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ON L^2-BETTI NUMBERS FOR ABELIAN GROUPS

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1. **Introduction.** Let M be a differentiable manifold which admits the free action of a group Γ with compact quotient $M' = M/\Gamma$. Suppose that the Γ action lifts to a Hermitian vector bundle $E \to M'$. If Γ leaves invariant a measure μ on M, then denote by $L^2(E)$ the completion of $C_0^\infty(E)$ with respect to the inner product $\langle f, g\rangle = \int_M \langle f, g\rangle_E \, d\mu$, for $f, g \in C_0^\infty(E)$.

Given a self-adjoint elliptic operator D on $L^2(E)$, let Ker D be the kernel of D. If $P: L^2(E) \to \text{Ker } D$ is orthogonal projection then P is given by a smooth kernel $P(x, y) \in C^\infty(\text{Hom}(E_y, E_x))$ [1]. Following M. F. Atiyah, one defines

$$\dim_\Gamma \text{Ker } D = \int_{\mathcal{D}} \text{Tr}(P(x, x)) \, d\mu(x).$$

Here $\mathcal{D}$ is a fundamental domain for Γ and Tr denotes the trace.

If Γ is finite then dim Ker D is easily seen to be a multiple of the reciprocal of the order of Γ. It is a question of current research interest to determine which values are assumed by $\dim_\Gamma$ Ker D for Γ of infinite order. Observe that Γ is necessarily finitely generated since it is a homomorphic image of $\pi_1(M')$, the fundamental group of the compact manifold M'.

In the present note we will establish

THEOREM 1.1. *Let Γ be abelian with torsion subgroup of order k. Then, for any self-adjoint elliptic operator D, $k \dim_\Gamma$ Ker D is an integer.*

If M is a Riemannian manifold, $E = \Lambda^p M$, and $D = \Delta_p$ is the Laplacian of M acting on p-forms, then Theorem I.1 is due to J. Cohen [3]. In fact, by the generalized DeRham–Hodge theory of J. Dodziuk [4], $\dim_\Gamma \text{Ker } \Delta_p = \beta_p(\Gamma)$. Here $\beta_p(\Gamma)$ is a topological invariant representing the Γ-dimension of the p'th L^2-cohomology group. Using Dodziuk's work, J. Cohen deduces Theorem 1.1 for Δ_p by algebraic methods.

The present paper was motivated by the desire to derive J. Cohen's results directly using the analytical framework of [1].

2. **Analytic families of compact operators.** The proof of Theorem 1.1 will require some preliminary lemmas concerning holomorphic families of compact operators $K(z)$, $z \in C^n$. These results are known for $n = 1$ [5, p. 371].

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One has

LEMMA 2.1. *Let $K(z): H \to H$, where H is a Hilbert space, be an analytic family of compact operators parameterized by $z \in \Omega$, where Ω is a connected open set in C^n. Then there exists a locally finite collection of analytic sets $A_\alpha \subset \Omega$, $A_\alpha \neq \Omega$, so that on $\Omega - U_\alpha A_\alpha$ the multiplicity of the eigenvalue 1 for $K(z)$ is constant.*

Proof. The result is local in nature, so it will suffice to establish our claim on a neighborhood U of each point in Ω. Here one needs the result that removing an analytic set from Ω will not disconnect Ω [6, p. 6].

As is well-known, the eigenvalues of a compact operator may accumulate only at zero. Using this fact, the projection technique of [5, p. 370] reduces the problem to the case where $\dim H < \infty$.

Let $\dim H = n < \infty$. The eigenvalues ζ of $K(z)$ are given by

$$\zeta^n + f_1(z)\zeta^{n-1} + \cdots + f_n(z) = 0 \tag{2.2}$$

where the functions $f_i(z)$ are holomorphic. The singular points (z, ζ) of the analytic set C defined by (2.2) form an analytic subset $B \subset C$, $B \neq C$. Consider the map $\pi(z, \zeta) = z$ from B into U. Since π is at most n to 1, the proper mapping theorem [6, p. 129] implies that $A_0 = \pi(B)$ is an analytic set in U.

In $U - A_0$, the eigenvalues of $K(z)$ are given by analytic functions $g_i(z)$, $i = 1, \ldots, n$. If $g_i(z)$ is not identically one, then $A_i = \{z \mid g_i(z) = 1\}$ is an analytic set in U, $A_i \neq U$. The result follows.

COROLLARY 2.3. *Let $Q \subset R^n$ be the set of real points in Ω, as in Lemma 2.1. Then the multiplicity of the eigenvalue 1 for $K(x)$, $x \in Q$, is constant almost everywhere in Q.*

The reason for considering families parameterized by C^n, rather than just R^n, is that a complex codimension one set cannot disconnect C^n, whereas a real codimension one set may disconnect R^n.

3. **Rationality for Abelian Groups.** Let Γ be a finitely generated abelian group. Denote $n = \text{rank } \Gamma$ and $k = \text{order torsion } \Gamma$. All unitary representations χ of Γ are one dimensional. Moreover, the set $\hat{\Gamma}$ of unitary representations admits the structure of a compact abelian Lie group, with multiplication given by $(\chi\chi')(\gamma) = \chi(\gamma)\chi'(\gamma)$ for $\chi, \chi' \in \hat{\Gamma}$, $\gamma \in \Gamma$. We may write $\hat{\Gamma} = T^n \times \Sigma_k$ where T^n is an n-torus and Σ_k is a collection of k discrete points. Let $\hat{\Gamma}$ be endowed with a $\hat{\Gamma}$-invariant measure of total measure one. Then each component of $\hat{\Gamma}$ has measure $1/k$.

We will need two lemmas concerning the compact abelian Lie group $\hat{\Gamma}$:

LEMMA 3.1 (Orthogonality Relations). *Let* $\gamma_1, \gamma_2 \in \Gamma$ *and* $\chi \in \hat{\Gamma}$. *Then*

$$\int_{\hat{\Gamma}} \chi(\gamma_1)\overline{\chi(\gamma_2)} = \begin{cases} 0 & \gamma_1 \neq \gamma_2 \\ 1 & \gamma_1 = \gamma_2 \end{cases}.$$

Proof. [8, pp. 221–222]

LEMMA 3.2 (Parseval's Formula). *Let* $f(\chi) \in L^2(\hat{\Gamma})$ *and* $a_\gamma(f)$ *its Fourier coefficients defined by* $a_\gamma(f) = \int_{\hat{\Gamma}} f(\chi)\overline{\chi(\gamma)}$, $\gamma \in \Gamma$. *Then*

$$\int_{\hat{\Gamma}} |f(\chi)|^2 = \sum_{\Gamma} |a_\gamma(f)|^2$$

Proof. [7, vol. I, p. 45].

Our most basic result is

THEOREM 3.3. *Let* D, Γ, M, M', E, E' *be as in Theorem* I.1 *and the introduction. Denote by* D_χ *the elliptic operator on* $L^2(E') \times L^2(F')$ *induced by D. Here* $F_\chi \to M'$ *is the flat bundle associated to the representation* χ *of* Γ.

The Hilbert space $L^2(E)$ *decomposes under the action of the group* Γ *as a direct integral:*

$$L^2(E) \to \int_{\hat{\Gamma}} L^2(E') \otimes L^2(F_\chi) \tag{3.4}$$

Moreover, U intertwines D with the direct integral:

$$D \to \int_{\hat{\Gamma}} D_\chi \tag{3.5}$$

Consequently, one has

$$\dim_\Gamma \operatorname{Ker} D = \int_{\hat{\Gamma}} \dim(\operatorname{Ker} D_\chi) \tag{3.6}$$

Proof. For $f \in C_0^\infty(E)$, we define

$$Uf(p, \chi) = \sum_{\Gamma} \overline{\chi(\gamma)}\gamma^* f(\gamma p) \tag{3.7}$$

where we identify $p \in M'$ with $p \in \mathcal{D}$, a fundamental domain for Γ.

Now

$$\|Uf\|_2^2 = \int_{\hat{\Gamma}} \int_{M'} \|Uf(p, \chi)\|^2$$

$$\|Uf\|_2^2 = \int_{\hat{\Gamma}} \int_{\mathcal{D}} \left\|\sum_{\Gamma} \overline{\chi(\gamma)}\gamma^* f(\gamma p)\right\|^2$$

$$\|Uf\|_2^2 = \sum_{\gamma \in \Gamma} \sum_{\mu \in \Gamma} \int_{\hat{\Gamma}} \bar{\chi}(\gamma)\chi(\mu) \times \int_{\mathcal{D}} \langle \gamma^* f(\gamma p), \mu^* f(\mu p)\rangle.$$

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Applying the orthogonality relations, Lemma 3.1, one obtains

$$\|Uf\|_2^2 = \sum_{\gamma\in\Gamma}\int_{\mathscr{D}} \|\gamma^* f(\gamma p)\|^2$$

$$\|Uf\|_2^2 = \int_M \|f(p)\|^2 = \|f\|_2^2.$$

This shows that U defined by (3.7) extends to an isometry of $L^2(E)$ into $\int_{\hat{\Gamma}} L^2(E')\otimes L^2(F_\chi)$.

One may easily check that the adjoint of U is given by

$$(U^*g)(\gamma p) = \int_\Gamma \chi(\gamma) g_\chi(p)$$

for $p\in\mathscr{D}$ and g a smooth element in $\int_{\hat{\Gamma}} L^2(E')\otimes L^2(F_\chi)$.

Moreover,

$$\|U^*g\|_2^2 = \int_M \|U^*g(p)\|^2$$

$$\|U^*g\|_2^2 = \sum_{\gamma\in\Gamma}\int_{\mathscr{D}} \left|\int_{\hat{\Gamma}} \chi(\gamma)g_\chi(p)\right|^2$$

$$\|U^*g\|_2^2 = \int_{\mathscr{D}} \sum_{\gamma\in\Gamma} \left|\int_{\hat{\Gamma}} \chi(\gamma)g_\chi(p)\right|^2.$$

Applying Parseval's formula, Lemma 3.2, we conclude that

$$\|U^*g\|_2^2 = \int_{\mathscr{D}}\int_\Gamma |g_\chi(p)|^2 = \|g\|_2^2.$$

Thus, U defined by (3.7) for $f\in C_0^\infty(E)$ extends to a unitary isomorphism $U: L^2(E)\to\int_{\hat{\Gamma}} L^2(E')\otimes L^2(F_\chi)$. It is clear from (3.7) that U intertwines D and $\int_{\hat{\Gamma}} D_\chi$. The orthogonal projection P onto ker D splits as $\int_{\hat{\Gamma}} P_\chi$, where P_χ is the projection of $L^2(E')\otimes L^2(F_\chi)$ onto ker D_χ. Thus

$$\dim_\Gamma \ker D = \int_{\mathscr{D}} \operatorname{Tr} P(x,x) = \int_{\mathscr{D}} \operatorname{Tr}\left(\int_{\hat{\Gamma}} P_\chi\right) = \int_{\hat{\Gamma}}\int_M \operatorname{Tr} P_\chi = \int_{\hat{\Gamma}} \dim\ker D_\chi.$$

We will now use (3.6) to prove Theorem I.1. Observe that $D_\chi f = 0$ if and only if $\exp(-D_\chi^* D_\chi)f = f$. Since M' is compact, $\exp(-D_\chi^* D_\chi)$ is a compact operator [2]. Moreover, by considering non-unitary characters, we may extend $\exp(-D_\chi^* D_\chi)$ to a holomorphic family of n complex variables. Applying Corollary 2.3, we conclude that ker D_χ has constant dimension d_i almost everywhere on each component $\hat{\Gamma}_i$ of $\hat{\Gamma}$, $i = 1, \ldots, k$. Then by (3.6):

$$\dim_\Gamma \ker D = \frac{1}{k}\sum_{i=1}^k d_i.$$

This demonstrates Theorem I.1.

Our consideration of the map U in (3.4) was motivated by the study of the Schrodinger Operator for periodic potentials in Euclidean space [7, Vol. IV, p. 285].

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