , thus ensuring that Zarhin's construction of a principal polarisation on $(A \times A^{\vee})^4$ is sufficiently functorial to give us a morphism $f_{g,D}$ for each D . Due to this choice, the morphisms $f_{g,D}$ are not unique.)

Thus our construction of morphisms from S_{2d} to a single Shimura variety component $S_{\#}$ is analogous to Zarhin's trick. The proof of Theorem 2.5 (from Proposition 2.4) plays the same role for the abelian varieties case as Lemma 4.3 and Proposition 4.4 do for the K3 case.

4.2 Moduli spaces of polarised K3 surfaces

We define a Shimura variety component S_{2d} as follows. Let Λ_{2d} denote the lattice

$$\Lambda_{2d} = E_8(-1)^{\oplus 2} \oplus U^{\oplus 2} \oplus \langle -2d \rangle.$$

The significance of this lattice is that if X is a K3 surface over \mathbb{C} and $\lambda \in H^2(X, \mathbb{Z}(1))$ is a polarisation of degree $2d$, then the orthogonal complement $\lambda^\perp \subset H^2(X, \mathbb{Z}(1))$ is a lattice isomorphic to Λ_{2d} (where $H^2(X, \mathbb{Z}(1))$ is equipped with the intersection pairing), see [Huy16, Example 14.1.11]. The lattice Λ_{2d} has signature $(2, 19)$, so it gives rise to an orthogonal Shimura datum $(\mathbf{SO}(\Lambda_{2d})_{\mathbb{Q}}, \Omega_{2d})$.

For any lattice Λ , the *dual lattice* is

$$\Lambda^\vee = \{v \in \Lambda_{\mathbb{Q}} : \psi(v, \Lambda) \subset \mathbb{Z}\}.$$

The *discriminant group* A_Λ is the quotient Λ^\vee/Λ . This is a finite abelian group. The orthogonal group $\mathbf{O}(\Lambda)$ acts on A_Λ . We define $\tilde{\mathbf{O}}(\Lambda) \subset \mathbf{O}(\Lambda)$ to be the kernel of the action on the discriminant group.

We write Ω_{2d} for the period domain $\Omega_{\Lambda_{2d}}$, as defined in §3.3. Pick a connected component $\Omega_{2d}^+ \subset \Omega_{2d}$. Let

$$\tilde{\mathbf{SO}}(\Lambda_{2d})_+ = \mathbf{SO}(\Lambda_{2d})_+ \cap \tilde{\mathbf{O}}(\Lambda_{2d}).$$

Let S_{2d} be the Shimura variety component whose complex points are given by $\tilde{\mathbf{SO}}(\Lambda_{2d})_+ \backslash \Omega_{2d}^+$. Knowing exactly which congruence subgroup is used to define S_{2d} will be important at the end of §4.3. As discussed in [MP15, §3.1], this Shimura variety component is defined over \mathbb{Q} .

As described in [Huy16, Corollary 6.4.3], the moduli space $M_{2d, \mathbb{C}}$ can be embedded as a Zariski open subset of the quotient $\tilde{\mathbf{O}}(\Lambda_{2d}) \backslash \Omega_{2d}$. This is a quotient of a Hermitian symmetric domain by an arithmetic group, and therefore is very similar to a Shimura variety. However the reductive group used in the definition of a Shimura datum is required to be connected and therefore we must use $\mathbf{SO}(\Lambda_{2d})_{\mathbb{Q}}$ rather than $\mathbf{O}(\Lambda_{2d})_{\mathbb{Q}}$. Hence $M_{2d, \mathbb{C}}$ does not itself embed in a Shimura variety component but rather we must use a double cover $\tilde{M}_{2d} \rightarrow M_{2d}$ (corresponding to the fact that $\tilde{\mathbf{SO}}(\Lambda_{2d})$ is an index-2 subgroup of $\tilde{\mathbf{O}}(\Lambda_{2d})$).

The double cover \tilde{M}_{2d} is defined to be the coarse moduli space of triples (X, λ, u) where X is a K3 surface, λ is a polarisation of X of degree $2d$ and u is an isometry

$$\det(P^2(X, \mathbb{Z}_2(1))) \rightarrow \det(\Lambda_{2d} \otimes_{\mathbb{Z}} \mathbb{Z}_2).$$

Here $P^2(X, \mathbb{Z}_2(1))$ denotes the orthogonal complement of the image of λ in the 2-adic cohomology $H^2(X, \mathbb{Z}_2(1))$.

There is an embedding of algebraic varieties $\tilde{M}_{2d} \rightarrow S_{2d}$ which realises \tilde{M}_{2d} as a Zariski open subset of S_{2d} . The fact that this embedding is defined over \mathbb{Q} was essentially first proved by Rizov ([Riz10, Theorem 3.9.1]; see also [MP15, Corollary 4.4]).

4.3 Construction of $S_{\#}$

We now construct a Shimura variety component $S_{\#}$ such that there is a finite morphism $f_{2d}: S_{2d} \rightarrow S_{\#}$ for every $d \in \mathbb{N}$. We do this by producing a lattice $\Lambda_{\#}$ with a primitive embedding $\Lambda_{2d} \rightarrow \Lambda_{\#}$ for every $d \in \mathbb{N}$, using the following theorem of Nikulin.²

² A simpler construction based on Lagrange's theorem was more recently used by Yiwei She in [She17, Lemma 3.3.1].

THEOREM 4.2 [Nik79, Corollary 1.12.3]. *Let Λ be an even lattice of signature (m_+, m_-) with discriminant group A_Λ . Let $s(A_\Lambda)$ be the minimum size of a generating set for A_Λ . There exists a primitive embedding of Λ into an even unimodular lattice of signature (n_+, n_-) if the following conditions are simultaneously satisfied:*

- (i) $n_+ - n_- \equiv 0 \pmod{8}$;
- (ii) $n_+ \geq m_+, n_- \geq m_-$;
- (iii) $(n_+ + n_-) - (m_+ + m_-) \geq s(A_\Lambda)$.

In the case $\Lambda = \Lambda_{2d}$ the discriminant group is $\mathbb{Z}/2d\mathbb{Z}$ and so $s(A_{\Lambda_{2d}}) = 1$. Thus in order to apply this theorem to obtain embeddings $\Lambda_{2d} \hookrightarrow \Lambda_\#$, the signature (n_+, n_-) of $\Lambda_\#$ must satisfy

$$n_+ - n_- \equiv 0 \pmod{8}; \quad n_+ \geq 2, n_- \geq 19; \quad n_+ + n_- \geq 22.$$

Furthermore, in order for $\Lambda_\#$ to give rise to an orthogonal Shimura variety component, we must have $n_+ = 2$. The conditions therefore reduce to $n_- \equiv 2 \pmod{8}$ and $n_- \geq 20$, so we can choose $n_- = 26$ to satisfy them.

According to [Ser70, ch. V, Theorem 5], there is a unique even unimodular lattice of signature $(2, 26)$, namely

$$\Lambda_\# = E_8(-1)^{\oplus 3} \oplus U^{\oplus 2}.$$

Since $\Lambda_\#$ is unimodular, we have $\mathrm{SO}(\Lambda_\#) = \widetilde{\mathrm{SO}}(\Lambda_\#)$. By Theorem 4.2, there is a primitive embedding $\iota_{2d}: \Lambda_{2d} \hookrightarrow \Lambda_\#$ for every $d \in \mathbb{N}$. These embeddings are not unique, we simply pick one for each d .

The embedding ι_{2d} induces an injection of special orthogonal groups over \mathbb{Q}

$$r_{2d}: \mathbf{SO}(\Lambda_{2d})_{\mathbb{Q}} \rightarrow \mathbf{SO}(\Lambda_\#)_{\mathbb{Q}},$$

extending isometries of $\Lambda_{2d, \mathbb{Q}}$ to $\Lambda_{\#, \mathbb{Q}}$ by letting them act trivially on the orthogonal complement of $\Lambda_{2d, \mathbb{Q}}$. Hence we get an injective morphism of Shimura data

$$(\mathbf{SO}(\Lambda_{2d})_{\mathbb{Q}}, \Omega_{2d}) \rightarrow (\mathbf{SO}(\Lambda_\#)_{\mathbb{Q}}, \Omega_\#),$$

where we write $\Omega_\# = \Omega_{\Lambda_\#}$. The map $\Omega_{2d} \rightarrow \Omega_\#$ commutes with complex conjugation, hence it induces a bijection $\pi_0(\Omega_{2d}) \xrightarrow{\sim} \pi_0(\Omega_\#)$. Thus we can choose Ω_{2d}^+ for each d in such a way that the images of all Ω_{2d}^+ in $\Omega_\#$ are contained in the same connected component. Let us call this component $\Omega_\#^+$.

Let $S_\#$ be the Shimura variety component whose \mathbb{C} -points are $\mathrm{SO}(\Lambda_\#)_+ \backslash \Omega_\#^+$.

According to [Huy16, Proposition 14.2.6], r_{2d} maps the congruence subgroup $\widetilde{\mathrm{SO}}(\Lambda_{2d})$ into $\mathrm{SO}(\Lambda_\#)$. Hence r_{2d} induces a morphism of Shimura variety components

$$f_{2d}: S_{2d} \rightarrow S_\#.$$

Because r_{2d} is an injective homomorphism of algebraic groups, f_{2d} is finite by [Pin90, Proposition 3.8(a)].

Note that r_{2d} does not map $\mathrm{SO}(\Lambda_{2d})$ into $\mathrm{SO}(\Lambda_\#)$, because the embedding $\Lambda_{2d} \hookrightarrow \Lambda_\#$ is not split over \mathbb{Z} . Thus it is important to us exactly which congruence subgroup is used in defining the Shimura variety component S_{2d} (namely $\widetilde{\mathrm{SO}}(\Lambda_{2d})_+$), and that r_{2d} maps this subgroup into $\mathrm{SO}(\Lambda_\#)_+$, so that we can deduce that the Shimura variety components S_{2d} (and not just covers of them) map into $S_\#$.

4.4 The transcendental lattice and $S_{\#}$

By Proposition 3.1, $S_{\#}$ has finitely many CM points defined over number fields of given degree n . However this is not enough to establish that

$$\bigcup_{d \in \mathbb{N}} \{x \in S_{2d} : x \text{ is a CM point and } x \text{ is defined over a number field of degree } n\}$$

is finite, because a single point of $S_{\#}$ might lie in the image of infinitely many S_{2d} .

Indeed, the union described above is infinite. To see this, consider any K3 surface X of CM type. Then $\text{NS}(\bar{X})$ has even rank and in particular has rank at least two. Pick two linearly independent polarisations λ_1, λ_2 of \bar{X} and choose a number field k over which both polarisations are defined. Now infinitely many integer combinations of these polarisations will be primitive elements of $\text{NS}(X_k)$. It follows that X_k has polarisations of arbitrarily large degree, giving rise to CM points in infinitely many varieties S_{2d} all defined over the same number field.

If X is a K3 surface over \mathbb{C} , its *transcendental lattice* $T(X)$ is the orthogonal complement of the Néron–Severi lattice $\text{NS}(X)$ in $H^2(X, \mathbb{Z}(1))$.

We shall show that if two polarised K3 surfaces give rise to the same point in $S_{\#}$, then their transcendental lattices are Hodge isometric, that is, isomorphic in the category of \mathbb{Z} -Hodge structures with a quadratic form. A result of Bridgeland and Maciocia [BM01] says that only finitely many K3 surfaces can have transcendental lattices in a given Hodge isometry class. Thus the points in all the moduli spaces \tilde{M}_{2d} which map to a single point in $S_{\#}$ can only be associated with finitely many isomorphism classes of complex K3 surfaces (forgetting the polarisations).

LEMMA 4.3. *Let (X, λ) and (X', λ') be polarised K3 surfaces over \mathbb{C} of degrees $2d$ and $2d'$ respectively. Let $x \in \tilde{M}_{2d}(\mathbb{C})$ and $x' \in \tilde{M}_{2d'}(\mathbb{C})$ be points whose images in the moduli spaces $M_{2d}(\mathbb{C})$ and $M_{2d'}(\mathbb{C})$ correspond to (X, λ) and (X', λ') , respectively. If $f_{2d}(x) = f_{2d'}(x')$ in $S_{\#}(\mathbb{C})$, then the transcendental lattices $T(X)$ and $T(X')$ are Hodge isometric.*

Proof. Choose points $\tilde{x} \in \Omega_{2d}^+$ and $\tilde{x}' \in \Omega_{2d'}^+$ which lift x and x' respectively. Let

$$\tilde{s} = (r_{2d})_*(\tilde{x}) \in \Omega_{\#}^+, \quad \tilde{s}' = (r_{2d'})_*(\tilde{x}') \in \Omega_{\#}^+.$$

The points $\tilde{x}, \tilde{x}', \tilde{s}$ and \tilde{s}' induce \mathbb{Z} -Hodge structures $H_x, H_{x'}, H_s$ and $H_{s'}$ which have underlying \mathbb{Z} -modules $\Lambda_{2d}, \Lambda_{2d'}, \Lambda_{\#}$ and $\Lambda_{\#}$, respectively.

By [Zar83, Theorem 1.4.1], $T(X)_{\mathbb{Q}}$ is an irreducible \mathbb{Q} -Hodge structure. By construction, $T(X)$ is a primitive lattice in $P^2(X, \mathbb{Z}(1))$. Consequently, if we choose an isometry of \mathbb{Z} -Hodge structures $P^2(X, \mathbb{Z}(1)) \cong H_x$, it will identify $T(X)$ with the smallest primitive sub- \mathbb{Z} -Hodge structure of H_x whose complexification contains $H_x^{-1,1}$.

Via the primitive embedding $\iota_{2d}: \Lambda_{2d} \rightarrow \Lambda_{\#}$, we can view H_x as a sub-Hodge structure of H_s . Because H_x and H_s are Hodge structures coming from orthogonal Shimura data, their $(-1, 1)$ -parts have dimension 1, and so $H_x^{-1,1} = H_s^{-1,1}$. Hence $T(X)$ is isometric to the smallest primitive sub- \mathbb{Z} -Hodge structure of H_s whose complexification contains $H_s^{-1,1}$.

Similarly, $T(X')$ is isometric to the smallest primitive sub- \mathbb{Z} -Hodge structure of $H_{s'}$ whose complexification contains $H_{s'}^{-1,1}$.

Because $f_{2d}(x) = f_{2d'}(x')$, there is a $\gamma \in \text{SO}(\Lambda_{\#})_+$ such that $\tilde{s}' = \gamma\tilde{s}$. It induces a Hodge isometry $H_s \rightarrow H_{s'}$. Therefore $T(X)$ is Hodge isometric to $T(X')$. \square

The following proposition is stated in [BM01] in the form ‘a K3 surface over \mathbb{C} has only finitely many Fourier–Mukai partners.’ It was shown by Orlov [Orl97, Theorem 3.3] that K3 surfaces

are Fourier–Mukai partners if and only if they have Hodge isometric transcendental lattices. In fact, the proof of the proposition in [BM01] is entirely in terms of lattices, and therefore we do not really need the notion of Fourier–Mukai partners at all. Since the proof is short and relies on similar techniques to the manipulations of lattices used elsewhere in this paper, we have reproduced a version of it here.

PROPOSITION 4.4 [BM01, Proposition 5.3]. *Let T be a \mathbb{Z} -Hodge structure with a quadratic form. There are finitely many isomorphism classes of K3 surfaces X over \mathbb{C} for which the transcendental lattice $T(X)$ is Hodge isometric to T .*

Proof. The Néron–Severi group $\text{NS}(X)$ is the orthogonal complement of $T(X)$ in $H^2(X, \mathbb{Z}(1))$, which is an even unimodular lattice. Thus the discriminant groups of $T(X)$ and $\text{NS}(X)$ are canonically isomorphic, whereas their discriminant forms differ by a sign [Nik79, Corollary 1.6.2]. According to [Cas78, ch. 9, Theorem 1.1] there are only finitely many isometry classes of lattices with given rank and discriminant. Hence there are finitely many choices for $\text{NS}(X)$.

We know that

$$T(X) \oplus \text{NS}(X) \subset H^2(X, \mathbb{Z}(1)) \subset (T(X) \oplus \text{NS}(X))^\vee.$$

The index of $T(X) \oplus \text{NS}(X)$ in $(T(X) \oplus \text{NS}(X))^\vee$ is finite. Hence for each possible isometry class of $\text{NS}(X)$, there are finitely many possibilities for $H^2(X, \mathbb{Z}(1))$ as a subgroup of the group $(T(X) \oplus \text{NS}(X))^\vee$.

Because $\text{NS}(X)$ is purely of Hodge type $(0, 0)$, the Hodge structure on $T(X)$ determines the Hodge structure on $(T(X) \oplus \text{NS}(X))^\vee$ and hence on $H^2(X, \mathbb{Z}(1))$.

Hence if we fix $T(X)$ up to Hodge isometry, then there are finitely many possibilities for $H^2(X, \mathbb{Z}(1))$ up to Hodge isometry. Finally, by the global Torelli theorem for unpolarised K3 surfaces ([Huy16, Theorem 7.5.3], building on [PS71]), the Hodge isometry class of $H^2(X, \mathbb{Z}(1))$ determines X . \square

To complete the proof of Theorem 4.1, note that if X is a K3 surface defined over a number field k of degree n , then X has a polarisation of some degree $2d$ defined over k and therefore gives rise to a point in $M_{2d}(k)$. We can lift this to a point $x \in \tilde{M}_{2d}(\overline{\mathbb{Q}})$ defined over a number field of degree $2n$. If X is a K3 surface of CM type, then x is a CM point in \tilde{M}_{2d} . It follows that $f_{2d}(x)$ is a CM point in $S_\#$ defined over a number field of degree $2n$. By Proposition 3.1, there are finitely many such CM points in $S_\#$. Combining Lemma 4.3 and Proposition 4.4 we see that each point in $S_\#$ comes from only finitely many $\overline{\mathbb{Q}}$ -isomorphism classes of K3 surfaces.

5. Brauer groups of forms and the Mumford–Tate conjecture

5.1 Mumford–Tate conjecture

Let X be a smooth, projective and geometrically integral variety over a field k that is finitely generated over \mathbb{Q} . We choose an embedding $k \hookrightarrow \mathbb{C}$ and define H as the quotient of $H^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ by its torsion subgroup. We write $H_{\mathbb{Q}} = H \otimes_{\mathbb{Z}} \mathbb{Q}$, $H_{\mathbb{R}} = H \otimes_{\mathbb{Z}} \mathbb{R}$, $H_{\mathbb{C}} = H \otimes_{\mathbb{Z}} \mathbb{C}$, and for a prime ℓ write $H_{\ell} = H \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$.

Let $\mathbf{GL}(H)$ be the group \mathbb{Z} -scheme such that for any commutative ring R we have $\mathbf{GL}(H)(R) = \text{GL}(H \otimes_{\mathbb{Z}} R)$. The generic fibre $\mathbf{GL}(H)_{\mathbb{Q}}$ is the algebraic group $\mathbf{GL}(H_{\mathbb{Q}})$ over \mathbb{Q} . The Mumford–Tate group $\mathbf{G}_{\mathbb{Q}} \subset \mathbf{GL}(H_{\mathbb{Q}})$ of the natural weight zero Hodge structure on H is the smallest connected algebraic group over \mathbb{Q} such that $\mathbf{G}_{\mathbb{R}}$ contains the image of the homomorphism $h : \mathbb{S} \rightarrow \mathbf{GL}(H)_{\mathbb{R}}$. It is well known that $\mathbf{G}_{\mathbb{Q}}$ is reductive so that $H_{\mathbb{Q}}$ is a semisimple $\mathbf{G}_{\mathbb{Q}}$ -module.

Let \mathbf{G} be the group \mathbb{Z} -scheme which is the Zariski closure of the Mumford–Tate group $\mathbf{G}_{\mathbb{Q}}$ in $\mathbf{GL}(H)$.

Let N be the quotient of $\mathrm{NS}(\overline{X})$ by its torsion subgroup. By the Lefschetz (1, 1)-theorem one has

$$(H_{\mathbb{Q}})^{\mathbf{G}_{\mathbb{Q}}} = H^{0,0} \cap H_{\mathbb{Q}} = N_{\mathbb{Q}}. \tag{5.1}$$

For a field L such that $k \subset L \subset \overline{k}$ we write Γ_L for the Galois group $\mathrm{Gal}(\overline{k}/L)$.

The comparison theorems between Betti and étale cohomology provide an isomorphism $H_{\ell} \cong H_{\mathrm{ét}}^2(\overline{X}, \mathbb{Z}_{\ell}(1))$. Let $\rho_{\ell} : \Gamma_k \rightarrow \mathbf{GL}(H)(\mathbb{Z}_{\ell})$ be the resulting continuous representation. We define $G_{k,\ell}$ to be the Zariski closure of $\rho_{\ell}(\Gamma_k)$ in $\mathbf{GL}(H)_{\mathbb{Z}_{\ell}}$. By a result of Serre there exists a finite field extension k^{conn} of k such that for every field $K \subset \overline{k}$ containing k^{conn} and every prime ℓ the group $G_{K,\ell,\mathbb{Q}_{\ell}}$ is connected, see [LP97].

Let us recall the Mumford–Tate conjecture, together with its integral and adelic variants. Let $\rho : \Gamma_k \rightarrow \mathbf{GL}(H)(\hat{\mathbb{Z}})$ be the continuous representation of Γ_k whose ℓ -adic component is ρ_{ℓ} .

The *Mumford–Tate conjecture* at a prime ℓ says that $\mathbf{G}_{\mathbb{Z}_{\ell}} = G_{k^{\mathrm{conn}},\ell}$. By theorems of Bogomolov [Bog80], Serre [Ser00] and Henniart [Hen82] this implies that $\rho_{\ell}(\Gamma_{k^{\mathrm{conn}}})$ is an open subgroup of $\mathbf{G}(\mathbb{Z}_{\ell})$ of finite index.

The *integral Mumford–Tate conjecture* says that there is a constant C such that for all primes ℓ the image $\rho_{\ell}(\Gamma_{k^{\mathrm{conn}}})$ is a subgroup of $\mathbf{G}(\mathbb{Z}_{\ell})$ of index at most C . This was conjectured by Serre to hold for any X , see [Ser77, Conjecture C.3.7] and [Ser94, 10.3].

The *adelic Mumford–Tate conjecture* says that $\rho(\Gamma_{k^{\mathrm{conn}}})$ is an open subgroup of $\mathbf{G}(\hat{\mathbb{Z}})$ and therefore (since $\mathbf{G}(\hat{\mathbb{Z}})$ is compact) has finite index. This conjecture can only be expected to hold if the Hodge structure on H is Hodge-maximal [CM15, 2.6], which is the case when X is a K3 surface [CM15, Proposition 6.2].

When X is an abelian variety, the *classical Mumford–Tate conjecture* for X is stated in terms of the natural Hodge structure on the first homology group $H_1 = H_1(X_{\mathbb{C}}, \mathbb{Z})$. Then $H_1 \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$ is identified with the ℓ -adic Tate module of X and so carries a natural Galois representation. The classical Mumford–Tate conjecture has integral and adelic versions.

It is clear that the adelic variant of the Mumford–Tate conjecture implies its integral variant, which implies the usual Mumford–Tate conjecture for any ℓ .

5.2 Brauer groups

Grothendieck [Gro68] defines the (cohomological) Brauer group of X as $\mathrm{Br}(X) = H_{\mathrm{ét}}^2(X, \mathbb{G}_m)$. We call $\mathrm{Br}(\overline{X})$ the *geometric Brauer group* of X . By [CS13, Théorème 2.1] the image of the natural map $\mathrm{Br}(X) \rightarrow \mathrm{Br}(\overline{X})$ is contained in $\mathrm{Br}(\overline{X})^{\Gamma_k}$ as a subgroup of finite index, which can be explicitly bounded when k is a number field [CS13, Théorèmes 2.2 et 4.3].

Grothendieck’s classical computation [Gro68, § 8] shows that $\mathrm{Br}(\overline{X})$ is a torsion abelian group which is an extension of a finite group by a divisible group $\mathrm{Br}(\overline{X})_{\mathrm{div}}$ isomorphic to $(\mathbb{Q}/\mathbb{Z})^{b_2-n}$, where $b_2 = \mathrm{rk}(H)$ and $n = \mathrm{rk}(\mathrm{NS}(\overline{X}))$. Let

$$T_{\ell}(\mathrm{Br}(\overline{X})) = \varprojlim \mathrm{Br}(\overline{X})[\ell^a], \quad V_{\ell}(\mathrm{Br}(\overline{X})) = T_{\ell}(\mathrm{Br}(\overline{X})) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}.$$

Then $\mathrm{Br}(\overline{X})_{\mathrm{div}}$ is the direct sum of its ℓ -primary torsion subgroups

$$\mathrm{Br}(\overline{X})_{\mathrm{div}}\{\ell\} = T_{\ell}(\mathrm{Br}(\overline{X})) \otimes_{\mathbb{Z}_{\ell}} \mathbb{Q}_{\ell}/\mathbb{Z}_{\ell}.$$

For $n \geq 1$ the Kummer exact sequence gives rise to exact sequences of Γ_k -modules

$$0 \rightarrow \mathrm{NS}(\overline{X})/\ell^n \rightarrow H_{\mathrm{ét}}^2(\overline{X}, \mu_{\ell^n}) \rightarrow \mathrm{Br}(\overline{X})[\ell^n] \rightarrow 0.$$

Let $N_{\ell} = N \otimes_{\mathbb{Z}} \mathbb{Z}_{\ell}$. Taking the projective limit in n we obtain the exact sequence of Γ_k -modules

$$0 \rightarrow N_{\ell} \rightarrow H_{\ell} \rightarrow T_{\ell}(\mathrm{Br}(\overline{X})) \rightarrow 0.$$

5.3 Uniform boundedness of Brauer groups

Recall that a variety Y over a field L such that $k \subset L \subset \bar{k}$ is called a (\bar{k}/L) -form of X if $Y \times_L \bar{k} \cong \bar{X}$.

DEFINITION. We say that the *Galois invariant subgroups of the geometric Brauer groups of forms of X are uniformly bounded* if for each positive integer n there exists a constant $C = C_{n,X}$ such that for every (\bar{k}/L) -form Y of X defined over a field extension L/k of degree $[L : k] \leq n$ we have $|\text{Br}(\bar{Y})^{\Gamma_L}| < C$.

The main result of this section is the following theorem.

THEOREM 5.1. *Let X be a smooth, projective and geometrically integral variety defined over a field k which is finitely generated over \mathbb{Q} . If the integral Mumford–Tate conjecture is true for X , then the Galois invariant subgroups of the geometric Brauer groups of forms of X are uniformly bounded.*

COROLLARY 5.2. *Let A be an abelian variety over a field k finitely generated over \mathbb{Q} for which the classical Mumford–Tate conjecture holds at a prime ℓ , for example an abelian variety of CM type. Then the Galois invariant subgroups of the geometric Brauer groups of forms of A are uniformly bounded.*

Proof. For an abelian variety, Cadoret and Moonen show in [CM15, Theorem A(i)] that the classical Mumford–Tate conjecture for one prime ℓ implies the integral classical Mumford–Tate conjecture. Let $\mathbf{G}_{1,\mathbb{Q}}$ be the Mumford–Tate group of A defined by the Hodge structure on H_1 . There is a natural surjective homomorphism $\mathbf{G}_1 \rightarrow \mathbf{G}$ and it is not hard to see that the integral classical Mumford–Tate conjecture implies the integral Mumford–Tate conjecture for H as stated above.

We note that the Mumford–Tate conjecture holds for abelian varieties of CM type. This was proved by Pohlmann [Poh68] for abelian varieties which are isogenous to a power of an absolutely simple abelian variety of CM type. For general abelian varieties of CM type, it follows from [Vas08, Theorem 1.3.1].

Thus we can conclude by appealing to Theorem 5.1. □

COROLLARY 5.3. *Let X be a K3 surface over a field k finitely generated over \mathbb{Q} . Then the Galois invariant subgroups of the geometric Brauer groups of forms of X are uniformly bounded.*

Proof. The adelic Mumford–Tate conjecture is true for K3 surfaces by [CM15, Theorem 6.6]. (This result builds on the proof of the Mumford–Tate conjecture for K3 surfaces by Tankeev [Tan95] and André [And96], as well as on previous work of Cadoret and Kret [CK16].) Thus the integral Mumford–Tate conjecture holds for X . □

In particular, for a K3 surface X over k , Corollary 5.3 allows one to recover the finiteness of $\text{Br}(\bar{X})^{\Gamma_k}$ established earlier in [SZ08].

5.4 Proof of Theorem 5.1

The proof of Theorem 5.1 has two steps. In Proposition 5.4 we show that it is possible to find a finite extension k' of L such that $\text{Br}(\bar{X}) \cong \text{Br}(\bar{Y})$ as $\Gamma_{k'}$ -modules and such that the degree $m = [k' : L]$ is bounded in terms of X . For this we can assume without loss of generality that $k = L$. Then we have $\text{Br}(\bar{Y})^{\Gamma_L} \subset \text{Br}(\bar{Y})^{\Gamma'} = \text{Br}(\bar{X})^{\Gamma'}$, where $\Gamma' = \Gamma_{k'}$. Hence it suffices to bound the size of $\text{Br}(\bar{X})^{\Gamma'}$ in terms of m . This is what we do in Proposition 5.5, which is the second step of the proof.

PROPOSITION 5.4. *Let X be a smooth, proper and geometrically integral variety defined over a field k of characteristic zero. Then there exists a constant $m = m_X$ such that for every (\bar{k}/k) -form Y of X we have an isomorphism $\text{Br}(\bar{Y})_{\text{div}} \cong \text{Br}(\bar{X})_{\text{div}}$ of $\Gamma_{k'}$ -modules, where k' is a field extension of k of degree $[k' : k] \leq m$.*

Proof. The varieties X and Y are defined over a subfield $k_0 \subset k$ which is finitely generated over \mathbb{Q} . Furthermore, k_0 can be chosen so that the isomorphism $\bar{Y} \cong \bar{X}$ is defined over a finite extension of k_0 . Thus without loss of generality we can assume that k is finitely generated over \mathbb{Q} . Let us choose an embedding $\bar{k} \hookrightarrow \mathbb{C}$. This allows us to define H as the quotient of $H^2(X_{\mathbb{C}}, \mathbb{Z}(1))$ by its torsion subgroup. The automorphism group $\text{Aut}(\bar{X})$ is a subgroup of $\text{Aut}(X_{\mathbb{C}})$, so we have a natural action of $\text{Aut}(\bar{X})$ on H .

The group $\text{Aut}(\bar{X})$ has a natural structure of a k -scheme which is locally of finite type. The action of Γ_k on $\text{Aut}(\bar{X})$ is continuous when $\text{Aut}(\bar{X})$ is given discrete topology, which means that the stabilisers of the elements of $\text{Aut}(\bar{X})$ are open subgroups of the profinite group Γ_k . By the comparison theorems between Betti and ℓ -adic étale cohomology, the action of $\text{Aut}(\bar{X})$ on H is compatible with the natural action of $\text{Aut}(\bar{X})$ on H_{ℓ} identified with the quotient of $H^2_{\text{ét}}(\bar{X}, \mathbb{Z}_{\ell}(1))$ by its torsion subgroup. Thus the images of both actions are canonically isomorphic.

We define $A(X)$ as the image of the natural action of $\text{Aut}(\bar{X})$ on H . Since $A(X)$ is identified with the image of the action of $\text{Aut}(\bar{X})$ on H_{ℓ} , the discrete group $A(X)$ inherits a continuous action of Γ_k so that the natural surjective homomorphism $\text{Aut}(\bar{X}) \rightarrow A(X)$ is Γ_k -equivariant. The action of Γ_k on $A(X)$ is a continuous homomorphism $\Gamma_k \rightarrow \text{Aut}(A(X))$, where the group of automorphisms $\text{Aut}(A(X))$ of the group $A(X)$ is given discrete topology. Since Γ_k is compact and $\text{Aut}(A(X))$ is discrete, this homomorphism has finite image. Replacing k by a finite field extension, we can assume that Γ_k acts trivially on $A(X)$.

Since Y is a (\bar{k}/k) -form of X , by Galois descent Y can be obtained by twisting X with a continuous 1-cocycle $c : \Gamma_k \rightarrow \text{Aut}(\bar{X})$. Composing c with $\text{Aut}(\bar{X}) \rightarrow A(X)$ we obtain a cocycle $\tilde{c} : \Gamma_k \rightarrow A(X)$. The action of $\text{Aut}(\bar{X})$ on $T_{\ell}(\text{Br}(\bar{X})) = H_{\ell}/N_{\ell}$ factors through $A(X)$. Hence the Γ_k -module $T_{\ell}(\text{Br}(\bar{Y}))$ is the twist of the Γ_k -module $T_{\ell}(\text{Br}(\bar{X}))$ by \tilde{c} composed with the natural homomorphism $A(X) \rightarrow \text{GL}(T_{\ell}(\text{Br}(\bar{X})))$.

We have arranged that the action of Γ_k on $A(X)$ is trivial, therefore the cocycle \tilde{c} is a homomorphism $\Gamma_k \rightarrow A(X)$. Since Γ_k is compact and $A(X)$ is discrete, the image $G = \tilde{c}(\Gamma_k)$ is finite. Let $k' \subset \bar{k}$ be the invariant field of the kernel of \tilde{c} . The group G is a finite subgroup of $\text{GL}(b_2, \mathbb{Z})$, where b_2 is the rank of H . Minkowski's lemma [Min87] says that there exists a constant m depending only on b_2 such that $|G| = [k' : k] \leq m$. The cocycle \tilde{c} is trivialised by the base change from k to k' , hence $\text{Br}(\bar{X})_{\text{div}}$ and $\text{Br}(\bar{Y})_{\text{div}}$ are isomorphic $\Gamma_{k'}$ -modules. \square

PROPOSITION 5.5. *Let X be a smooth, projective and geometrically integral variety defined over a field k which is finitely generated over \mathbb{Q} . Assume that the integral Mumford–Tate conjecture is true for X . Then for each positive integer m there exists a constant $C' = C'_{m,X}$ such that for every subgroup $\Gamma' \subset \Gamma_k$ of index m we have $|\text{Br}(\bar{X})^{\Gamma'}| < C'$.*

In order to prove Proposition 5.5, it is clear that we can replace k by a finite field extension, so we assume that $k = k^{\text{conn}}$. Since $\text{Br}(\bar{X})$ is an extension of a finite abelian group by $\text{Br}(\bar{X})_{\text{div}}$, the proposition follows from the following two claims.

CLAIM 1. *There is a positive $\ell_0 = \ell_0(X, m)$ such that $\text{Br}(\bar{X})_{\text{div}}[\ell]^{\Gamma'} = 0$ for $\ell \geq \ell_0$.*

CLAIM 2. *For each ℓ there is a positive integer $a = a(X, m, \ell)$ such that $\ell^a \text{Br}(\bar{X})_{\text{div}}\{\ell\}^{\Gamma'} = 0$.*

Note that ℓ_0 and a do not depend on Γ' but only on the index $m = [\Gamma_k : \Gamma']$.

Proof of Claim 1. Since the integral Mumford–Tate conjecture is true for X , there is a constant C such that for all primes ℓ the image $\rho_\ell(\Gamma_k)$ is a subgroup of $\mathbf{G}(\mathbb{Z}_\ell)$ of index at most C . The isomorphism of Γ -modules $\text{Br}(\overline{X})[\ell] = T_\ell(\text{Br}(\overline{X}))/\ell$ shows that to prove Claim 1, it is enough to prove that there exists ℓ_0 such that $(T_\ell(\text{Br}(\overline{X}))/\ell)^S = 0$ for any $\ell \geq \ell_0$ and any subgroup $S \subset \mathbf{G}(\mathbb{Z}_\ell)$ of index at most mC .

The generic fibre of $\mathbf{G} \rightarrow \text{Spec}(\mathbb{Z})$ is the connected algebraic group $\mathbf{G}_\mathbb{Q}$. By [EGAIV, Proposition 9.7.8] we can assume that ℓ_0 is large enough so that for any prime $\ell \geq \ell_0$ the fibre $\mathbf{G} \times_{\mathbb{Z}} \text{Spec}(\mathbb{F}_\ell)$ is a connected algebraic group over \mathbb{F}_ℓ . We identify $H/N \cong \mathbb{Z}^r$ with the \mathbb{Z} -points of the affine space $\mathbb{A}_{\mathbb{Z}}^r$ over $\text{Spec}(\mathbb{Z})$. Then $T_\ell(\text{Br}(\overline{X}))/\ell = (H/N)/\ell$ is identified with $\mathbb{A}_{\mathbb{Z}}^r(\mathbb{F}_\ell)$. The representation $\mathbf{G} \rightarrow \mathbf{GL}(H_\mathbb{Q}/N_\mathbb{Q})$ extends to a natural action of \mathbf{G} on $\mathbb{A}_{\mathbb{Z}}^r$. Let us denote the corresponding morphism by

$$\sigma: \mathbf{G} \times \mathbb{A}_{\mathbb{Z}}^r \rightarrow \mathbb{A}_{\mathbb{Z}}^r.$$

Consider the morphism

$$(\sigma, p_2): \mathbf{G} \times \mathbb{A}_{\mathbb{Z}}^r \longrightarrow \mathbb{A}_{\mathbb{Z}}^r \times \mathbb{A}_{\mathbb{Z}}^r.$$

Let W be the closed subscheme of $\mathbf{G} \times \mathbb{A}_{\mathbb{Z}}^r$ defined as the inverse image of the diagonal in $\mathbb{A}_{\mathbb{Z}}^r \times \mathbb{A}_{\mathbb{Z}}^r$. The geometric fibres of $p_2: W \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ are the stabilisers of geometric points. The geometric points $\bar{x} \in \mathbb{A}_{\mathbb{Z}}^r$ that are fixed by the action of \mathbf{G} are those for which the fibre of $p_2: W \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ is the whole of $\mathbf{G} \times_{\mathbb{Z}} \text{Spec}(k(\bar{x}))$.

We write P for the product of all primes less than ℓ_0 . Then $\mathbf{G} \times_{\mathbb{Z}} \text{Spec}(\mathbb{F}_\ell)$ is a connected algebraic group over \mathbb{F}_ℓ whenever $(\ell, P) = 1$. Thus a geometric point $\bar{x} \in \mathbb{A}_{\mathbb{Z}[1/P]}^r$ is a fixed point if and only if the dimension of its stabiliser is $d = \dim(\mathbf{G})$.

By [EGAIV, Théorème 13.1.3], the dimension of the fibres of $p_2: W \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ is upper semi-continuous on the source W . Since the identity section of $\mathbf{G} \times \mathbb{A}_{\mathbb{Z}}^r \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ factors through W , we can pull back by this section to conclude that the dimension of the fibres of $p_2: W \rightarrow \mathbb{A}_{\mathbb{Z}}^r$ is upper semi-continuous on the target $\mathbb{A}_{\mathbb{Z}}^r$. Therefore we have a closed subscheme $F \subset \mathbb{A}_{\mathbb{Z}[1/P]}^r$ defined as the scheme of points whose stabilisers have maximal dimension d . The geometric points of F are the fixed geometric points for the action of $\mathbf{G} \times_{\mathbb{Z}} \mathbb{Z}[1/P]$.

By (5.1) we have $(H_{\mathbb{C}})^{\mathbf{G}_{\mathbb{C}}} = N_{\mathbb{C}}$. Since $H_{\mathbb{C}}$ is a semisimple $\mathbf{G}_{\mathbb{C}}$ -module, this implies $(H_{\mathbb{C}}/N_{\mathbb{C}})^{\mathbf{G}_{\mathbb{C}}} = 0$. Thus the generic fibre $F_{\mathbb{Q}} \subset \mathbb{A}_{\mathbb{Q}}^r$ is one point $\{0\}$. The scheme F has finite type, so it has only finitely many irreducible components. Exactly one of them dominates $\text{Spec}(\mathbb{Z}[1/P])$, namely, the image of the section $\text{Spec}(\mathbb{Z}[1/P]) \rightarrow \mathbb{A}_{\mathbb{Z}[1/P]}^r$ corresponding to the origin in $\mathbb{A}^r(\mathbb{Z}[1/P])$. Let us write $\mathbb{A}_{\mathbb{Z}[1/P]}^r \setminus \{0\}$ for the complement to the image of this section. Other irreducible components of F do not meet the generic fibre $F_{\mathbb{Q}}$, so they are contained in the fibres of the structure morphism $F \rightarrow \text{Spec}(\mathbb{Z}[1/P])$. Therefore, after increasing ℓ_0 we can assume that the stabiliser of every geometric point $\bar{x} \in \mathbb{A}_{\mathbb{Z}[1/P]}^r \setminus \{0\}$ is a subgroup of $\mathbf{G} \times_{\mathbb{Z}} \text{Spec}(k(\bar{x}))$ of dimension at most $d - 1$.

Let W' be the inverse image of $\mathbb{A}_{\mathbb{Z}[1/P]}^r \setminus \{0\}$ in $W \times_{\mathbb{Z}} \mathbb{Z}[1/P]$, and let

$$\pi: W' \longrightarrow \mathbb{A}_{\mathbb{Z}[1/P]}^r \setminus \{0\}$$

be the natural projection. The number of geometric connected components of the fibres of π is a constructible function [EGAIV, Corollaire 9.7.9], hence there exists a constant h such that for any $M \in \mathbb{A}_{\mathbb{Z}[1/P]}^r(\mathbb{F}_\ell)$, $M \neq 0$, the fibre $W_M = \pi^{-1}(M)$ has at most h geometric connected

components. By a result of Nori [Nor87, Lemma 3.5], the number of \mathbb{F}_ℓ -points of a connected algebraic group G over \mathbb{F}_ℓ satisfies

$$(\ell - 1)^{\dim(G)} \leq |G(\mathbb{F}_\ell)| \leq (\ell + 1)^{\dim(G)}.$$

By the choice of ℓ_0 each geometric fibre of π has dimension at most $d - 1$. Thus $|W_M(\mathbb{F}_\ell)| \leq h(\ell + 1)^{d-1}$. On the other hand, we have $|\mathbf{G}(\mathbb{F}_\ell)| \geq (\ell - 1)^d$. After increasing ℓ_0 we obtain that there exists an $\varepsilon > 0$ such that for any prime $\ell \geq \ell_0$ and any \mathbb{F}_ℓ -point $M \in \mathbb{A}_{\mathbb{Z}[1/P]}^r$, $M \neq 0$, the index of the stabiliser of M in $\mathbf{G}(\mathbb{F}_\ell)$ is greater than $\varepsilon\ell$. Take $\ell_0 > \varepsilon^{-1}mC$. Then no non-zero point of $T_\ell(\mathrm{Br}(\bar{X}))/\ell$ is fixed by a subgroup $S \subset \mathbf{G}(\mathbb{F}_\ell)$ of index at most mC , hence $(T_\ell(\mathrm{Br}(\bar{X}))/\ell)^S = 0$. This finishes the proof of Claim 1. \square

Proof of Claim 2. We now fix ℓ . By the Mumford–Tate conjecture $\rho_\ell(\Gamma)$ is a subgroup of finite index in $\mathbf{G}(\mathbb{Z}_\ell)$. Since $\mathbf{G}(\mathbb{Z}_\ell)$ is a compact ℓ -adic analytic Lie group, by Lazard’s theory it is a topologically finitely generated profinite group (see, for example, [DdSMS91, Corollary 9.36]). Then $\mathbf{G}(\mathbb{Z}_\ell)$ has only finitely many open subgroups of fixed index [DdSMS91, Proposition 1.6]. Thus $\rho_\ell(\Gamma)$ has only finitely many subgroups S of index at most m . It suffices to show that $\mathrm{Br}(\bar{X})\{\ell\}^S$ is finite for each of these subgroups S .

It is well known that $\mathrm{Br}(\bar{X})_{\mathrm{div}}\{\ell\}^S$ is finite if $V_\ell(\mathrm{Br}(\bar{X}))^S = 0$. Indeed, if $\mathrm{Br}(\bar{X})\{\ell\}^S$ is infinite, then $\mathrm{Br}(\bar{X})$ has an S -stable element of order ℓ^n for each positive integer n . The limit of a projective system of non-empty finite sets is non-empty. Applying this to the limit of the sets of elements of order ℓ^n in $\mathrm{Br}(\bar{X})^S$ we obtain a non-zero element of $T_\ell(\mathrm{Br}(\bar{X}))^S$, hence a non-zero element of $V_\ell(\mathrm{Br}(\bar{X}))^S$.

We claim that we have the following equalities

$$V_\ell(\mathrm{Br}(\bar{X}))^S = (H_{\mathbb{Q}_\ell}/N_{\mathbb{Q}_\ell})^S = (H_{\mathbb{Q}_\ell}/N_{\mathbb{Q}_\ell})^{\mathbf{G}_{\mathbb{Q}_\ell}} = (H_{\mathbb{Q}_\ell})^{\mathbf{G}_{\mathbb{Q}_\ell}}/N_{\mathbb{Q}_\ell} = 0.$$

The second one is due to the fact that S is a Zariski dense subset of the algebraic group $\mathbf{G}_{\mathbb{Q}_\ell}$. Since $\mathbf{G}_{\mathbb{Q}_\ell}$ is reductive, the $\mathbf{G}_{\mathbb{Q}_\ell}$ -module $H_{\mathbb{Q}_\ell}$ is semisimple, and this implies the third equality. The last equality follows from (5.1). This proves Claim 2, and so finishes the proof of Theorem 5.1. \square

Remark. The same proof can be used to prove the following statement. Let A be an abelian variety over a field k finitely generated over \mathbb{Q} for which the classical Mumford–Tate conjecture holds at a prime ℓ . For each positive integer n there exists a constant $C = C_{n,A}$ such that for every abelian variety B over a field L , where $k \subset L \subset \bar{k}$ and $[L : k] \leq n$, if B is a (\bar{k}/L) -form of A , then $|B(L)_{\mathrm{tors}}| < C$.

ACKNOWLEDGEMENTS

The authors are grateful to François Charles for a very useful discussion and to Anna Cadoret for her comments and suggestions that helped to improve the results and the presentation of this paper. They thank Wessel Bindt, Yuri Zarhin and the referee for their comments. Both authors have been supported by the EPSRC grant EP/M020266/1.

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Martin Orr m.orr@imperial.ac.uk

Department of Mathematics, South Kensington Campus,
Imperial College London, London SW7 2BZ, UK

Alexei N. Skorobogatov a.skorobogatov@imperial.ac.uk

Department of Mathematics, South Kensington Campus,
Imperial College London, London SW7 2BZ, UK

and

Institute for the Information Transmission Problems,
Russian Academy of Sciences, 19 Bolshoi Karetnyi,
Moscow 127994, Russia