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A CR Analogue of Yau's Conjecture on Pseudoharmonic Functions of Polynomial Growth

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Abstract. In this paper, we first derive the CR volume doubling property, CR Sobolev inequality, and the mean value inequality. We then apply them to prove the CR analogue of Yau's conjecture on the space consisting of all pseudoharmonic functions of polynomial growth of degree at most d in a complete noncompact pseudohermitian $(2n+1)$-manifold. As a by-product, we obtain the CR analogue of the volume growth estimate and the Gromov precompactness theorem.

1 Introduction

S.-Y. Cheng [16] and S.-T. Yau [44] derived the well-known gradient estimate for positive harmonic functions and obtained the classical Liouville theorem, stating that any bounded harmonic function is constant in complete noncompact Riemannian manifolds with nonnegative Ricci curvature. Let $\mathcal{H}^d(M)$ be the space of harmonic functions of polynomial growth of degree at most d in a complete noncompact Riemannian manifold M^m. Yau conjectured that the dimension $h^d(M)$ of $\mathcal{H}^d(M)$ is finite for each positive integer d and satisfies the estimate $h^d(M) \le h^d(\mathbf{R}^m)$. Colding and Minicozzi [19] affirmatively answered the first question and proved that

$$h^d(M) \le C_0 d^{m-1}$$

for manifolds of nonnegative Ricci curvature with C_0 depending on the Neumann–Poincaré inequality and the volume doubling constant. Later, Li [33] produced an elegant and shorter proof requiring only the manifold to satisfy the volume doubling property (see Definition 1.2) and the mean value inequality (see Definition 1.10). For the latter question, the sharp upper bound estimate is still missing except for the special cases $m = 2$ or $d = 1$ obtained by Li and Tam [34, 35]) and Kasue [28], and the rigidity part is only known for the special case $d = 1$ obtained by Li [32] and Cheeger, Colding, and Minicozzi [15]. By modifying the arguments of Yau [44], Cheng and Yau [16], and Chang, Kuo, and Lai [12], Chang, Kuo, and Tie [13] derived a sub-gradient

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estimate for positive pseudoharmonic functions in a complete noncompact pseudohermitian $(2n+1)$-manifold (M, J, θ). This sub-gradient estimate can serve as the CR version of Yau's gradient estimate. As an application of the sub-gradient estimate, the CR analogue of Liouville-type theorem holds for positive pseudoharmonic functions.

In the current paper, we study the CR analogue of Yau's conjecture on the space $\mathcal{H}^d(M)$ consisting of all pseudoharmonic functions of polynomial growth of degree at most d in a complete noncompact pseudohermitian $(2n+1)$-manifold (see Definition 1.1). We will show that the first part of the CR Yau conjecture holds for pseudoharmonic functions of polynomial growth as in Theorem 1.12 via Li's method, which is more adaptable to the case of pseudohermitian geometry.

Throughout the paper, we assume that (M, J, θ) is a complete pseudohermitian $(2n+1)$-manifold. A piecewise smooth curve $\gamma: [0,1] \to M$ is said to be horizontal if $\dot{\gamma}(t) \in \xi$ whenever $\dot{\gamma}(t)$ exists. We denote by $C_{p,q}$ the set of all horizontal curves joining p, q. The Carnot–Carathéodory distance between two points $p, q \in M$ is defined by $d_{cc}(p,q) = \inf\{l(\gamma) : \gamma \in C_{p,q}\}$, where the length of γ is $l(\gamma) = \int_0^1 \langle \dot{\gamma}(t), \dot{\gamma}(t)\rangle_{L_\theta}^{1/2}\, dt$.

Since M is a complete pseudohermitian $(2n+1)$-manifold, we know that d_{cc} exists by one of Chow's theorems [17].

Definition 1.1 Let $\mathcal{H}^d(M)$ be the space of all polynomial growth pseudoharmonic real-valued functions defined on M of order at most d, *i.e.*,

$$\mathcal{H}^d(M) = \left\{ f \;\middle|\; \Delta_b f = 0 \text{ and } |f(x)| \leq Cr(x)^d \text{ for some constant } C \right\},$$

where $r(x)$ is the Carnot–Carathéodory distance from x to a fixed point p in M. We also denote the dimension of the vector space $\mathcal{H}^d(M)$ by $h^d(M)$.

Definition 1.2 A complete pseudohermitian $(2n+1)$-manifold M has the *volume doubling property* $(\mathcal{V}_\mu)$ for some $\mu > 1$ if there exists a universal constant $C_{\mathcal{V}} > 0$ such that for any $\sigma > 1$, $0 < \rho < \infty$, $x \in M$, the volume of the geodesic ball centered at x satisfies the inequality $V_x(\sigma\rho) \leq C_{\mathcal{V}}\sigma^\mu V_x(\rho)$. Here $V_x(\rho)$ is the volume of the geodesic ball $B_x(\rho)$ centered at x with radius ρ.

In Section 2, for $X = X^\alpha Z_\alpha$, we will define the pseudohermitian Ricci tensors $R_{\alpha\overline{\beta}}$ and the pseudohermitian torsion tensors $A_{\alpha\beta}$ as follows:

$$\begin{aligned} \operatorname{Ric}(X,X) &= R_{\alpha\overline{\beta}} X^\alpha X^{\overline{\beta}}, \\ \operatorname{Tor}(X,X) &= i\sum_{\alpha,\beta}(A_{\overline{\alpha}\overline{\beta}} X^{\overline{\alpha}} X^{\overline{\beta}} - A_{\alpha\beta} X^\alpha X^\beta) = 2\operatorname{Re}(iA_{\overline{\alpha}\overline{\beta}} X^{\overline{\alpha}} X^{\overline{\beta}}), \\ (\operatorname{div} A)^2(X,X) &= A_{\alpha\gamma,\overline{\gamma}} A_{\overline{\sigma}\overline{\beta},\sigma} X^\alpha X^{\overline{\beta}}. \end{aligned}$$

We observe that the pseudohermitian Ricci tensors $R_{\alpha\overline{\beta}}$ (or Ric) are the mixed type component of the Tanaka–Webster Ricci curvature tensors, and the pseudohermitian torsion tensors $i(n-1)A_{\alpha\beta}$ (or $R_{\alpha\beta}$) are the purely holomorphic part of the Tanaka–Webster Ricci curvature tensors. Hence, in order to make geometric assumptions for the Tanaka–Webster Ricci curvature tensors, we make the equivalent assumptions $(*)$

and $(**)$ below on the pseudohermitian Ricci tensors $R_{\alpha\overline{\beta}}$ and the pseudohermitian torsion tensors $A_{\alpha\beta}$, respectively.

Theorem 1.3 *Assume that a complete pseudohermitian $(2n+1)$-manifold M satisfies the curvature assumptions*

$$(*) \qquad \operatorname{Ric}(X,X) \geq k_0 \langle X,X\rangle_{L_\theta},$$

and

$$(**) \qquad \sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty, \quad \sup_{i,j\in I_n} |A_{ij,\overline{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$, $I_n = \{1,2,\dots,n\}$ and where k_0, k_1, k_2 are constants with $k_1, k_2 \geq 0$. Then for any $\sigma > 1$, there exist positive constants κ, C_1, C_8, C_9 such that

$$V_x(\sigma\rho) \leq C_1 \sigma^{2C_9} e^{(C_1\sigma^2 + C_8)k\rho^2} V_x(\rho) \tag{1.1}$$

with $\kappa = 2[-k_0 + |n-2|k_1 + \epsilon k_2] \geq 0$, for a positive constant ϵ and where C_9 is the constant in (3.8).

Remark 1.4 (i) If a complete pseudohermitian $(2n+1)$-manifold M has the properties $\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty$ and $\sup_{i,j\in I_n} |A_{ij,\overline{i}}|^2 \leq k_2 < \infty$, then the curvature assumptions can be replaced by

$$\begin{aligned}[2\operatorname{Ric} - (n-2)\operatorname{Tor} - 2\epsilon(\operatorname{div} A)^2](X,X) \\ \geq -2(-k_0 + |n-2|k_1 + \epsilon k_2)\langle X,X\rangle_{L_\theta} := -\kappa\langle X,X\rangle_{L_\theta}\end{aligned}$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$, a positive constant ϵ.

(ii) The nonnegativity of the pseudohermitian Ricci curvature *i.e.*, $\kappa = 0$ is equivalent to saying $\operatorname{Ric}(X,X) \geq (|n-2|k_1 + \epsilon k_2)\langle X,X\rangle_{L_\theta}$, which is

$$[2\operatorname{Ric}(X,X) \geq [(n-2)\operatorname{Tor} + 2\epsilon(\operatorname{div} A)^2](X,X),$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and a positive constant ϵ.

In particular, M satisfies the following volume doubling property $(\mathcal{V}_\mu)$ for some $\mu = 2C_9$.

Corollary 1.5 *A complete pseudohermitian $(2n+1)$-manifold M satisfies*

$$\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty \quad \textit{and} \quad \sup_{i,j\in I_n} |A_{ij,\overline{i}}|^2 \leq k_2 < \infty,$$

where k_1, k_2 are nonnegative constants. Assume also the nonnegativity of the pseudohermitian Ricci curvature $\operatorname{Ric}(X,X) \geq (|n-2|k_1 + \epsilon k_2)\langle X,X\rangle_{L_\theta}$, for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and where ϵ is a positive constant. Then for any $\sigma > 1$, there exists a positive constant $C_{\mathcal{V}} = C_{\mathcal{V}}(n)$ such that $V_x(\sigma\rho) \leq C_{\mathcal{V}}\sigma^{2C_9} V_x(\rho)$.

Furthermore, we obtain the following CR volume growth estimate.

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Corollary 1.6 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with* $\mathrm{Ric}(X, X) \geq k_0 \langle X, X\rangle_{L_\theta}$, *and*

$$\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty, \quad \sup_{i,j\in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and where k_0, k_1, k_2 are constants with $k_1, k_2 \geq 0$. Then for any $\sigma > 1$, there exists a positive constant $\widetilde{C}_1$ such that $V_x(\sigma) \leq \widetilde{C}_1 V_x(1)\sigma^{2C_9} e^{C_1 k\sigma^2}$. In particular, assume that the nonnegativity of the pseudohermitian Ricci curvature

$$\mathrm{Ric}(X, X) \geq (|n-2|k_1 + \epsilon k_2)\langle X, X\rangle_{L_\theta},$$

for a positive constant ϵ. Then $V_x(\sigma) \leq \widetilde{C}_1 V_x(1)\sigma^{2C_9}$.

Remark 1.7 (i) For the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}_n$, it has been shown [38] that

$$V_x(\sigma) \leq \widetilde{C}_1 V_x(1)\sigma^{2n+2}, \tag{1.2}$$

where $2n+2$ is the homogeneous dimension.

(ii) The best constant is expected to be $C_9 = n+1$, the same as in the Heisenberg group $\mathbf{H}_n$. However, at this step we only can show $C_9 = mn(1 + \frac{6}{\rho_2}) = n+3$ when the torsion is vanishing. In general, it follows from (3.8) and (3.10) that we can choose $n+3 < C_9 < n+3+\varepsilon$ for any fixed small $\varepsilon > 0$.

Now, applying the volume doubling constant and upper bound estimate of the heat kernel as in Proposition 3.3, we can obtain the CR analogue of the Sobolev inequality.

Theorem 1.8 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with*

$$\mathrm{Ric}(X, X) \geq k_0 \langle X, X\rangle_{L_\theta},$$

and

$$\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty, \quad \sup_{i,j\in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and where k_0, k_1, k_2 are constants with $k_1, k_2 \geq 0$. Then

$$\Big(\int_{B_x(\rho)} |\varphi|^{\frac{2Q}{Q-2}}\Big)^{\frac{Q-2}{Q}} \leq C_s \rho^2 e^{Ck\rho^2} V_x(\rho)^{-\frac{2}{Q}} \Big(\int_{B_x(\rho)} |\nabla_b \varphi|^2 \, d\mu + \rho^{-2} \int_{B_x(\rho)} \varphi^2 \, d\mu\Big),$$

for any $\varphi \in C_0^\infty(B_x(\rho))$, $x \in M$. Here $Q = 3C_9$.

Theorem 1.8 and Moser's iteration method will yield the mean value inequality.

Theorem 1.9 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with*

$$\mathrm{Ric}(X, X) \geq k_0 \langle X, X\rangle_{L_\theta},$$

and

$$\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty, \quad \sup_{i,j\in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and where k_0, k_1, k_2 are constants with $k_1, k_2 \geq 0$. Then there exists a constant $C_2 > 0$ such that for any $\rho > 0, x \in M$, and any nonnegative subpseudoharmonic function f defined on M, we have

$$[f(x)]^2 \leq C_2 V_x^{-1}(\rho) e^{C_2 k \rho^2} \int_{B_x(\rho)} f(y)^2 \, d\mu.$$

As a consequence, M satisfies the mean value inequality $(\mathcal{M})$.

Definition 1.10 A complete pseudohermitian $(2n+1)$-manifold M satisfies the *mean value inequality* $(\mathcal{M})$ if there exists a universal constant $C_{\mathcal{M}} > 0$ such that for any point $x \in M$, $0 < \rho < \infty$, and any non-negative subpseudoharmonic function $f(x)$ defined on M, we have

$$[f(x)]^2 \leq \frac{C_{\mathcal{M}}}{V_x(\rho)} \int_{B_x(\rho)} f(y)^2 \, d\mu.$$

Corollary 1.11 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with*

$$\sup_{i,j \in I_n} |A_{ij}| \leq k_1 < \infty \quad \text{and} \quad \sup_{i,j \in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$ and k_1, k_2 are nonnegative constants. Assume the nonnegativity of the pseudohermitian Ricci curvature

$$\mathrm{Ric}(X, X) \geq (|n-2|k_1 + \epsilon k_2)\langle X, X\rangle_{L_\theta},$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$, where ϵ is a positive constant. Then there exists a constant $C_{\mathcal{M}} > 0$ such that for any $\rho > 0$, $x \in M$, and any nonnegative subpseudoharmonic function f defined on M, we have

$$[f(x)]^2 \leq C_{\mathcal{M}} V_x^{-1}(\rho) \int_{B_x(\rho)} f(y)^2 \, d\mu.$$

As a consequence of Corollary 1.5 and Corollary 1.11, we have the following main result on a CR analogue of Yau's conjecture for pseudoharmonic functions of polynomial growth.

Theorem 1.12 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with*

$$\sup_{i,j \in I_n} |A_{ij}| \leq k_1 < \infty \quad \text{and} \quad \sup_{i,j \in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$, where k_1, k_2 are nonnegative constants. Assume the nonnegativity of the pseudohermitian Ricci curvature

$$\mathrm{Ric}(X, X) \geq (|n-2|k_1 + \epsilon k_2)\langle X, X\rangle_{L_\theta},$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$, where ϵ is a positive constant. Then the dimension of $\mathcal{H}^d(M)$ is finite. Moreover, there exists a constant $C_0 = C(C_{\mathcal{M}}, C_{\mathcal{V}}) > 0$ such that

$$h^d(M) \leq C_0 d^{\frac{2n}{2n+1}(2C_9 - 1)}, \tag{1.3}$$

for all $d \geq 1$. Here C_9 is the constant in Corollary 1.5.

As a consequence of Theorem 1.12 and [13], we have the folowing.

Corollary 1.13 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold of vanishing torsion. Assume that $\mathrm{Ric}(X, X) \geq 0$ for $X = X^\alpha Z_\alpha \in T_{1,0}M$. Then there exists a constant $C_0 = C(C_\mathcal{M}, C_\mathcal{V}) > 0$ such that*

$$h^d(M) \leq C_0 d^{\frac{2n}{2n+1}(2C_9-1)}$$

for all $d \geq 1$. In particular, $h^d(M) = 1$ for $d < 1$.

Remark 1.14 (i) In the $(2n+1)$-dimensional Heisenberg group $\mathbf{H}_n$, we compute explicitly (see Section 7) that

$$h^d(\mathbf{H}_n) \approx \frac{d^{2n}}{(2n)!}. \tag{1.4}$$

On the other hand, it follows from Corollary 1.6 and (1.2) that the power $2C_9$ of volume in ρ is $2n+2$,and then $(2C_9 - 1) = 2n+1$ in $\mathbf{H}_n$. Therefore, the power of the right-hand side in (1.3) will be $\frac{2n}{2n+1}(2C_9 - 1) = 2n$ which is sharp as in (1.4) for the Heisenberg group $\mathbf{H}_n$.

(ii) It is not clear that if a complete pseudohermitian $(2n+1)$-manifold with nonnegative pseudohermitian Ricci curvature and vanishing torsion admits nonconstant pseudoharmonic functions of polynomial growth. We refer to [37, 43] for the related topic.

(iii) However, similar results hold for a complete pseudohermitian $(2n+1)$-manifold of nonnegative pseudohermitian Ricci and vanishing torsion (M, J_0, θ_0). We refer to [2] for more details.

(iv) By applying the volume doubling property (1.1) together with methods of M. Gromov [26] and C. Villani [42], we have the following CR analogue of the Gromov precompactness theorem. Let $\mathcal{M}(k_0, k_1, k_2, D)$ be the space of compact pseudohermitian $(2n+1)$-manifolds (M, J, θ) equipped with their Carnot–Carathéodory geodesic distance and volume measure and

(a) $\mathrm{Ric}(X, X) \geq k_0 \langle X, X\rangle_{L_\theta}$, $\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty$, and $\sup_{i,j\in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty$, for $X = X^\alpha Z_\alpha \in T_{1,0}M$, where k_0, k_1, k_2 are constants with $k_1,\ k_2 \geq 0$.

(b) $\mathrm{diam}(M) \leq D$. Then $\mathcal{M}(k_0, k_1, k_2, D)$ is precompactness in the measured Gromov–Hausdorff topology.

We will pursue this topic further in a forthcoming paper.

In Käehler geometry, one of the primary goals is to generalize the classical uniformization theorem in higher dimensions. Siu and Yau [41] initiated a program of using holomorphic functions of polynomial growth to holomorphically embed a complete Kähler manifold into the complex Euclidean space. More precisely, the well-known uniformization conjecture asks if every complete noncompact Kähler manifold with positive holomorphic bisectional curvature is biholomorphic to a complex Euclidean space. For this reason, Yau [45] proposed to study the spaces of holomorphic functions of polynomial growth on complete noncompact Kähler manifolds with nonnegative holomorphic bisectional curvature. In our forthcoming paper, we will investigate the related CR analogue of Yau's conjecture on the spaces of CR holomorphic functions of polynomial growth in a complete noncompact pseudohermitian manifold with nonnegative pseudohermitian bisectional curvature.

The rest of the paper is organized as follows. In Section 2, we introduce some basic notions of a pseudohermitian $(2n+1)$-manifold. In Section 3, we derive a CR curvature-dimension inequality and the heat kernel estimate in a pseudohermitian $(2n+1)$-manifold under some specific assumptions on the pseudohermitian Ricci curvature tensor and the torsion tensor. From these estimates, we obtain the CR volume doubling property. Then the volume doubling property and the heat kernel estimate yield the CR Sobolev inequality. In Section 4, by applying the volume doubling property and CR Sobolev inequality obtained in Section 3, we derive the mean value inequality via Moser's iteration method. In Section 5, following Li's method, we use the CR volume doubling property and mean value inequality to prove the CR analogue of Yau's conjecture for the space of pseudoharmonic functions of polynomial growth of degree at most d in a complete noncompact pseudohermitian $(2n+1)$-manifold. As a by-product, we obtain the CR volume growth estimate. In the last section, we study the pseudoharmonic polynomials on the Heisenberg group and obtain a precise estimate of the dimension of the linear space of the pseudoharmonic polynomials of degree at most d.

2 Preliminaries

We introduce some basic notions of a pseudohermitian manifold (see [12, 20, 30] for more details). Let (M,ξ) be a $(2n+1)$-dimensional, orientable, contact manifold with contact structure ξ. A *CR structure compatible with* ξ is an endomorphism $J:\xi\to\xi$ such that $J^2=-1$. We also assume that J satisfies the integrability condition: if X and Y are in ξ, then so are $[JX,Y]+[X,JY]$ and $J([JX,Y]+[X,JY])=[JX,JY]-[X,Y]$. Let $\{\mathbf{T},Z_\alpha,Z_{\overline{\alpha}}\}$ be a frame of $TM\otimes\mathbb{C}$, where Z_α is any local frame of $T_{1,0}$, $Z_{\overline{\alpha}}=\overline{Z_\alpha}\in T_{0,1}$ and $\mathbf{T}$ is the characteristic vector field. Then $\{\theta,\theta^\alpha,\theta^{\overline{\alpha}}\}$, the coframe dual to $\{\mathbf{T},Z_\alpha,Z_{\overline{\alpha}}\}$, satisfies $d\theta=ih_{\alpha\overline{\beta}}\theta^\alpha\wedge\theta^{\overline{\beta}}$ for some positive definite hermitian matrix of functions $(h_{\alpha\overline{\beta}})$. If we have this contact structure, we also call such M a strictly pseudoconvex CR $(2n+1)$-manifold. The Levi form $\langle\cdot,\cdot\rangle_{L_\theta}$ is the Hermitian form on $T_{1,0}$ defined by $\langle Z,W\rangle_{L_\theta}=-i\langle d\theta,Z\wedge\overline{W}\rangle$. We can extend $\langle\cdot,\cdot\rangle_{L_\theta}$ to $T_{0,1}$ by defining $\langle\overline{Z},\overline{W}\rangle_{L_\theta}=\overline{\langle Z,W\rangle}_{L_\theta}$ for all $Z,W\in T_{1,0}$. The Levi form induces naturally a Hermitian form on the dual bundle of $T_{1,0}$, denoted by $\langle\cdot,\cdot\rangle_{L_\theta^*}$, and hence on all the induced tensor bundles. Integrating the Hermitian form (when acting on sections) over M with respect to the volume form $d\mu=\theta\wedge(d\theta)^n$, we get an inner product on the space of sections of each tensor bundle.

The pseudohermitian connection of (J,θ) is the connection ∇ on $TM\otimes\mathbb{C}$ (and extended to tensors) given in terms of a local frame $Z_\alpha\in T_{1,0}$ by

$$\nabla Z_\alpha=\omega_\alpha{}^\beta\otimes Z_\beta,\quad \nabla Z_{\overline{\alpha}}=\omega_{\overline{\alpha}}{}^{\overline{\beta}}\otimes Z_{\overline{\beta}},\quad \nabla T=0$$

and

$$d\theta^\beta=\theta^\alpha\wedge\omega_\alpha{}^\beta+\theta\wedge\tau^\beta,\quad 0=\tau_\alpha\wedge\theta^\alpha,\quad 0=\omega_\alpha{}^\beta+\omega_{\overline{\beta}}{}^{\overline{\alpha}},\tag{2.1}$$

where $\omega_\alpha{}^\beta$ are the 1-forms uniquely determined by equations (2.1). We can write (by Cartan's lemma) $\tau_\alpha=A_{\alpha\gamma}\theta^\gamma$ with $A_{\alpha\gamma}=A_{\gamma\alpha}$. The curvature of the Tanaka–Webster

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connection, expressed in terms of the coframe $\{\theta = \theta^0, \theta^\alpha, \theta^{\overline{\alpha}}\}$, is

$$\Pi_\beta{}^\alpha = \overline{\Pi_{\overline{\beta}}{}^{\overline{\alpha}}} = d\omega_\beta{}^\alpha - \omega_\beta{}^\gamma \wedge \omega_\gamma{}^\alpha, \quad \Pi_0{}^\alpha = \Pi_\alpha{}^0 = \Pi_0{}^{\overline{\beta}} = \Pi_{\overline{\beta}}{}^0 = \Pi_0{}^0 = 0.$$

Webster showed that $\Pi_\beta{}^\alpha$ can be written as

$$\Pi_\beta{}^\alpha = R_\beta{}^\alpha{}_{\rho\overline{\sigma}}\theta^\rho \wedge \theta^{\overline{\sigma}} + W_\beta{}^\alpha{}_\rho \theta^\rho \wedge \theta - W^\alpha{}_{\beta\overline{\rho}}\theta^{\overline{\rho}} \wedge \theta + i\theta_\beta \wedge \tau^\alpha - i\tau_\beta \wedge \theta^\alpha,$$

where the coefficients satisfy $R_{\beta\overline{\alpha}\rho\overline{\sigma}} = \overline{R_{\alpha\overline{\beta}\sigma\overline{\rho}}} = R_{\overline{\alpha}\beta\overline{\sigma}\rho} = R_{\rho\overline{\alpha}\beta\overline{\sigma}}$, and $W_{\beta\overline{\alpha}\gamma} = W_{\gamma\overline{\alpha}\beta}$. Here, $R_\gamma{}^\delta{}_{\alpha\overline{\beta}}$ is the pseudohermitian curvature tensor, $R_{\alpha\overline{\beta}} = R_\gamma{}^\gamma{}_{\alpha\overline{\beta}}$ is the pseudohermitian Ricci curvature tensor, and $A_{\alpha\beta}$ is the pseudohermitian torsion. The Webster scalar curvature is $R = R_\alpha{}^\alpha = h^{\alpha\overline{\beta}}R_{\alpha\overline{\beta}}$. Moreover, we have

$$R_{\alpha\beta} = i(n-1)A_{\alpha\beta} \quad \text{and} \quad R_{\alpha 0} = R_{00} = 0.$$

Furthermore, we define the bi-sectional curvature $R_{\alpha\overline{\alpha}\beta\overline{\beta}}(X,Y) = R_{\alpha\overline{\alpha}\beta\overline{\beta}}X_\alpha X_{\overline{\alpha}} Y_\beta Y_{\overline{\beta}}$ and the bi-torsion tensor $T_{\alpha\overline{\beta}}(X,Y) := i(A_{\overline{\beta}\overline{\rho}}X^{\overline{\rho}}Y_\alpha - A_{\alpha\rho}X^\rho Y_{\overline{\beta}})$, where $X = X^\alpha Z_\alpha$, $Y = Y^\beta Z_\beta$ in $T_{1,0}$. In particular, we have $\mathrm{Ric}(X,Y) = R_{\alpha\overline{\beta}}X^\alpha Y^{\overline{\beta}}$ and

$$\mathrm{Tor}(X,Y) := h^{\alpha\overline{\beta}}T_{\alpha\overline{\beta}}(X,Y) = i\sum_{\alpha,\beta}(A_{\overline{\alpha}\overline{\beta}}X^{\overline{\alpha}}Y^{\overline{\beta}} - A_{\alpha\beta}X^\alpha Y^\beta).$$

We also define the tensor A^2 by $A^2(X,Y) = \sum_\gamma (A_{\alpha\gamma}A_{\overline{\beta}\overline{\gamma}}X^\alpha Y^{\overline{\beta}})$.

We will denote the components of covariant derivatives with indices preceded by a comma; thus we write $A_{\alpha\beta,\gamma}$. The indices $\{0, \alpha, \overline{\alpha}\}$ indicate derivatives with respect to $\{T, Z_\alpha, Z_{\overline{\alpha}}\}$. For derivatives of a scalar function, we will often omit the comma; for instance, $u_\alpha = Z_\alpha u$, $u_{\alpha\overline{\beta}} = Z_{\overline{\beta}}Z_\alpha u - \omega_\alpha{}^\gamma(Z_{\overline{\beta}})Z_\gamma u$. In particular, we have

$$|\nabla_b u|^2 = 2\sum_\alpha u_\alpha u_{\overline{\alpha}}, \quad |\nabla_b^2 u|^2 = 2\sum_{\alpha,\beta}(u_{\alpha\beta}u_{\overline{\alpha}\overline{\beta}} + u_{\alpha\overline{\beta}}u_{\overline{\alpha}\beta}),$$
$$\Delta_b u = Tr((\nabla^H)^2 u) = \sum_\alpha (u_{\alpha\overline{\alpha}} + u_{\overline{\alpha}\alpha}).$$

Next we recall some commutation relations [30]. Let φ be a scalar function and $\sigma = \sigma_\alpha\theta^\alpha$ a $(1,0)$ form. Let $\varphi_0 = \mathbf{T}\varphi$; then we have

$$\begin{aligned}
\varphi_{\alpha\beta} &= \varphi_{\beta\alpha},\\
\varphi_{\alpha\overline{\beta}} - \varphi_{\overline{\beta}\alpha} &= ih_{\alpha\overline{\beta}}\varphi_0,\\
\varphi_{0\alpha} - \varphi_{\alpha 0} &= A_{\alpha\beta}\varphi^\beta,\\
\sigma_{\alpha,0\beta} - \sigma_{\alpha,\beta 0} &= \sigma_{\alpha,\overline{\gamma}}A^{\overline{\gamma}}{}_\beta - \sigma^{\overline{\gamma}}A_{\alpha\beta,\overline{\gamma}},\\
\sigma_{\alpha,0\overline{\beta}} - \sigma_{\alpha,\overline{\beta}0} &= \sigma_{\alpha,\gamma}A^\gamma{}_{\overline{\beta}} + \sigma^{\overline{\gamma}}A_{\overline{\gamma}\overline{\beta},\alpha}.
\end{aligned}$$

Finally, we provide the real version of pseudohermitian geometry for completeness. Write $Z_\beta = \frac{1}{2}(e_\beta - ie_{n+\beta})$ for real vectors $e_\beta, e_{n+\beta}, \beta = 1,\dots,n$. It follows that $e_{n+\beta} = Je_\beta$. We also write $\varphi_{e_\beta} = e_\beta\varphi$ and $\nabla_b\varphi = \frac{1}{2}(\varphi_{e_\beta}e_\beta + \varphi_{e_{n+\beta}}e_{n+\beta})$. Moreover, we have $\varphi_{e_je_k} = e_ke_j\varphi - \nabla_{e_k}e_j\varphi$ and $\Delta_b\varphi = \frac{1}{2}\sum_\beta(\varphi_{e_\beta e_\beta} + \varphi_{e_{n+\beta}e_{n+\beta}})$.

3 The CR Volume Doubling Property and Sobolev Inequality

In this section, we first derive a CR curvature-dimension inequality and the heat kernel estimate in a pseudohermitian $(2n+1)$-manifold when the pseudohermitian Ricci curvature tensor and the torsion tensor satisfy some specific conditions. Then we obtain the CR volume doubling property via the heat kernel estimate. Finally, by applying the volume doubling property and the heat kernel estimate, we prove the above CR Sobolev inequality.

One of the key steps of Li and Yau's proof of gradient estimates of the heat equation on a Riemannian manifold is the Bochner formula in terms of the Riemannian Ricci curvature tensors. In the CR analogue of the Li–Yau gradient estimate, the crucial step is the following CR Bochner formula [24]:

$$\begin{aligned}(3.1)\quad \frac{1}{2}\Delta_b|\nabla_b f|^2 = |\operatorname{Hess}(f)|^2 + \langle \nabla_b f, \nabla_b(\Delta_b f)\rangle + 2\langle J\nabla_b f, \nabla_b f_0\rangle \\ + (2\operatorname{Ric} - (n-2)\operatorname{Tor})((\nabla_b f)_C, (\nabla_b f)_C),\end{aligned}$$

where $(\nabla_b f)_c$ is the $T_{1,0}M$-component of $(\nabla_b f)$. Note that the right-hand side involves a term $\langle J\nabla_b f, \nabla_b f_0\rangle$ that has no analogue in the Riemannian case. In the case of vanishing torsion tensors, we are able to deal with the extra term $\langle J\nabla_b f, \nabla_b f_0\rangle$ as in [11,12] by using the arguments of [36]. In this paper, we will use gradient estimates and some related results in a complete pseudohermitian manifold with nonvanishing torsion tensor.

On the other hand, Bakry and Emery [1] pioneered the approach to generalize curvature in the context of gradient estimates by the so-called curvature-dimension inequality. We will define the CR version of the curvature-dimension inequality that was first introduced by Baudoin, Bonnefont, and Garofalo [2] in the context of sub-Riemannian geometry.

Definition 3.1 Let (M, J, θ) be a smooth pseudohermitian $(2n+1)$-manifold and a real frame $\{e_\beta, e_{n+\beta}, \mathbf{T}\}$ spanning the tangent space TM. For $\rho_1 \in \mathbb{R}, \rho_2 > 0, d \geq 0$, and $m > 0$, we say that M satisfies *the CR curvature-dimension inequality*, $\mathrm{CD}(\rho_1, \rho_2, d, m)$, if

$$\frac{1}{m}(\Delta_b f)^2 + \Big(\rho_1 - \frac{d}{\nu}\Big)\Gamma(f,f) + \rho_2\Gamma^Z(f,f) \leq \Gamma_2(f,f) + \nu\Gamma_2^Z(f,f)$$

for any smooth function $f \in C^\infty(M)$ and $\nu > 0$.

Here we have

$$\Gamma(f,\ f) := \sum_{j\in I_{2n}} |e_j f|^2, \quad \Gamma^Z(f,\ f) := |\mathbf{T}f|^2,$$

$$\Gamma_2(f,\ f) := \frac{1}{2}\Big[\Delta_b(\Gamma(f,\ f)) - 2\sum_{j\in I_{2n}}(e_j f)(e_j\Delta_b f)\Big],$$

$$\Gamma_2^Z(f,\ f) := \frac{1}{2}\big[\Delta_b(\Gamma^Z(f,\ f)) - 2(\mathbf{T}f)(\mathbf{T}\Delta_b f)\big].$$

Note that

$$\Gamma_2(f,\,f) = \sum_{i,j\in I_{2n}} |e_i e_j f|^2 + \sum_{j\in I_{2n}} (e_j f)([\Delta_b, e_j]f),$$
$$\Gamma_2^Z(f,\,f) = \sum_{i\in I_{2n}} |e_i \mathbf{T} f|^2 + (\mathbf{T}f)([\Delta_b, \mathbf{T}]f).$$

Now we proceed to derive a CR curvature-dimension inequality in a closed pseudohermitian $(2n+1)$-manifold under some specific assumptions on the pseudohermitian Ricci curvature tensor and the torsion tensor. We will first prove the the following key lemma.

Lemma 3.2 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with $\mathrm{Ric}(X, X) \geq k_0\langle X, X\rangle_{L_\theta}$, $\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty$, and $\sup_{i,j\in I_n} |A_{ij,\bar{\imath}}|^2 \leq k_2 < \infty$, for $X = X^\alpha Z_\alpha \in T_{1,0}M$, $I_n = \{1, 2, \dots, n\}$, and where k_0, k_1, k_2 are constants with $k_1, k_2 \geq 0$. Then M satisfies the CR curvature-dimension inequality $\mathrm{CD}(\rho_1, \rho_2, 4, 2mn)$ for $1 < m < +\infty$, and $N > 0$ such that*

$$\rho_1 := -\kappa \quad \textit{and} \quad \rho_2 := \frac{2n}{m} - \frac{8n^3N^2}{\epsilon} - \frac{2mn^2N^2{k_1}^2}{m-1} > 0,$$

for $0 < \nu \leq N$ and a positive constant ϵ with $\kappa = 2[-k_0 + |n-2|k_1 + \epsilon k_2] \geq 0$.

Proof The CR Bochner formula (3.1) implies

$$\frac{1}{4}\Gamma_2(f,f) = |\,\mathrm{Hess}(f)|^2 + \big(2\,\mathrm{Ric} - (n-2)\,\mathrm{Tor}\big)\big((\nabla_b f)_c, (\nabla_b f)_c\big) + 2\langle J\nabla_b f, \nabla_b f_0\rangle.$$

With the equality $\frac{1}{2}\Gamma_2^Z(f,f) = |\nabla_b f_0|^2 + f_0[\Delta_b, T]f$, we have

(3.2)
$$\Gamma_2(f,f) + \nu\Gamma_2^Z(f,f) = 4\Big[|\,\mathrm{Hess}(f)|^2 + (2\,\mathrm{Ric} - (n-2)\,\mathrm{Tor})((\nabla_b f)_c, (\nabla_b f)_c) + 2\langle J\nabla_b f, \nabla_b f_0\rangle\Big] + 2\nu|\nabla_b f_0|^2 + 2\nu f_0[\Delta_b, T]f.$$

On the other hand, we have

$$|\,\mathrm{Hess}(f)|^2 = 2\Big(\sum_{i,j\in I_n} |f_{ij}|^2 + \sum i, j \in I_n |f_{i\bar{j}}|^2\Big) \geq \frac{1}{2n}|\Delta_b f|^2 + \frac{n}{2}|f_0|^2 \tag{3.3}$$

and

$$\langle J\nabla_b f, \nabla_b f_0\rangle \geq -\frac{|\nabla_b f|^2}{\nu} - \frac{\nu}{4}|\nabla_b f_0|^2. \tag{3.4}$$

Finally, it follows from the commutation relation [12] that

$$\Delta_b f_0 = (\Delta_b f)_0 + 2\big[\,(A_{\alpha\beta} f^\alpha)^\beta + (A_{\bar{\alpha}\bar{\beta}} f^{\bar{\alpha}})^{\bar{\beta}}\big]. \tag{3.5}$$

Then (3.2), (3.4), and (3.5) yield that

$$\begin{aligned}\Gamma_2(f,\ f)+\nu\Gamma_2^Z(f,\ f)\geq 8\Big[\sum_{\alpha,\beta}(|f_{\alpha\beta}|^2+|f_{\alpha\overline{\beta}}|^2)\Big]&\\ +\Big[4(k_0-(n-2)k_1)-\frac{8}{\nu}\Big]|\nabla_b f|^2&\\ -8\nu|f_0|\sum_{\alpha,\beta}|(A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta+A_{\overline{\alpha}\overline{\beta}}f_{\beta\alpha})|.&\end{aligned}$$

Next Young's inequality implies that

$$|f_0|(|A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta|+|A_{\overline{\alpha}\overline{\beta}}f_{\beta\alpha}|)\leq\frac{|f_0|^2}{4\varepsilon_1}+\varepsilon_1|A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta|^2+\frac{|f_0|^2}{4\varepsilon_2}+\varepsilon_2|A_{\overline{\alpha}\overline{\beta}}f_{\beta\alpha}|^2,$$

for any $\varepsilon_1,\varepsilon_2>0$. We can choose $\varepsilon_2=\frac{m-1}{mNk_1^2}$ for $m>1$ and N with $\nu\leq N$. This implies that $(1-N\varepsilon_2k_1^2)=\frac{1}{m}$.

It follows from (3.3) that for $\epsilon=4N\varepsilon_1 n$

$$\begin{aligned}\Gamma_2(f,\ f)+\nu\Gamma_2^Z(f,\ f)&\geq 8\sum_{\alpha,\beta}|f_{\alpha\overline{\beta}}|^2+8\sum_{\alpha,\beta}(1-\nu\varepsilon_2|A_{\overline{\alpha}\overline{\beta}}|^2)|f_{\beta\alpha}|^2\\ &\quad+\Big[4(k_0-(n-2)k_1)-\frac{8}{\nu}\Big]|\nabla_b f|^2\\ &\quad-2\nu\sum_{\alpha,\beta}\Big(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)|f_0|^2-8\nu\varepsilon_1\sum_{\alpha,\beta}|A_{\overline{\alpha}\overline{\beta},\alpha}f_\beta|^2\\ &\geq\frac{8}{m}\sum_{\alpha,\beta}(|f_{\alpha\overline{\beta}}|^2+|f_{\beta\alpha}|^2)+\Big(-2\kappa-\frac{8}{\nu}\Big)|\nabla_b f|^2\\ &\quad-2n^2N\Big(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)|f_0|^2\\ &\geq\frac{4}{m}\Big(\frac{1}{2n}|\Delta_b f|^2+\frac{n}{2}|f_0|^2\Big)+\Big(-2\kappa-\frac{8}{\nu}\Big)|\nabla_b f|^2\\ &\quad-2n^2N\Big(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_2}\Big)|f_0|^2\\ &\geq\frac{1}{2mn}(Lf)^2+\Big(-\kappa-\frac{4}{\nu}\Big)\Gamma(f,f)\\ &\quad+\Big(\frac{2n}{m}-\frac{2n^2N}{\varepsilon_1}-\frac{2mn^2N^2k_1^2}{m-1}\Big)\Gamma^Z(f,f).\end{aligned}$$

Now we can choose N sufficiently small so that either

$$\frac{2n}{m}-\frac{2n^2N}{\varepsilon_1}-\frac{2mn^2N^2k_1^2}{m-1}>0\quad\text{or}\quad\frac{2n}{m}-\frac{8n^3N^2}{\epsilon}-\frac{2mn^2N^2k_1^2}{m-1}>0.$$

This completes the proof of the lemma. ∎

We now apply the CR curvature-dimension inequality CD as in Lemma 3.2 and the subgradient estimate in [3, 11] for the semigroup solution $u(x,t)=P_tf(x)$ of the heat flow (3.9), and prove the following crucial estimate for the symmetric heat kernel $p(x,y,t)>0$ associated with the heat semigroup P_t. We also refer to [10, 11] for similar results for the general solution of the heat flow.

Proposition 3.3 *Let (M, J, θ) be a complete pseudohermitian $(2n+1)$-manifold with*

$$\mathrm{Ric}(X, X) \geq k_0 \langle X, X\rangle_{L_\theta},$$
$$\sup_{i,j\in I_n} |A_{ij}| \leq k_1 < \infty, \quad \text{and} \quad \sup_{i,j\in I_n} |A_{ij,\bar{i}}|^2 \leq k_2 < \infty,$$

for $X = X^\alpha Z_\alpha \in T_{1,0}M$, where k_0, k_1, k_2 are constants with $k_1, k_2 \geq 0$.

(i) *There exist positive constants $C_3(\rho_2)$, $C_4(\rho_2)$, $C_5(\rho_2)$ such that for $x, y \in M$ and $t > 0$,*

$$p(x, y, t) \leq \frac{C_3}{V_x(\sqrt{t})^{\frac{1}{2}} V_y(\sqrt{t})^{\frac{1}{2}}} \exp\Big(-C_5 \frac{d_{cc}^2(x, y)}{t} + C_4 \kappa t\Big). \tag{3.6}$$

(ii) *There exist positive constants $C_6(\rho_2)$, $C_7(\rho_2)$, $C_8(\rho_2)$ such that for $x, y \in M$ and $t > 0$,*

$$p(x, y, t) \geq \frac{C_6}{V_x(\sqrt{t})} \exp\Big[-C_7 \frac{{d_{cc}}^2(x, y)}{t} - C_8 \kappa(t + d_{cc}^2(x, y))\Big]. \tag{3.7}$$

(iii) *There exist positive constants $C_9(\rho_2)$, $C_{10}(\rho_2)$ such that for $0 < s < t$,*

$$\frac{p(x, x, s)}{p(x, x, t)} \leq \Big(\frac{t}{s}\Big)^{C_9} e^{C_{10}\kappa(t-s)}. \tag{3.8}$$

Here $C_9 = mn(1 + \frac{6}{\rho_2})$, $\rho_2 > 0$ and $\kappa = \kappa(k_0, k_1, k_2) \geq 0$ are constants as in Lemma 3.2. In addition, if the torsion is vanishing, $C_9 = n + 3$ with $\rho_2 = 2n$ and $m = 1$.

Proof We will use the semigroup method from [3]. It is known that the heat semigroup $(P_t)_{t\geq 0}$ is given by $P_t = \int_0^\infty e^{-\lambda t}\, dE_\lambda$ for the spectral decomposition of $\Delta_b = -\int_0^\infty \lambda\, dE_\lambda$ in $L^2(M)$. It is a one-parameter family of bounded operators on $L^2(M)$. We define $P_t f(x) = \int_M p(x, y, t) f(y)\, d\mu(y)$. Here $p(x, y, t) > 0$ is the so-called symmetric heat kernel associated with P_t. Due to the hypoellipticity of Δ_b, the function $(x, t) \to P_t f(x)$ is smooth on $M \times (0, \infty)$, $f \in C_0^\infty(M)$. Moreover $u(x, t) = P_t f(x)$ is a solution of the CR heat equation

$$\begin{aligned} \frac{\partial}{\partial t} u &= \Delta_b u, \\ u(x, 0) &= f(x). \end{aligned} \tag{3.9}$$

In [3, 11], the subgradient estimate was derived for the solution $P_t f(x)$ of the CR heat equation (3.9) on $M \times [0, T)$ for arbitrary T with $\nu = (T - t)^3$, if M satisfies the CR curvature-dimension inequality CD . However, we choose t sufficiently close to the arbitrary T so that $0 < \nu \leq N$. It follows from Lemma 3.2 that M satisfies the Baudoin–Garofalo type curvature-dimension inequality $\mathrm{CD}(\rho_1, \rho_2, 4, 2mn)$ for $1 < m < +\infty$, and smaller $N > 0$ such that

$$\rho_2 := \frac{2n}{m} - \frac{8n^3N^2}{\epsilon} - \frac{2mn^2N^2k_1^2}{m-1} > 0. \tag{3.10}$$

It follows from [3] that we have the desired upper and lower estimates for the CR heat kernel on $M \times [0, T)$, as stated in the proposition. ∎

As a consequence of the previous estimate, we will derive the CR volume doubling property.

Theorem 3.4 *Under the hypotheses of Proposition 3.3, for any $\sigma > 1$, there exists a positive constant C_1 such that $V_x(\sigma\rho) \leq C_1\sigma^{2C_9}e^{(C_1\sigma^2+C_8)\kappa\rho^2}V_x(\rho)$.*

Proof It follows from Proposition 3.3 that $p(x,x,t) \leq \frac{C_3}{V_x(\sqrt{t})}e^{C_4\kappa t}$. On the other hand, by applying (3.8) for $0 < \tau < t$, we have

$$p(x,x,\tau) \leq p(x,x,t)\Big(\frac{t}{\tau}\Big)^{C_9}e^{C_{10}\kappa(t-\tau)}.$$

Accordingly,

$$p(x,x,\tau) \leq \Big(\frac{t}{\tau}\Big)^{C_9}\frac{C_3}{V_x(\sqrt{t})}e^{C_{10}\kappa(t-\tau)+C_4\kappa t} \leq \Big(\frac{t}{\tau}\Big)^{C_9}\frac{C_3}{V_x(\sqrt{t})}e^{(C_{10}+C_4)\kappa t},$$

and then for $\rho_0 < \rho$, we have $V_x(\rho) \leq (\frac{\rho}{\rho_0})^{2C_9}\frac{C_3}{p(x,x,\rho_0^2)}e^{(C_{10}+C_4)\kappa\rho^2}$. This implies

$$V_x(\sigma\rho) \leq (\sigma)^{2C_9}\frac{C_3}{p(x,x,\rho^2)}e^{(C_{10}+C_4)\kappa\sigma^2\rho^2}.$$

But from (3.7), we have $p(x,x,\rho^2) \geq \frac{C_6}{V_x(\rho)}e^{-C_8\kappa\rho^2}$. Accordingly,

$$V_x(\sigma\rho) \leq (\sigma)^{2C_9}\frac{C_3}{C_6}e^{[(C_{10}+C_4)\sigma^2+C_8]\kappa\rho^2}V_x(\rho),$$

and then $V_x(\sigma\rho) \leq C_1\sigma^{2C_9}e^{(C_1\sigma^2+C_8)\kappa\rho^2}V_x(\rho)$. The proof of the theorem is, therefore, complete. ∎

Let $H^{B_{x_0},D}(x,y,t)$ be the Dirichlet heat kernel on the geodesic ball $B_{x_0} = B_{x_0}(r)$ with $x, y \in B_{x_0}(r)$.

Theorem 3.5 *Under the hypothesis of Proposition 3.3, for $r^2 < T$, we have*

$$H^{B_{x_0},D}(x,y,t) \leq \frac{C'}{V_{x_0}(r)}r^Q t^{-\frac{Q}{2}}e^{C'\kappa r^2},$$

where $Q = 3mn(1+\frac{6}{p_2})$.

Proof It follows from the volume doubling property that

$$\begin{aligned}V_x(\sqrt{t}) &\leq V_y(\sqrt{t}+d_{cc}(x,y))\\ &\leq C_1e^{[2C_1(1+\frac{d_{cc}^2(x,y)}{t})+C_8]\kappa t}\Big(1+\frac{d_{cc}(x,y)}{\sqrt{t}}\Big)^{2C_9}V_y(\sqrt{t})\\ &\leq C_1e^{C(d_{cc}^2(x,y)+t)\kappa}\Big(1+\frac{d_{cc}(x,y)}{\sqrt{t}}\Big)^{2C_9}V_y(\sqrt{t}).\end{aligned}$$

On the other hand, (3.6) yields that

$$p(x,y,t) \leq \frac{C_3}{V_x(\sqrt{t})^{\frac{1}{2}}V_y(\sqrt{t})^{\frac{1}{2}}}\exp\Big(-C_5\frac{d_{cc}^2(x,y)}{t}+C_4\kappa t\Big).$$

All these imply that

$$p(x,y,t) \le C_1 \frac{1}{V_x(\sqrt{t})}\Big(1+\frac{d_{cc}(x,y)}{\sqrt{t}}\Big)^{C_9} e^{C\kappa(t+d_{cc}^2(x,y))-C_5\frac{d_{cc}^2(x,y)}{t}}.$$

Theorem 12.2 of [31] implies that

$$H^{B_{x_0},D}(x,y,t) \le C_1 \frac{1}{V_x(\sqrt{t})}\Big(1+\frac{d_{cc}(x,y)}{\sqrt{t}}\Big)^{C_9} e^{C\kappa(t+d_{cc}^2(x,y))-C_5\frac{d_{cc}^2(x,y)}{t}}. \tag{3.11}$$

Now for $t \le r^2$ and $x, y \in B_{x_0}(r)$, the doubling property yields

$$V_{x_0}(r) \le V_x(2r) \le C_1 e^{C\kappa r^2}\Big(\frac{r}{\sqrt{t}}\Big)^{2C_9} V_x(\sqrt{t}). \tag{3.12}$$

From (3.11) and (3.12), we have

$$\begin{aligned} H^{B_{x_0},D}(x,y,t) &\le \frac{C'}{V_{x_0}(r)} r^{2C_9} t^{-C_9} e^{C'\kappa r^2}\Big(1+\frac{d_{cc}(x,y)}{\sqrt{t}}\Big)^{C_9} \\ &\le \frac{C'}{V_{x_0}(r)} r^{3C_9} t^{-\frac{3C_9}{2}} e^{C'\kappa r^2} \\ &\le \frac{C'}{V_{x_0}(r)} r^{Q} t^{-\frac{Q}{2}} e^{C'\kappa r^2}, \end{aligned}$$

where $Q := 3C_9 = 3mn(1+\frac{6}{\rho_2})$. This completes the proof of the theorem. ∎

Finally, applying the volume doubling property and the upper bound estimate of the heat kernel, we can prove the CR analogue of the Sobolev inequality based on the methods of Li [31] and Saloff-Coste [39].

Theorem 3.6 *Under the hypothesis of Proposition 3.3, for any $\varphi \in C_0^\infty(B_x(r))$, $x \in M$, we have*

$$\begin{aligned} &\Big(\frac{1}{V_x(r)}\int_{B_x(r)} |\varphi|^{\frac{2Q}{Q-2}}\,d\mu\Big)^{\frac{Q-2}{Q}} \\ &\qquad \le C r^2 e^{C\kappa r^2}\Big[\frac{1}{V_x(r)}\Big(\int_{B_x(r)} |\nabla_b\varphi|^2\,d\mu + r^{-2}\int_{B_x(r)} \varphi^2\,d\mu\Big)\Big], \end{aligned}$$

where $Q = 3mn(1+\frac{6}{\rho_2})$.

Proof Let $\{\phi_i\}$ be the set of orthonormal eigenfunctions with eigenvalues $\{\mu_i\}$ with respect to the sub-Laplacian operator. We have $\varphi = \sum_{i=1}^\infty a_i\phi_i$, for any $\varphi \in C_0^\infty(B_x(r))$, $x \in M$, and then $\triangle_b\varphi = -\sum_{i=1}^\infty \mu_i a_i \phi_i$. We define the pseudo-differential operator $(-\triangle_b)^\alpha$ by $(-\triangle_b)^\alpha\varphi = \sum_{i=1}^\infty \mu_i^\alpha a_i\phi_i$, for any $\alpha \in R$. So we have

$$\int_{B_x(r)} |\nabla_b\varphi|^2\,d\mu = -\int_{B_x(r)} \varphi \triangle_b \varphi\,d\mu = \sum_{i=1}^\infty \mu_i a_i^2 = \int_{B_x(r)} |(-\triangle_b)^{\frac12}\varphi|^2\,d\mu.$$

Let $A_1 = \frac{C}{V_x(r)} r^Q e^{C\kappa r^2}$; that is $H^{B_{x_0},D}(x,y,t) \le A_1 t^{-\frac{Q}{2}}$. In the following, we first prove that

$$\int_{B_x(r)} |(-\triangle_b)^{\frac{-1}{2}}\varphi|^2\,d\mu = \int_{B_x(r)} |\nabla_b\varphi|^2\,d\mu \ge C_1 A_1^{-\frac{2}{Q}}\Big(\int_{B_x(r)} |\varphi|^{\frac{2Q}{Q-2}}\,d\mu\Big)^{\frac{Q-2}{Q}}. \tag{3.13}$$

Let $\psi = (-\triangle_b)^{\frac{1}{2}}\varphi$. Then formally $\varphi = (-\triangle_b)^{\frac{-1}{2}}\psi$. It suffices to prove the following:

$$\int_{B_x(r)} |\psi|^2\, d\mu \geq C_1 A_1^{-\frac{2}{Q}} \Big(\int_{B_x(r)} |(-\triangle_b)^{-\frac{1}{2}}\psi|^{\frac{2Q}{Q-2}}\, d\mu \Big)^{\frac{Q-2}{Q}}.$$

That is, the operator $(-\triangle_b)^{-\frac{1}{2}}: L^2(B_x(r)) \to L^{\frac{2Q}{Q-2}}(B_x(r))$ is bounded and satisfies the inequality (3.13).

Let $\psi = \sum_{i=1}^{\infty} b_i\phi_i$. Then we have

$$\begin{aligned}
(-\triangle_b)^{-\frac{1}{2}}\psi(y) &= \sum_{i=1}^{\infty} \mu_i^{-\frac{1}{2}} b_i\phi_i(y) = \int_{B_x(r)} \sum_{i=1}^{\infty} \mu_i^{-\frac{1}{2}}\phi_i(y)\phi_i(z)\psi(y)\, d\mu_z \\
&= B\int_0^{\infty}\int_{B_x(r)} t^{-\frac{1}{2}} e^{-\mu_i t}\phi_i(y)\phi_i(z)\psi(z)\, d\mu_z dt \\
&= B\int_0^{\infty} t^{-\frac{1}{2}} \int_{B_x(r)} H^{B_0,D}(y,z,t)\psi(z)\, d\mu_z dt,
\end{aligned}$$

where $B^{-1} = \int_0^{\infty} s^{-\frac{1}{2}} e^{-s}\, ds = \Gamma(\frac{1}{2})$. For $0 < T < \infty$, define the functions T_1 and T_2 by

$$T_1(\psi) = B\int_0^{T} t^{-\frac{1}{2}} \int_{B_x(r)} H^{B_{x_0},D}(y,z,t)\, d\mu_z dt$$

and

$$T_2(\psi) = B\int_T^{\infty} t^{-\frac{1}{2}} \int_{B_x(r)} H^{B_{x_0},D}(y,z,t)\psi(z)\, d\mu_z dt,$$

respectively. Hence, we have $(-\triangle_b)^{-\frac{1}{2}} = T_1 + T_2$. Then, similar to part of the process in the proof of [31, Theorem 11.4], we can prove that the inequality (3.13) holds.

To prove the Sobolev inequality, let ξ be a non-negative cut-off function with the following properties:

$$|\nabla_b\xi|^2 \leq Cr^{-2} \quad \text{and} \quad \xi(y) = \begin{cases} 1 & \text{if } y \in B_x(\frac{3r}{4}), \\ 0 & \text{if } y \in M\backslash B_x(r). \end{cases}$$

Then (3.13) yields

$$\begin{aligned}
2\int_{B_x(r)} |\nabla_b\varphi|^2\, d\mu + 2Cr^{-2}\int_{B_x(r)} \varphi^2\, d\mu &\geq \int_{B_x(r)} \xi^2|\nabla_b\varphi|^2\, d\mu + 2\int_{B_x(r)} |\xi\varphi|^2\, d\mu \\
&\geq \int_{B_x(r)} |\nabla_b(\xi\varphi)|^2\, d\mu \\
&\geq C_1 A_1^{-\frac{2}{Q}} \Big(\int_{B_x(r)} |\xi\varphi|^{\frac{2Q}{Q-2}}\, d\mu \Big)^{\frac{Q-2}{Q}} \\
&\geq C_1 A_1^{-\frac{2}{Q}} \Big(\int_{B_x(\frac{3r}{4})} |\varphi|^{\frac{2Q}{Q-2}}\, d\mu \Big)^{\frac{Q-2}{Q}}.
\end{aligned}$$

If $y \in \partial B_x(\frac{r}{2})$, then $\{B_y(\frac{3r}{4}) : y \in \partial B_x(\frac{r}{2})\}$ forms an open cover of the closure of $B_x(r)$. There exists a positive integer l such that $B_x(r) \subset \bigcup_{i=1}^{l} B_{y_i}(\frac{3r}{4})$. For any $\varphi \in C_0^{\infty}(B_x(r))$, we have

$$2\int_{B_x(r)} |\nabla_b\varphi|^2\, d\mu + 2Cr^{-2}\int_{B_x(r)} \varphi^2\, d\mu \geq C_1 A_1^{-\frac{2}{Q}} \Big(\int_{B_x(r)\cap B_{y_i}(\frac{3r}{4})} |\varphi|^{\frac{2Q}{Q-2}}\, d\mu \Big)^{\frac{Q-2}{Q}}.$$

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Therefore, we have

$$2l \int_{B_x(r)} |\nabla_b \varphi|^2 \, d\mu + 2lCr^{-2} \int_{B_x(r)} \varphi^2 d\mu \geq C_1 A_1^{-\frac{2}{Q}} \sum_{i=1}^{l} \Big(\int_{B_x(r) \cap B_{y_i}(\frac{3r}{4})} |\varphi|^{\frac{2Q}{Q-2}} \, d\mu \Big)^{\frac{Q-2}{Q}}$$
$$\geq C_1 A_1^{-\frac{2}{Q}} \Big(\int_{B_x(r)} |\varphi|^{\frac{2Q}{Q-2}} \, d\mu \Big)^{\frac{Q-2}{Q}}.$$

Finally, we conclude that

$$\int_{B_x(r)} |\nabla_b \varphi|^2 \, d\mu + r^{-2} \int_{B_x(r)} \varphi^2 \, d\mu \geq \widetilde{C} A_1^{-\frac{2}{Q}} \Big(\int_{B_x(r)} |\varphi|^{\frac{2Q}{Q-2}} \, d\mu \Big)^{\frac{Q-2}{Q}}.$$

This completes the proof of this theorem. ∎

4 A Mean Value Inequality

In this section, we will apply the volume doubling estimate (1.1) and the CR Sobolev inequality to obtain the following mean value inequality through the method of Moser's iteration [40].

Theorem 4.1 *Under the hypothesis of Proposition 3.3, there exists a constant $C_2 > 0$ such that for any $\rho > 0$, $x \in M$, and any nonnegative subpseudoharmonic function f defined on M, we have*

$$[f(x)]^2 \leq C_2 V_x^{-1}(\rho) e^{C_2 \kappa \rho^2} \int_{B_x(\rho)} f(y)^2 \, d\mu.$$

Proof For any nonnegative subpseudoharmonic function f on M, we have

$$\int_{B_x(\rho)} \nabla_b f \nabla_b \phi \, d\mu \leq 0,$$

where ϕ is a nonnegative function in $C_0^\infty(B_x(\rho))$. For $\phi = \psi^2 f$ and $\psi \in C_0^\infty(B_x(\rho))$, we have

$$0 \geq \int_{B_x(\rho)} \nabla_b f \nabla_b (\psi^2 f) \, d\mu = \int_{B_x(\rho)} \psi^2 |\nabla_b f|^2 \, d\mu + 2 \int_M f \psi \nabla_b f \nabla_b \psi \, d\mu.$$

Thus, we have

$$\int_{B_x(\rho)} \psi^2 |\nabla_b f|^2 \, d\mu \leq 2 \Big| \int_M f \psi \nabla_b f \nabla_b \psi \, d\mu \Big|$$
$$\leq 2 \int_{B_x(\rho)} |\nabla_b \psi|^2 f^2 \, d\mu + \frac{1}{2} \int_M \psi^2 |\nabla_b f|^2 \, d\mu.$$

It follows that

$$\int_{B_x(\rho)} \psi^2 |\nabla_b f|^2 \, d\mu \leq 4 \int_{B_x(\rho)} |\nabla_b \psi|^2 f^2 \, d\mu \leq 4 \|\nabla_b \psi\|_\infty^2 \int_{\mathrm{supp}(\psi)} f^2 \, d\mu.$$

We choose ψ such that

$$0 \leq \psi \leq 1, \quad \mathrm{supp}(\psi) \subset \sigma B_x(\rho), \quad \psi = 1 \text{ in } \sigma' B_x(\rho), \quad |\nabla_b \psi| \leq 2((\sigma - \sigma')\rho)^{-1},$$

where $0 < \sigma' < \sigma < 1$, so we have

$$\int_{\sigma' B_x(\rho)} |\nabla_b f|^2 \, d\mu \leq C(\rho\omega)^{-2} \int_{\sigma B_x(\rho)} f^2 \, d\mu, \quad \omega = \sigma - \sigma'.$$

Using the CR Sobolev inequality, set $q = \frac{Q}{Q-2}$. From the Hölder inequality, we have

$$\int_{B_x(\rho)} f^{2(1+\frac{2}{Q})}\, d\mu \le \Big(\int_{B_x(\rho)} f^{2q}\, d\mu \Big)^{\frac{1}{q}} \Big(\int_{B_x(\rho)} f^2\, d\mu \Big)^{\frac{2}{Q}}$$
$$\le \Big(\int_{B_x(\rho)} f^2\, d\mu \Big)^{\frac{2}{Q}} \Big(E(B) \int_{B_x(\rho)} [|\nabla_b f|^2 + r^{-2}|f|^2]\, d\mu \Big),$$

for any $u \in C_0^\infty(B_x(\rho))$ and $E(B) = C_s \rho^2 e^{C\kappa\rho^2} V_x(\rho)^{-\frac{2}{Q}}$. Hence, we have

$$\int_{\sigma' B_x(\rho)} f^{2(1+\frac{2}{Q})}\, d\mu \le E(B) \Big(C(\rho\omega)^{-2} \int_{\sigma B_x(\rho)} f^2\, d\mu \Big)^{(1+\frac{2}{Q})}.$$

Setting $\theta = 1 + \frac{2}{Q}$, we have

$$\int_{\sigma' B_x(\rho)} f^{2\theta}\, d\mu \le E(B) \Big(C(\rho\omega)^{-2} \int_{\sigma B_x(\rho)} f^2\, d\mu \Big)^{\theta}. \tag{4.1}$$

On the other hand, for $p \ge 1$, we have

$$-\triangle_b f^p = -p f^{p-1} \triangle_b f - p(p-1) f^{p-2} |\nabla_b f|^2 \le 0.$$

Then for the function f^p in (4.1), we have

$$\int_{\sigma' B_x(\rho)} f^{2p\theta}\, d\mu \le E(B) \Big(C(\rho\omega)^{-2} \int_{\sigma B_x(\rho)} f^{2p}\, d\mu \Big)^{\theta}. \tag{4.2}$$

Let $\omega_i = (1-\delta)2^{-i}$ such that $\sum_{i=1}^{\infty} \omega_i = 1-\delta$. Set $\sigma_0 = 1$ and $\sigma_{i+1} = \sigma_i - \omega_i = 1 - \sum_{i=1}^{i} \omega_j$. Applying (4.2) with $p = p_i = \theta^i$, $\sigma = \sigma_i$ and $\sigma' = \sigma_{i+1}$, we have

$$\int_{\sigma_{i+1} B_x(\rho)} f^{2\theta^{i+1}}\, d\mu \le E(B) \Big(C^{i+1} (\rho(1-\delta))^{-2} \int_{\sigma_i B_x(\rho)} f^{2\theta^i}\, d\mu \Big)^{\theta}, \quad C \ge 4$$

and

$$\Big(\int_{\sigma_{i+1} B_x(\rho)} f^{2\theta^{i+1}}\, d\mu \Big)^{\theta^{-i-1}}$$
$$\le E(B)^{\sum_{j=1}^{i+1} \theta^{-j-1}} C^{\sum_{j=1}^{i+1} (j+1)\theta^{-j-1}} (\rho(1-\delta))^{-2\sum_{j=1}^{i+1} \theta^{-j}} \int_{B_x(\rho)} f^2\, d\mu.$$

Letting i tend to ∞, we have

$$\sup_{\delta B_x(\rho)} \{f^2\} \le C E(B)^{\frac{Q}{2}} [(1-\delta)]^{-Q} \rho^{-Q} \int_{B_x(\rho)} f^2\, d\mu.$$

Taking $\delta = \frac{1}{2}$, we have $f^2(x) \le C_2 V_x^{-1}(\rho) e^{C_2\kappa\rho^2} \int_{B_x(\rho)} f^2(y)\, d\mu$. ∎

5 Polynomial Growth Pseudoharmonic Functions

In this section, we will prove our main result. We first recall [31, Lemma 28.3].

Lemma 5.1 *Let K be a k-dimensional linear space of sections of a vector bundle E over M. Assume that M has polynomial volume growth of order at most μ,* i.e., *$V_p(\rho) \le C\rho^\mu$ for $p \in M$ and $\rho \to \infty$. Suppose each section $u \in K$ is of polynomial growth at most degree d, such that $|u|(x) \le Cr^d(x)$, where $r(x)$ is the Carnot–Carathéodory distance to the fixed point $p \in M$. For any $\beta > 1$, $\delta > 0$, and $\rho_0 > 0$, there exists $\rho > \rho_0$ such that*

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if $\{u_i\}_{i=1}^k$ *is an orthonormal basis of* K *with respect to the inner product* $A_{\beta\rho}(u,v) = \int_{B_p(\beta\rho)} \langle u, v\rangle \, d\mu$, *then* $\sum_{i=1}^k \int_{B_p(\rho)} |u_i|^2 \, d\mu \geq k\beta^{-(2d+\mu+\delta)}$.

In the following, we prove the main result by applying the volume doubling property and the mean value inequality.

Theorem 5.2 *Assume the hypothesis of Proposition 3.3 with* $\kappa = 0$. *Suppose* E *is a rank-*m *vector bundle over* M. *Let* $S_d(M,E) \subset \Gamma(E)$ *be a linear subspace of sections* u *of* E *satisfying* $\triangle_b |u| \geq 0$ *and* $|u|(x) \leq O(r^d(x))$ *as* $r(x) \to \infty$.

Then the dimension of $S_d(M,E)$ *is finite. Moreover, there exists a constant* $C > 0$ *depending only on* C_9 *such that*

$$\dim S_d(M,E) \leq mCC_{\mathcal{M}} d^{\frac{2n}{2n+1}(2C_9-1)} \tag{5.1}$$

for all $d \geq 1$.

Proof From the volume doubling property, we have the comparison inequality

$$V_p(\rho_2) \leq C_1 V_p(\rho_1) \Big(\frac{\rho_2}{\rho_1}\Big)^{2C_9}, \tag{5.2}$$

and then we have

$$V_p(\rho) \leq C\rho^{2C_9}. \tag{5.3}$$

On the other hand, we also have the mean value inequality

$$f^2(x) \leq C_{\mathcal{M}} V_p^{-1}(\rho) \int_{B_p(\rho)} f^2(y)\, d\mu.$$

Let K be a finite-dimensional linear subspace of $S_d(M,E)$ with $\dim K = k$ and let $\{u_i\}_{i=1}^k$ be any basis of K. Then for $p \in M$, $\rho > 0$, and any $0 < \epsilon < 1$, to complete the proof of the theorem, it suffices to show that

$$\sum_{i=1}^k \int_{B_p(\rho)} |u_i|^2 \, d\mu \leq mCC_{\mathcal{M}} \sup_{u \in \{\langle A,U\rangle\}} \int_{B_p((1+\epsilon)\rho)} |u|^2 \, d\mu, \tag{5.4}$$

where the supremum is taken over all $u \in K$ of the form $u = \langle A, U\rangle$ for some unit vector $A = (a_1,\dots,a_k) \in R^k$ with $U = (u_1,\dots,u_k)$. We will prove (5.4) later. To finish the proof of Theorem 5.2, let $\{u_i\}_{i=1}^k$ be an $A_{\beta\rho}$-orthonormal basis of any finite-dimensional subspace $K \subset S_d(M,E)$. By applying (5.3) and Lemma 5.1, there exists a $\rho > 0$ such that

$$\sum_{i=1}^k \int_{B_p(\rho)} |u_i|^2 \, d\mu \geq k\beta^{-(2d+2C_9+\delta)}. \tag{5.5}$$

Since $\int_{B((1+\epsilon)\rho)} |u|^2 = 1$ for all $u \in \{\langle A, U\rangle\}$, it follows from the inequality (5.4) that by setting $\beta = 1+\epsilon$, we have $\sum_{i=1}^k \int_{B_p(\rho)} |u_i|^2 d\mu \leq mCC_{\mathcal{M}}\epsilon^{-(2C_9-1)}$. For $d \geq 1$, setting

$$\epsilon = (2d)^{-\frac{2n}{2n+1}} \tag{5.6}$$

combined with (5.5) gives us $\sum_{i=1}^k \int_{B_p(\rho)} |u_i|^2 d\mu \geq Ck$. Therefore, the estimate (5.1) on k follows easily. Note that extra care for the power in (5.6) is used to obtain the order of the power $\frac{2n}{2n+1}(2C_9 - 1)$ in Theorem 5.2, because this is sharp on the Heisenberg group $\mathbf{H}_n$, as shown in the next section.

Finally, we prove 5.4 by following the method in [31]. For completeness, we will outline it here. We first observe that for any $x \in B_p(\rho)$, there exists a subspace $K_x \subset K$ which is of, at most, codimension m, such that $u(x) = 0$ for all $u \in K_x$. Hence, by an orthonormal change of basis, we can assume that $u_i \in K_x$ for $m+1 \le i \le k$ and $\sum_{i=1}^k |u_i|^2(x) = \sum_{i=1}^n |u_i|^2(x)$. Since $\triangle_b |u_i| \ge 0$, it follows from the CR mean value inequality that

$$
\begin{aligned}
\sum_{i=1}^{k} |u_i|^2(x) &= \sum_{i=1}^{n} |u_i|^2(x) \\
&\le C_{\mathcal{M}} V_x^{-1}((1+\epsilon)\rho - r(x)) \sum_{i=1}^{m} \int_{B_x((1+\epsilon)\rho - r(x))} |u_i|^2 \, d\mu \\
&\le C_{\mathcal{M}} V_x^{-1}((1+\epsilon)\rho - r(x)) \sup_{u \in \{\langle A, U\rangle\}} \int_{B_x((1+\epsilon)\rho - r(x))} |u|^2 \, d\mu.
\end{aligned} \tag{5.7}
$$

The volume doubling property (5.2) and the fact that $r(x) \le \rho$ imply that

$$
\begin{aligned}
C_1 V_p((1+\epsilon)\rho - r(x)) &\ge V_p(2\rho)\Big(\frac{(1+\epsilon)\rho - r(x)}{2\rho}\Big)^{2C_9} \\
&\ge V_p(\rho)\Big(\frac{(1+\epsilon)\rho - r(x)}{2\rho}\Big)^{2C_9}.
\end{aligned} \tag{5.8}
$$

From (5.7) and (5.8), we have

$$
\begin{aligned}
&\sum_{i=1}^{k} \int_{B_p(\rho)} |u_i|^2 \, d\mu \\
&\le \frac{mC_1 C_{\mathcal{M}} 2^{2C_9}}{V_p(\rho)} \sup_{u \in \{\langle A, U\rangle\}} \int_{B_p((1+\epsilon)\rho)} u^2 \, d\mu \int_{B_p(\rho)} \Big(\frac{(1+\epsilon)\rho - r(x)}{\rho}\Big)^{-2C_9} d\mu.
\end{aligned} \tag{5.9}
$$

Now we define $f(r) = ((1+\epsilon) - \rho^{-1} r)^{-2C_9}$. It follows that $f'(r) \ge 0$ and then

$$
\begin{aligned}
\int_{B_p(\rho)} \Big(\frac{(1+\epsilon)\rho - r(x)}{\rho}\Big)^{-2C_9} d\mu &= \int_0^{\rho} A_p(t) f(t) dt \\
&= [f(t)V_p(t)]|_0^{\rho} - \int_0^{\rho} f'(t) V_p(t)\, dt \\
&\le [f(t)V_p(t)]|_0^{\rho} - \rho^{-2C_9} V_p(\rho) \int_0^{\rho} f'(t) t^{2C_9}\, dt \\
&\le [f(t)V_p(t)]|_0^{\rho} - \rho^{-2C_9} V_p(\rho) \\
&\qquad \times \Big([f(t)t^{2C_9}]|_0^{\rho} - 2C_9 \int_0^{\rho} f(t) t^{2C_9 - 1}\, dt \Big) \\
&\le 2\rho^{-1} V_p(\rho) C_9 \int_0^{\rho} ((1+\epsilon) - t\rho^{-1})^{-2C_9}\, dt \\
&\le \frac{2C_9}{2C_9 - 1} V_p(\rho) \epsilon^{-(2C_9 - 1)}.
\end{aligned} \tag{5.10}
$$

Hence, (5.4) follows from (5.9) and (5.10). ∎

Corollary 5.3 *Under the hypotheses of Theorem 5.2, we conclude that the dimension of $\mathcal{H}^d(M)$ is finite. Moreover, there exists a constant $C_0 = C(C_{\mathcal{M}}, C_{\mathcal{V}}) > 0$ such that $h^d(M) \le C_0 d^{\frac{2n}{2n+1}(2C_9-1)}$, for all $d \ge 1$.*

6 Pseudoharmonic Functions of Polynomial Growth on Heisenberg Groups

We start with the most general definition of the Heisenberg group. The non-isotropic Heisenberg group $\mathbf{H}_n$ is the Lie group with underlying manifold

$$\mathbf{C}^n \times \mathbf{R} = \{[\mathbf{z}, t] : \mathbf{z} \in \mathbf{C}^n,\ t \in \mathbf{R}\}$$

and the multiplication law

$$[\mathbf{z}, t] \cdot [\mathbf{w}, s] = \Big[\mathbf{z} + \mathbf{w}, t + s + 2\mathrm{Im} \sum_{j=1}^{n} a_j z_j \overline{w}_j\Big], \tag{6.1}$$

where $\mathbf{a} = (a_1, a_2, \dots, a_n) \in \mathbf{R}_+^n$.

It is easy to check that the multiplication (6.1) does indeed make $\mathbf{C}^n \times \mathbf{R}$ into a group whose identity is the origin $e = [\mathbf{0}, 0]$, and where the inverse is given by $[\mathbf{z}, t]^{-1} = [-\mathbf{z}, -t]$.

The Lie algebra $\mathfrak{h}_n$ of $\mathbf{H}_n$ is a vector space that, together with a Lie bracket operation defined on it, represents the infinitesimal action of $\mathbf{H}_n$. Let $\mathfrak{h}_n$ denote the vector space of left-invariant vector fields on $\mathbf{H}_n$. Note that this linear space is closed with respect to the bracket operation $[\mathbf{V}_1,\ \mathbf{V}_2] = \mathbf{V}_1\mathbf{V}_2 - \mathbf{V}_2\mathbf{V}_1$. The space $\mathfrak{h}_n$, equipped with this bracket, is referred to as the Lie algebra of $\mathbf{H}_n$.The Lie algebra structure of $\mathfrak{h}_n$ is most readily understood by describing it in terms of the following basis:

$$\mathbf{X}_j = \frac{\partial}{\partial x_j} + 2a_j y_j \frac{\partial}{\partial t}, \quad \mathbf{Y}_j = \frac{\partial}{\partial y_j} - 2a_j x_j \frac{\partial}{\partial t}, \quad \text{and} \quad \mathbf{T} = \frac{\partial}{\partial t},$$

where $j = 1, 2, \dots, n$, $\mathbf{z} = (z_1, z_2, \dots, z_n) \in \mathbf{C}^n$ with $z_j = x_j + i y_j$; $t \in \mathbf{R}$.

Note that we have the commutation relations

$$[\mathbf{Y}_j, \mathbf{X}_k] = 4a_j \delta_{jk} \mathbf{T} \quad \text{for } j,\ k = 1, 2, \dots, n. \tag{6.2}$$

Next we define the complex vector fields

$$\begin{aligned} \overline{\mathbf{Z}}_j &= \frac{1}{2}(\mathbf{X}_j + i\mathbf{Y}_j) = \frac{\partial}{\partial \overline{z}_j} - i a_j z_j \frac{\partial}{\partial t}, \\ \mathbf{Z}_j &= \frac{1}{2}(\mathbf{X}_j - i\mathbf{Y}_j) = \frac{\partial}{\partial z_j} + i a_j \overline{z}_j \frac{\partial}{\partial t}, \end{aligned} \tag{6.3}$$

for $j = 1, 2, \dots, n$. Here, as usual, we have

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\Big) \quad \text{and} \quad \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}\Big(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\Big).$$

The commutation relations (6.2) then become $[\overline{\mathbf{Z}}_j, \mathbf{Z}_k] = 2i a_j \delta_{jk} \mathbf{T}$, with all other commutators among the $\mathbf{Z}_j$, $\overline{\mathbf{Z}}_k$ and $\mathbf{T}$ vanishing.

The Heisenberg sub-Laplacian is the differential operator

$$\mathcal{L}_{\mathbf{a},\gamma} = -\frac{1}{2}\sum_{j=1}^{n}(\mathbf{Z}_j\overline{\mathbf{Z}}_j + \overline{\mathbf{Z}}_j\mathbf{Z}_j) + \gamma\mathbf{T} = -\frac{1}{4}\sum_{j=1}^{n}(\mathbf{X}_j^2 + \mathbf{Y}_j^2) + i\gamma\mathbf{T}$$

with $\mathbf{Z}_j$ and $\overline{\mathbf{Z}}_j$ given by (6.3). In the case of $a_j = 1$ for all j', the operator $\mathcal{L}_\gamma$ was first introduced by Folland and Stein [22] in the study of $\overline{\partial}_b$ complex on a non-degenerate CR manifold. They found the fundamental solution of $\mathcal{L}_\gamma$. Beals and Greiner [4] solved the case that a_j' may be different. Readers can also consult [7, 8] for detailed discussions.

For functions $f,\ g \in \mathcal{S}(\mathbf{H}_n)$, the Heisenberg convolution is given by

$$f * g(\mathbf{x}) = \int_{\mathbf{H}_n} f(\mathbf{y})g(\mathbf{y}^{-1}\mathbf{x})dV(\mathbf{y}).$$

Here $dV(\mathbf{y})$ is the Haar measure on $\mathbf{H}_n$ that is exactly the Euclidean measure on $\mathbf{R}^{2n+1}$.

6.1 The Fundamental Solution

The fundamental solution and heat kernel of $\mathcal{L}_{\mathbf{a},\gamma}$ can be derived using Laguerre calculus. Following [5], we introduce the complex distance and volume element on the Heisenberg group:

$$g(s;\mathbf{z},t) = \sum_{j=1}^{n} a_j s|z_j|^2\coth(2a_j s) - it \quad \text{and} \quad v(s) = \prod_{j=1}^{n}\frac{2a_j}{\sinh(2a_j s)}.$$

The fundamental solution of $\mathcal{L}_{\mathbf{a},\gamma}$ can be written in a closed form:

$$\Psi(\mathbf{z},t) = \frac{2(n-1)!}{(2\pi)^{n+1}}\int_{-\infty}^{\infty} e^{2\gamma s}\frac{v(s)\,ds}{[g(s;\mathbf{z},t)]^n}. \tag{6.4}$$

If $|\mathbf{z}| = 0$ and $t \neq 0$, then $g(s;\mathbf{z},t) = -it$. The integrand of (6.4) is not integrable at $s = 0$. To regularize the integration we must deform its path of integration from $(-\infty,\infty)$ to

$$(-\infty + i\varepsilon\operatorname{sgn} t, \infty + i\varepsilon\operatorname{sgn} t), \quad \text{where} \quad 0 < \varepsilon < \min_{1\le j\le n}\frac{\pi}{2a_j}.$$

We refer to [4] for the exact definition of this path. Finally, we have the formula

$$\Psi(\mathbf{z},t) = \frac{2(n-1)!}{(2\pi)^{n+1}}\int_{-\infty+i\varepsilon\operatorname{sgn}t}^{\infty+i\varepsilon\operatorname{sgn}t} e^{2\gamma s}\frac{v(s)\,ds}{[g(s;\mathbf{z},t)]^n}.$$

6.2 The Heat Kernel

In the isotropic case, the heat kernel was independently studied by Gaveau [23] via a probability method and by Hulanicki [27] using the Fourier transform on $\mathbf{H}_n$ and the basis of Laguerre functions. Later, Beals and Greiner [4] solved the general case by the geometric mechanics method. We also refer readers to the the paper by Calin, Chang, and Tie [9] that used a different method:

(6.5)

$$h_s(\mathbf{z},t)=\frac{1}{2(\pi s)^{n+1}}\int_{-\infty}^{\infty}\Big[\prod_{j=1}^{n}\frac{a_j\tau}{\sinh(a_j\tau)}\Big]\exp\Big\{\frac{it\tau}{s}+\gamma\tau-\frac{1}{s}\sum_{j=1}^{n}\frac{a_j\tau}{\tanh(a_j\tau)}|z_j|^2\Big\}\,d\tau.$$

In this case, we set $a_1=a$, $n=1$, and $\gamma=0$ in (6.5) and the heat kernel then has the form

$$h_s(z,t)=\frac{1}{2(\pi s)^2}\int_{-\infty}^{\infty}\frac{a\tau}{\sinh(a\tau)}\exp\Big\{-\frac{\tau}{s}\Big[\frac{a}{\tanh(a\tau)}|z|^2-it\Big]\Big\}\,d\tau.$$

6.3 Spherical Harmonics on $\mathbf{H}_1$

Greiner [25] initiated the study of $\mathcal{L}_\gamma$-harmonic polynomials, *i.e.*, $\mathcal{L}_\gamma p=0$ on $\mathbf{H}_1$. He found a basis and proved that the linear space $\mathcal{H}_m^{(\gamma)}$ of H-homogeneous $\mathcal{L}_\gamma$-harmonic polynomials of degree m, $m=0,1,2,\dots$, has dimension $m+1$. Dunkl [21] derived the general formulas of $\mathcal{L}_\gamma$-harmonic polynomials for the isotropic Heisenberg group $\mathbf{H}_n$. Below, we will follow Dunkl's formulation of $\mathcal{H}_m^{(\gamma)}$. A basis of $\mathcal{H}_m^{(\gamma)}$ can be defined as follows. First, we define the *generalized ultraspherical polynomial* $C_N^{(\alpha,\beta)}(z)$ of degree N with the index (α,β) by its generating formula

$$(1-\rho\bar{z})^{-\alpha}(1-\rho z)^{-\beta}=\sum_{N=0}^{\infty}r^N C_N^{(\alpha,\beta)}(z)\quad\text{for } |\rho z|<1. \tag{6.6}$$

From the generating function (6.6), we have

$$C_N^{(\alpha,\beta)}(z)=\sum_{j=0}^{N}\frac{(\alpha)_j(\beta)_{N-j}}{j!(N-j)!}\bar{z}^j z^{N-j},\qquad N\in\mathbf{N},$$

where the shifted factorial $(a)_N$ is defined by

$$(a)_0=1,\quad (a)_{j+1}=(a)_j(a+j)=a(a+1)(a+2)\cdots(a+j)=\frac{\Gamma(a+j+1)}{\Gamma(a)}.$$

For the problem we are working on, a is always a nonnegative integer and we have

$$(a)_0=1\quad\text{and}\quad (a)_k=a(a+1)\cdots(a+k-1)=\frac{(a+k-1)!}{(a-1)!}.$$

Then for $k,l\in\mathbf{N}$, we can define $V_{k,l}$ to be the set of harmonic and homogeneous polynomials on $\mathbf{C}^n$ of bidegree (k,l), *i.e.*, $p(z,\bar{z})$ satisfies $p(cz)=c^k\bar{c}^l p(z)$ for all $c\in\mathbf{C}$, and $\sum_{j=1}^{n}\frac{\partial^2 p}{\partial z_j\partial\bar{z}_j}=0$. So $V_{k,l}$ is an irreducible $U(n)$-module of dimension:

$$v_{k,l}=\frac{k+l+n-1}{n-1}\cdot\frac{(n-1)_k(n-1)_l}{k!l!}.$$

For all $p\in V_{k,l}$, we have $\mathcal{L}_\gamma([p(z)C_N^{(\alpha,\beta)}(t+i|z|^2)])=0$, where $\alpha=\frac{n-\gamma}{2}$, $\beta=\frac{n+\gamma}{2}$, $N\in\mathbf{N}$, and every $\mathcal{L}_\gamma$-harmonic polynomial is a linear combination of such terms for all $k,l,N\in\mathbf{N}$.

For $p\in V_{k,l}$, $p(z)C_N^{(\alpha,\beta)}(t+i|z|^2)$ is H-homogeneous of degree $2N+k+l$. We will find the dimension of the linear space $\mathcal{H}_m^{(\gamma)}$ of H-homogeneous $\mathcal{L}_\gamma$-harmonic polynomials of degree m for $n > 1$ from the dimension of $V_{k,l}$. The problem is to find how many $(N, k, l) \in \mathbf{N}^3$ satisfy $2N + k + l = m$ for a fixed $m \in \mathbf{N}$.

Koranyi and Stanton [29] proved that if $|\gamma| < n$ and a $\mathcal{L}_\gamma$-harmonic function f on $\mathbf{H}_n$ is majorized by a polynomial, then f must be a polynomial.

We give an outline of the computations of the $\mathcal{L}_\gamma$-harmonic polynomials. Because the space of $\mathcal{L}_\gamma$-harmonic polynomials is a $U(n)$-module, it can be decomposed into copies of $V_{k,l}$, with $k, l \in \mathbf{N}$. And then every $\mathcal{L}_\gamma$-harmonic function is a sum of the terms like $p(z)g_1(z,t)$, where $p \in V_{k,l}$ and g_1 is invariant under $U(n)$. Hence, one has $g_1(z,t) = g_2(|z|^2, t)$ for some $g_2(s,t)$. Furthermore, Greiner observed that $g_2(|z|^2, t) = g(t + i|z|^2)$ for some g. Then some simple calculations yield that for any $p(z) \in V_{k,l}$, $\mathcal{L}_\gamma(p(z)g(t + i|z|^2)) = 0$ if and only if $g(t + i|z|^2)$ satisfies

$$\Big((\zeta - \overline{\zeta})\frac{\partial^2}{\partial\zeta\partial\overline{\zeta}} - (\alpha + l)\frac{\partial}{\partial\zeta} + (\beta + k)\frac{\partial}{\partial\overline{\zeta}}\Big) g(\zeta) = 0, \tag{6.7}$$

where $\alpha = \frac{n-\gamma}{2}$ and $\beta = \frac{n+\gamma}{2}$. Here equation (6.7) can be derived from the change of variable $\zeta = t + i|z|^2$. Then polynomial solutions can be split by degree of homogeneity. Let $g(\zeta) = \sum_{j=0}^N a_j \zeta^j \overline{\zeta}^{N-j}$. Then (6.7) leads to the two term recurrence relation

$$(j+1)(\beta + k + N - j - 1)a_{j+1} - (N - j)(\alpha + l + j)a_j = 0,$$

which has a unique solution

$$a_j = c\frac{(\alpha + l)_j(\beta + k)_{N-j}}{j!(N-j)!}$$

for any constant c. This yields that the polynomial solutions for g are arbitrary linear combinations of $C_N^{(\alpha,\beta)}(\zeta)$.

Equation (6.7) has an interesting non-polynomial solution:

$$g(\zeta) = (c - \overline{\zeta})^{-\alpha-l}(c - \zeta)^{-\beta-k}, \quad c \in \mathbf{C}.$$

This can be verified by direct computations. This type of solution is also not smooth.

When $n = 1$, the dimension D_m of the linear space of $\mathcal{L}_\gamma$-harmonic polynomials of H-homogeneous degree m is $m + 1$. Hence the dimension of $\mathcal{L}_\gamma$-harmonic polynomials of H-homogeneous degree less than or equal to d is

$$1 + 2 + 3 + \cdots + (d+1) = \frac{(d+1)(d+2)}{2} \approx \frac{d^2}{2}.$$

Next we consider the case $n \geq 2$. We first compute the dimension D_m of the linear space of $\mathcal{L}_\gamma$-harmonic polynomials of H-homogeneous degree m. Since any $\mathcal{L}_\gamma$-harmonic polynomial of H-homogeneous degree m is a linear combination of $p(z)C_N^{(\alpha,\beta)}(t + i|z|^2)$ with $p(z) \in V_{k,l}$ and $k, l, N \in \mathbf{N}$ satisfying $k + l + 2N = m$, we have

$$D_m = \sum_{N=0}^{[m/2]} \sum_{k+l=m-2N} \nu_{k,l} = \sum_{N=0}^{[m/2]} \frac{m - 2N + n - 1}{n-1} \sum_{k+l=m-2N} \frac{(n-1)_k(n-1)_l}{k!l!}. \tag{6.8}$$

Here

$$\Big[\frac{m}{2}\Big] = \text{integer part of } \frac{m}{2} = \begin{cases} \frac{m}{2} & \text{when } m \text{ is even,} \\ \frac{m-1}{2} & \text{when } m \text{ is odd.} \end{cases}$$

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We will first find the sum over $k+l=p=m-2N$ by applying a trick from the binomial formula for $(1-x)^{-(n-1)}$. First we observe that

$$(1-x)^{-(n-1)}=\sum_{k=0}^{\infty}\frac{(n-1)n(n+1)\cdot(n-2+k)}{k!}x^k=\sum_{k=0}^{\infty}\frac{(n-1)_k}{k!}x^k.$$

This implies that the sum over $k+l=p=m-2N$ in (6.8) is the coefficient of x^p of the Taylor series of $(1-x)^{-2(n-1)}$, *i.e.*,

$$\begin{aligned}\sum_{k+l=m-2N}\frac{(n-1)_k(n-1)_l}{k!l!}&=\frac{(2n-2)(2n-1)\cdots(2n-2+m-2N-1)}{(m-2N)!}\\&=\frac{(2n-2)_{m-2N}}{(m-2N)!}.\end{aligned}$$

This is because $(1-x)^{-(n-1)}(1-x)^{-(n-1)}=(1-x)^{-2(n-1)}$. The above formula is the result of the product formula for the Taylor series.

Hence, we have

$$\begin{aligned}D_m&=\sum_{N=0}^{[m/2]}\frac{m-2N+n-1}{n-1}\cdot\frac{(2n-2)_{m-2N}}{(m-2N)!}\\&=\sum_{N=0}^{[m/2]}\frac{m-2N+n-1}{n-1}\cdot\frac{(2n-2+m-2N-1)!}{(2n-3)!(m-2N)!}\\&=\sum_{N=0}^{[m/2]}\frac{(m-2N)+n-1}{n-1}\cdot\frac{(m-2N+1)(m-2N+2)\cdots(m-2N+2n-3)}{(2n-3)!}.\end{aligned}$$

6.4 Estimate of the Dimension $H^d(\mathbf{H}_n)$

Recall that the dimension of the linear space of $\mathcal{L}_\gamma$-harmonic polynomials of H-homogeneous degree m is

$$D_m=\frac{2}{(2n-2)!}\sum_{N=0}^{[m/2]}(m-2N+n-1)\cdot[(m-2N+1)(m-2N+2)\cdots(m-2N+2n-3)].$$

The term $(m-2N+n-1)\cdot[(m-2N+1)(m-2N+2)\cdots(m-2N+2n-3)]$ can be written as a polynomial of $(m-2N)$ of degree $2n-2$ with coefficients being polynomials in n, *i.e.*,

$$\begin{aligned}&((m-2N)+n-1)\cdot[(m-2N+1)(m-2N+2)\cdots(m-2N+2n-3)]\\&\qquad=\sum_{k=0}^{2n-2}E_k(n)(m-2N)^{2n-2-k}.\end{aligned}$$

In particular, we have $E_0(n)=1$ and $E_{2n-2}(n)=(2n-3)!\cdot(n-1)=\frac{1}{2}(2n-2)!$. This implies that

$$D_m=\frac{2}{(2n-2)!}\sum_{N=0}^{[m/2]}\sum_{k=0}^{2n-2}E_k(n)(m-2N)^{2n-2-k}=\sum_{k=0}^{2n-2}E_k(n)\sum_{N=0}^{[m/2]}(m-2N)^{2n-2-k}.$$

The dimension of the linear space of $\mathcal{L}_\gamma$-harmonic polynomials of H-homogeneous degree less than or equal to d is

$$\begin{aligned}
D = \sum_{i=0}^{d} D_i &= \frac{2}{(2n-2)!} \sum_{i=0}^{d} \sum_{N=0}^{[i/2]} \sum_{k=0}^{2n-2} E_k(n)(i-2N)^{2n-2-k} \\
&= \frac{2}{(2n-2)!} \sum_{k=0}^{2n-2} E_k(n) \sum_{i=0}^{d} \sum_{N=0}^{[i/2]} (i-2N)^{2n-2-k} \\
&= \frac{2}{(2n-2)!} \sum_{k=0}^{2n-2} E_k(n) \sum_{N=0}^{[d/2]} \sum_{i=2N}^{d} (i-2N)^{2n-2-k} \\
&= \frac{2}{(2n-2)!} \sum_{k=0}^{2n-2} E_k(n) \sum_{N=0}^{[d/2]} \sum_{i=0}^{d-2N} i^{2n-k-2}.
\end{aligned}$$

Here, we have exchanged the order of the last two sums. Now we apply Faulhaber's classical formula (see [6] for a proof)

$$1^k + 2^k + \cdots + N^k = \frac{1}{k+1} N^{k+1} + \frac{1}{2} N^k + \frac{k}{12} N^{k-1} + O(N^{k-3})$$

to the above to estimate the sum over i to get

$$\sum_{i=0}^{d-2N} i^{2n-2-k} = \frac{(d-2N)^{2n-k-1}}{2n-k-1} + \frac{(d-2N)^{2n-k-2}}{2} + O((d-2N)^{2n-k-3}).$$

This yields

$$\begin{aligned}
D = \frac{2}{(2n-2)!} \sum_{k=0}^{2n-2} E_k(n) \sum_{N=0}^{[d/2]} &\frac{(d-2N)^{2n-k-1}}{2n-k-1} \\
&+ \frac{(d-2N)^{2n-k-1}}{2} + O((d-2N)^{2n-k-3}).
\end{aligned}$$

To get the leading term, we need to consider the term $k=0$ and note that $E_0(n) = 1$:

$$D = \frac{2}{(2n-2)!} \sum_{N=0}^{[\frac{d}{2}]} \frac{(d-2N)^{2n-1}}{2n-1} + O((d-2N)^{2n-2}).$$

When $d = 2\nu$ is even, we can reduce the above to

$$\begin{aligned}
D &= \frac{2^{2n}}{(2n-1)!} \sum_{N=0}^{\nu} (\nu-N)^{2n-1} + O((\nu-N)^{2n-2}) \\
&= \frac{2^{2n}}{(2n-1)!} \frac{\nu^{2n}}{2n} + O((2\nu)^{2n-1}) = \frac{d^{2n}}{(2n)!} + O((2\nu)^{2n-1}).
\end{aligned}$$

When $d = 2\nu + 1$ is odd, we can reduce the above to

$$\begin{aligned} D &= \frac{2}{(2n-1)!}\sum_{N=0}^{\nu}\Big(\big(2(\nu-N)+1\big)\Big)^{2n-1} + O\Big(\big(2\nu-2N+1\big)^{2n-2}\Big) \\ &= \frac{2}{(2n-1)!}\Big[\sum_{N=1}^{2\nu} n^{2n-1} - \sum_{N=1}^{\nu}(2N)^{2n-1}\Big] + O\big((2\nu+1)^{2n-1}\big) \\ &= \frac{2}{(2n-1)!}\Big[\frac{(2\nu)^{2n}}{2n} + \frac{(2\nu)^{2n-1}}{2} - 2^{2n-1}\Big(\frac{\nu^{2n}}{2n} + \frac{\nu^{2n-1}}{2}\Big)\Big] + O\big((2\nu+1)^{2n-1}\big) \\ &= \frac{d^{2n}}{(2n)!} + O(d^{2n-1}). \end{aligned}$$

Hence, the leading term of D is $\frac{d^{2n}}{(2n)!}$. This also coincides with the case $n = 1$. We can also determine the lower terms by increasing the values of k. Note that $D = 2n+1$ when $d = 1$.

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