Proceedings of the Royal Society of Edinburgh: Section A Mathematics, 1–25, *2024*
DOI:10.1017/prm.2024.135

Finely quasiconformal mappings

Panu Lahti
Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing 100190, PR China (panulahti@amss.ac.cn)

(Received 13 September 2024; revised 29 November 2024; accepted 2 December 2024)

We introduce a relaxed version of the metric definition of quasiconformality that is natural also for mappings of low regularity, including $W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$-mappings. Then we show on the plane that this relaxed definition can be used to prove Sobolev regularity, and that these 'finely quasiconformal' mappings are in fact quasiconformal.

Keywords: finely open set; quasiconformal mapping; Sobolev mapping; capacity; fine differentiability

2020 *Mathematics Subject Classification:* 30C65; 46E35; 31C40

1. Introduction

A homeomorphism $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is said to be quasiconformal if $K_f(x) \le K < \infty$ for all $x \in \mathbb{R}^n$, where

$$K_f(x) := \limsup_{r\to 0} \left(\frac{\operatorname{diam} f(B(x,r))^n}{|f(B(x,r))|} \right)^{1/(n-1)}.$$

We always consider $n \ge 2$, and we use $|\cdot|$ for the Euclidean norm as well as for the Lebesgue measure. There are several equivalent definitions of quasiconformality; the above is a 'metric' definition. As part of an 'analytic definition', it is known that quasiconformal mappings are in the Sobolev class $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$.

There has been wide interest in showing that if quasiconformality is assumed in some relaxed sense, it follows that the mapping in question is in fact quasiconformal, or at least has some lower regularity, such as $W^{1,1}_{\mathrm{loc}}$-regularity. For example, Koskela–Rogovin [15, corollary 1.3] show that if $f\colon \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, $K_f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, and $K_f < \infty$ outside a set of σ-finite $\mathcal{H}^{n-1}$-measure, then $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$. Many results in the same vein have been proven starting from Gehring [8, 9], see also Balogh–Koskela [2], Fang [7], Heinonen–Koskela–Shanmugalingam–Tyson [10], Kallunki–Koskela [12], and Margulis–Mostow [20]. Several works study specifically the issue of $W^{1,1}_{\mathrm{loc}}$-regularity, see Balogh–Koskela–Rogovin [3], Kallunki–Martio [13], and Williams [22].

The quantity K_f^{n-1} can be essentially thought of as '$|\nabla f|^n$ divided by the Jacobian determinant'. Indeed, for a quasiconformal mapping $f\colon \mathbb{R}^n \to \mathbb{R}^n$, we know that

$$K_f(x)^{n-1}|\det \nabla f(x)| = \frac{2^n}{\omega_n}\|\nabla f(x)\|^n \quad \text{for a.e. } x \in \mathbb{R}^n, \tag{1.1}$$

where ω_n is the Lebesgue measure of the unit ball, $\|\cdot\|$ is the maximum norm, and ∇f can be understood to be either the classical gradient or the weak gradient. With the latter interpretation, all of the quantities in (1.1) make sense also for mappings of lower Sobolev regularity, but the equality can fail already for $W^{1,n}_{\mathrm{loc}}$-mappings—let alone $W^{1,1}_{\mathrm{loc}}$-mappings—since for them $\operatorname{diam} f(B(x,r))$ can easily be ∞ for every $x \in \mathbb{R}^n$ and $r > 0$; see example 3.1. The problem is that the quantity K_f is very sensitive to oscillations and essentially tailored to mappings f that have better than $W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$-regularity. We wish to find a quantity that corresponds to '$|\nabla f|^n$ divided by the Jacobian determinant' in the case of $W^{1,1}_{\mathrm{loc}}$-mappings. Hence we define the relaxed quantities

$$K_{f,U}(x,r) := \left(\frac{\operatorname{diam} f(B(x,r)\cap U)^n}{|f(B(x,r))|}\right)^{1/(n-1)} \quad \text{and} \quad K^{\mathrm{fine}}_f(x) := \inf \limsup_{r\to 0} K_{f,U}(x,r),$$

where the infimum is taken over *1-finely open* sets $U \ni x$; we give definitions in §2. In the following analog of (1.1), f^* is the so-called precise representative of f.

THEOREM 1.2. *For every* $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, *we have*

$$K^{\mathrm{fine}}_{f^*}(x)^{n-1}|\det \nabla f(x)| \le \frac{2^n}{\omega_n}\|\nabla f(x)\|^n \quad \text{for a.e. } x \in \mathbb{R}^n,$$

with the interpretation $\infty \times 0 = 0$ *if* $\det \nabla f(x) = 0$.

This shows that K^{fine}_f is generally much smaller than K_f. On the other hand, the mapping we give in the aforementioned example 3.1 is by no means a homeomorphism. Thus one can ask: for a homeomorphism f, are conditions on K^{fine}_f enough to prove Sobolev regularity, or even quasiconformality? Our main result is the following analog on the plane of the aforementioned Koskela–Rogovin [15] and of other similar results.

THEOREM 1.3. *Let* $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ *be a homeomorphism. Let* $1 \le p \le 2$. *Suppose* $K^{\mathrm{fine}}_f \in L^{p^*/2}_{\mathrm{loc}}(\mathbb{R}^2)$ *and* $K^{\mathrm{fine}}_f < \infty$ *outside a set* E *of* σ*-finite* $\mathcal{H}^1$*-measure. Then* $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2;\mathbb{R}^2)$, *and in the case* $p = 2$ *we obtain that* f *is quasiconformal and that* $K^{\mathrm{fine}}_f(x) = K_f(x)$ *for a.e.* $x \in \mathbb{R}^2$.

Here $p^* = 2p/(2-p)$ when $1 \le p < 2$, and $p^* = \infty$ when $p = 2$. In the case $1 \le p < 2$, this theorem can be viewed as a statement about 'finely quasiconformal' mappings of low regularity. The condition on the size of the exceptional set E is known to be quite sharp, as noted, e.g., in remark 1.9 of Williams [22]; the same is true in our setting since the set where $K^{\mathrm{fine}}_f = \infty$ is of course smaller than the set where $K_f = \infty$. In the case $p = 2$, we get the following corollary saying that 'finely quasiconformal' mappings are in fact quasiconformal.

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Corollary 1.4. *Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$ be a homeomorphism and suppose that $K_f^{\mathrm{fine}}(x) \le K < \infty$ for every $x \in \mathbb{R}^2$. Then f is quasiconformal.*

2. Preliminaries

Our definitions and notation are standard, and the reader may consult, e.g., the monograph Evans–Gariepy [6] for more background. We will work in the Euclidean space $\mathbb{R}^n$ with $n \ge 2$. We denote the n-dimensional Lebesgue outer measure by $\mathcal{L}^n$. We denote the s-dimensional Hausdorff content by $\mathcal{H}^s_R$ and the Hausdorff measure by $\mathcal{H}^s$, with $0 < R \le \infty$ and $0 \le s \le n$. If a property holds outside a set of Lebesgue measure zero, we say that it holds almost everywhere, or 'a.e.'.

We denote the characteristic function of a set $A \subset \mathbb{R}^n$ by $\chi_A\colon \mathbb{R}^n \to \{0,1\}$. We denote by $|v|$ the Euclidean norm of $v \in \mathbb{R}^n$, and we also write $|A| := \mathcal{L}^n(A)$ for a set $A \subset \mathbb{R}^n$. We write $B(x,r)$ for an open ball in $\mathbb{R}^n$ with centre $x \in \mathbb{R}^n$ and radius $r > 0$, that is, $B(x,r) = \{y \in \mathbb{R}^n : |y-x| < r\}$. We will sometimes use the notation $2B(x,r) := B(x,2r)$. For matrices $A \in \mathbb{R}^{n\times n}$, we consider the Euclidean norm $|A|$ as well as the maximum norm

$$\|A\| := \max_{v\in\mathbb{R}^n,\,|v|=1} |Av|.$$

By 'measurable', we mean $\mathcal{L}^n$-measurable, unless otherwise specified. If a function u is in $L^1(D)$ for some measurable set $D \subset \mathbb{R}^n$ of nonzero and finite Lebesgue measure, we write

$$u_D := \fint_D u(y)\,d\mathcal{L}^n(y) := \frac{1}{\mathcal{L}^n(D)} \int_D u(y)\,d\mathcal{L}^n(y)$$

for its mean value in D.

We will always denote by $\Omega \subset \mathbb{R}^n$ an open set, and we consider $1 \le p < \infty$. Let $l \in \mathbb{N}$. The Sobolev space $W^{1,p}(\Omega;\mathbb{R}^l)$ consists of mappings $f \in L^p(\Omega;\mathbb{R}^l)$ whose first weak partial derivatives $\partial f_j/\partial x_k$, $j = 1,\ldots,l$, $k = 1,\ldots,n$, belong to $L^p(\Omega)$. We will only consider $l = 1$ or $l = n$. The weak partial derivatives form the matrix $(\nabla f)_{jk}$. The Dirichlet space $D^p(\Omega;\mathbb{R}^l)$ is defined in the same way, except that the integrability requirement for the mapping itself is relaxed to $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^l)$. The Sobolev norm is

$$\|f\|_{W^{1,p}(\Omega;\mathbb{R}^l)} := \|f\|_{L^p(\Omega;\mathbb{R}^l)} + \|\nabla f\|_{L^p(\Omega;\mathbb{R}^{l\times n})},$$

where the L^p norms are defined with respect to the Euclidean norm.

Consider a homeomorphism $f\colon \Omega \to \Omega'$, with $\Omega, \Omega' \subset \mathbb{R}^n$ open. In addition to the Jacobian determinant $\det \nabla f(x)$, we also define the Jacobian

$$J_f(x) := \limsup_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|}, \quad x \in \Omega. \tag{2.1}$$

Note that J_f is the density of the pullback measure

$$f_{\#}\mathcal{L}^n(A) := \mathcal{L}^n(f(A)), \quad \text{for Borel } A \subset \Omega.$$

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By well-known results on densities, see e.g. [6, p. 42], we know the following: $J_f(x)$ exists as a limit for a.e. $x \in \Omega$, is a Borel function, and

$$\int_\Omega J_f \, d\mathcal{L}^n \le |f(\Omega)|. \tag{2.2}$$

Equality holds if f is absolutely continuous in measure, that is, if $|A| = 0$ implies $|f(A)| = 0$. We will use the following 'analytic' definition of quasiconformality. For the equivalence of different definitions of quasiconformality, including the metric definition used in the introduction, see e.g. [10, theorem 9.8].

Definition 2.3. *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be open sets. A homeomorphism $f \in W^{1,n}_{\mathrm{loc}}(\Omega; \Omega')$ is said to be quasiconformal if*

$$\|\nabla f(x)\|^n \le K |\det \nabla f(x)| \quad \text{for a.e. } x \in \Omega, \tag{2.4}$$

for some constant $K < \infty$.

Here we understand ∇f to be the weak gradient. However, as a homeomorphism, f is locally *monotone*, and combining this with the fact that $f \in W^{1,n}_{\mathrm{loc}}(\Omega; \Omega')$, by, e.g., Malý [18, theorems 3.3 and 4.3] we know that f is differentiable a.e. Moreover, by [19, corollary B] and [18, theorem 3.4], such f is absolutely continuous in measure and satisfies the area formula, implying that

$$\int_W J_f \, d\mathcal{L}^n = |f(W)| = \int_W |\det \nabla f| \, d\mathcal{L}^n$$

for every open $W \subset \Omega$, and so $|\det \nabla f| = J_f$ a.e. in Ω. Thus in (2.4), we could equivalently replace $|\det \nabla f|$ with J_f.

We will need the following Vitali–Carathéodory theorem; for a proof see e.g. [11, p. 108].

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be open and let $h \in L^1(\Omega)$ be nonnegative. Then there exists a sequence $\{h_i\}_{i=1}^\infty$ of lower semicontinuous functions on Ω such that $h \le h_{i+1} \le h_i$ for all $i \in \mathbb{N}$, and $h_i \to h$ in $L^1(\Omega)$.*

The theory of BV mappings that we present next can be found in the monograph Ambrosio–Fusco–Pallara [1]. As before, let $\Omega \subset \mathbb{R}^n$ be an open set. Let $l \in \mathbb{N}$. A mapping $f \in L^1(\Omega; \mathbb{R}^l)$ is of bounded variation, denoted $f \in \mathrm{BV}(\Omega; \mathbb{R}^l)$, if its weak derivative is an $\mathbb{R}^{l \times n}$-valued Radon measure with finite total variation. This means that there exists a (unique) Radon measure Df such that for all $\varphi \in C^1_c(\Omega)$, the integration-by-parts formula

$$\int_\Omega f_j \frac{\partial \varphi}{\partial x_k} \, d\mathcal{L}^n = -\int_\Omega \varphi \, d(Df_j)_k, \quad j = 1, \ldots, l, \ k = 1, \ldots, n,$$

holds. The total variation of Df is denoted by $|Df|$. The BV norm is defined by

$$\|f\|_{\mathrm{BV}(\Omega)} := \|f\|_{L^1(\Omega)} + |Df|(\Omega).$$

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We denote by ∇f the density of the absolutely continuous part of *Df*. If we do not know a priori that a mapping $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^l)$ is a BV mapping, we consider

$$\mathrm{Var}(f,\Omega) := \sup\left\{\sum_{j=1}^{l}\int_\Omega f_j \operatorname{div}\varphi_j \, d\mathcal{L}^n,\ \varphi \in C_c^1(\Omega;\mathbb{R}^{l\times n}),\ |\varphi| \le 1\right\}. \tag{2.6}$$

If $\mathrm{Var}(f,\Omega) < \infty$, then the $\mathbb{R}^{l\times n}$-valued Radon measure Df exists and $\mathrm{Var}(f,\Omega) = |Df|(\Omega)$ by the Riesz representation theorem, and $f \in \mathrm{BV}(\Omega)$ provided that $f \in L^1(\Omega;\mathbb{R}^l)$. If $E \subset \mathbb{R}^n$ with $\mathrm{Var}(\chi_E,\mathbb{R}^n) < \infty$, we say that E is a set of finite perimeter.

The coarea formula states that for a function $u \in \mathrm{BV}(\Omega)$, we have

$$|Du|(\Omega) = \int_{-\infty}^{\infty} |D\chi_{\{u>t\}}|(\Omega)\, dt. \tag{2.7}$$

Here we abbreviate $\{u > t\} := \{x \in \Omega \colon u(x) > t\}$.

The relative isoperimetric inequality states that for every set of finite perimeter $E \subset \mathbb{R}^n$ and every ball $B(x,r)$, we have

$$\min\{\mathcal{L}^n(B(x,r)\cap E), \mathcal{L}^n(B(x,r)\setminus E)\} \le C_I r |D\chi_E|(B(x,r)), \tag{2.8}$$

where the constant $C_I \ge 1$ only depends on n. The following relative isoperimetric inequality holds on the plane: for every set of finite perimeter $E \subset \mathbb{R}^2$ and every disk $B(x,r)$, we have

$$\min\{\mathcal{L}^2(B(x,r)\cap E), \mathcal{L}^2(B(x,r)\setminus E)\} \le r|D\chi_E|(B(x,r)). \tag{2.9}$$

For $f \in L^1_{\mathrm{loc}}(\Omega)$, we define the precise representative by

$$f^*(x) := \limsup_{r\to 0} \fint_{B(x,r)} f \, d\mathcal{L}^n, \quad x \in \Omega. \tag{2.10}$$

For $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, we let $f^*(x) := (f_1^*(x), \ldots, f_n^*(x))$.

For basic results in the one-dimensional case $n=1$, see [1, Section 3.2]. If $\Omega \subset \mathbb{R}$ is an open interval, we define the pointwise variation of $f\colon \Omega \to \mathbb{R}^n$ by

$$\mathrm{pV}(f,\Omega) := \sup \sum_{j=1}^{N-1} |f(x_j) - f(x_{j+1})|, \tag{2.11}$$

where the supremum is taken over all collections of points $x_1 < \cdots < x_N$ in Ω. For a general open $\Omega \subset \mathbb{R}$, we define $\mathrm{pV}(f,\Omega)$ to be $\sum \mathrm{pV}(f,I)$, where the sum runs over all connected components I of Ω. For every pointwise defined $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, we have $\mathrm{Var}(f,\Omega) \le \mathrm{pV}(f,\Omega)$.

Denote by $\pi_n\colon \mathbb{R}^n \to \mathbb{R}^{n-1}$ the orthogonal projection onto $\mathbb{R}^{n-1}$: for $x = (x_1,\dots,x_n) \in \mathbb{R}^n$,

$$\pi_n((x_1,\dots,x_n)) := (x_1,\dots,x_{n-1}). \tag{2.12}$$

For $z \in \pi_n(\Omega)$, we denote the slices of an open set $\Omega \subset \mathbb{R}^n$ by

$$\Omega_z := \{t \in \mathbb{R}\colon (z,t) \in \Omega\}.$$

We also denote $f_z(t) := f(z,t)$ for $z \in \pi_n(\Omega)$ and $t \in \Omega_z$. For any continuous $f \in L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, we know that $\mathrm{Var}(f,\Omega)$ is at most the sum of

$$\int_{\pi_n(\Omega)} \mathrm{pV}(f_z,\Omega_z)\, d\mathcal{L}^{n-1}(z) \tag{2.13}$$

and the analogous quantities for the other $n-1$ coordinate directions, see [1, theorem 3.103].

The (Sobolev) 1-capacity of a set $A \subset \mathbb{R}^n$ is defined by

$$\mathrm{Cap}_1(A) := \inf \|u\|_{W^{1,1}(\mathbb{R}^n)},$$

where the infimum is taken over Sobolev functions $u \in W^{1,1}(\mathbb{R}^n)$ satisfying $u \ge 1$ in a neighbourhood of A.

Given sets $A \subset W \subset \mathbb{R}^n$, where W is open, the relative p-capacity is defined by

$$\mathrm{cap}_1(A,W) := \inf \int_W |\nabla u|\, d\mathcal{L}^n,$$

where the infimum is taken over functions $u \in W^{1,1}_0(W)$ satisfying $u \ge 1$ in a neighbourhood of A. The class $W^{1,1}_0(W)$ is the closure of $C^1_c(W)$ in the $W^{1,p}(\mathbb{R}^n)$-norm.

By [5, theorem 3.3], given a function $u \in \mathrm{BV}(\Omega)$, there is a sequence $\{u_j\}_{j=1}^\infty$ of functions in $W^{1,1}(\Omega)$ such that

$$u_j \to u \text{ in } L^1(\Omega), \quad |Du_j|(\Omega) \to |Du|(\Omega), \text{ and } \ u_j^\vee(x) \ge u^\vee(x) \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \Omega. \tag{2.14}$$

If $B(x,r)$ is a ball with $0 < r \le 1$, and F is a measurable set with $\mathcal{L}^n(F \cap B(x,r)) \le \frac{1}{2}\mathcal{L}^n(B(x,r))$ and $|D\chi_F|(B(x,r)) < \infty$, then by combining, e.g., theorem 5.6 and theorem 5.15(iii) of [6], we get

$$|D\chi_{B(x,r)\cap F}|(\mathbb{R}^n) \le C\|\chi_F\|_{\mathrm{BV}(B(x,r))}$$

for some constant C depending only on n, r. On the other hand, by the relative isoperimetric inequality (2.8), we have

$$\begin{aligned}\|\chi_F\|_{\mathrm{BV}(B(x,r))} = \mathcal{L}^n(F \cap B(x,r)) + |D\chi_F|(B(x,r)) &\le (C_I r + 1)|D\chi_F|(B(x,r))\\ &\le 2C_I|D\chi_F|(B(x,r)),\end{aligned}$$

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since $r \le 1$ and $C_I \ge 1$. Combining these, we get

$$|D\chi_{B(x,r)\cap F}|(\mathbb{R}^n) \le C|D\chi_F|(B(x,r)), \tag{2.15}$$

and by a scaling argument we see that in fact C only depends on n, not on r.

By [4, proposition 6.16], we know that for a ball $B(x,r)$ and $A \subset B(x,r)$, we have

$$\frac{\mathrm{Cap}_1(A)}{C'(1+r)} \le \mathrm{cap}_1(A, B(x,2r)), \tag{2.16}$$

where C' is a constant depending only on n.

We denote $\omega_n := |B(0,1)|$.

LEMMA 2.17. *Suppose $x \in \mathbb{R}^n$, $0 < r < 1$, and $A \subset B(x,r)$. Then we have*

$$\frac{\mathcal{L}^n(A)}{\mathcal{L}^n(B(x,r))} \le \frac{2C_I}{\omega_n}\frac{\mathrm{Cap}_1(A)}{r^{n-1}} \quad \text{and} \quad \mathrm{cap}_1(A, B(x,2r)) \le C\,\mathrm{Cap}_1(A),$$

where C_I is the constant in the relative isoperimetric inequality (2.8), and C is a constant depending only on n.

Proof. For both inequalities, we can assume that $\mathrm{Cap}_1(A) < \infty$. Let $\varepsilon > 0$. We can choose a function $u \in W^{1,1}(\mathbb{R}^n)$ such that $u \ge 1$ in a neighbourhood of A, and

$$\|u\|_{W^{1,1}(\mathbb{R}^n)} < \mathrm{Cap}_1(A) + \varepsilon.$$

By the coarea formula (2.7), we then find $0 < t < 1$ such that $\{u > t\}$ contains a neighbourhood of A and

$$|D\chi_{\{u>t\}}|(\mathbb{R}^n) \le |Du|(\mathbb{R}^n) \le \|u\|_{W^{1,1}(\mathbb{R}^n)} < \mathrm{Cap}_1(A) + \varepsilon.$$

Denote $F := \{u > t\}$.

Case 1: Suppose $\mathcal{L}^n(F \cap B(x,r)) \ge \frac{1}{2}\mathcal{L}^n(B(x,r))$. We find $R \ge r$ such that $\mathcal{L}^n(F \cap B(x,R)) = \frac{1}{2}\mathcal{L}^n(B(x,R))$. By the relative isoperimetric inequality (2.8), we have

$$\begin{aligned}
\mathrm{Cap}_1(A) + \varepsilon > |D\chi_F|(\mathbb{R}^n) \ge |D\chi_F|(B(x,R)) &\ge C_I^{-1}\frac{1}{2}R^{-1}\mathcal{L}^n(B(x,R)) \\
&= \frac{\omega_n}{2C_I}R^{n-1} \\
&\ge \frac{\omega_n}{2C_I}r^{n-1} \\
&\ge \frac{\omega_n}{2C_I}r^{n-1}\frac{\mathcal{L}^n(F\cap B(x,r))}{\mathcal{L}^n(B(x,r))} \\
&\ge \frac{\omega_n}{2C_I}r^{n-1}\frac{\mathcal{L}^n(A)}{\mathcal{L}^n(B(x,r))}.
\end{aligned} \tag{2.18}$$

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Letting $\varepsilon \to 0$, we get the first result. Defining the cut-off function

$$\eta(y) := \max\left\{0, 1 - \frac{1}{r}\operatorname{dist}(y, B(x,r))\right\}, \quad y \in \mathbb{R}^n, \tag{2.19}$$

for which $\eta = 1$ in $B(x,r)$ and $\eta = 0$ in $\mathbb{R}^n \setminus B(x,2r)$, we get

$$\operatorname{cap}_1(A, B(x,2r)) \le \int_{\mathbb{R}^n} |\nabla \eta|\, d\mathcal{L}^n \le \frac{\omega_n (2r)^n}{r} \le 2^{n+1} C_I (\operatorname{Cap}_1(A) + \varepsilon)$$

by the first three lines of (2.18). Letting $\varepsilon \to 0$, we get the second result with $C = 2^{n+1} C_I$.

Case 2: Suppose $\mathcal{L}^n(F \cap B(x,r)) < \frac{1}{2}\mathcal{L}^n(B(x,r))$. By the relative isoperimetric inequality,

$$\begin{aligned}\operatorname{Cap}_1(A) + \varepsilon \ge |D\chi_F|(\mathbb{R}^n) \ge |D\chi_F|(B(x,r)) &\ge \frac{1}{C_I r}\mathcal{L}^n(F \cap B(x,r)) \\ &\ge \frac{1}{C_I r}\mathcal{L}^n(A) \\ &\ge \frac{\omega_n}{C_I}\mathcal{L}^n(A)\frac{r^{n-1}}{\mathcal{L}^n(B(x,r))}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get the first result.

By (2.15), we get

$$|D\chi_{B(x,r)\cap F}|(\mathbb{R}^n) \le C|D\chi_F|(B(x,r)) \le C\operatorname{Cap}_1(A) + C\varepsilon. \tag{2.20}$$

By (2.14), we find a sequence $\{u_j\}_{j=1}^\infty$ in $W^{1,1}(\mathbb{R}^n)$ such that $u_j \to \chi_{B(x,r)\cap F}$ in $L^1(\mathbb{R}^n)$, $|Du_j|(\mathbb{R}^n) \to |D\chi_{B(x,r)\cap F}|(\mathbb{R}^n)$, and $u_j \ge 1$ a.e. in a neighbourhood of A. Consider the cut-off function η from (2.19). We have $u_j\eta \to \chi_{B(x,r)\cap F}$ in $L^1(\mathbb{R}^n)$, $|D(u_j\eta)|(\mathbb{R}^n) \to |D\chi_{B(x,r)\cap F}|(\mathbb{R}^n)$, and $u_j\eta \ge 1$ a.e. in a neighbourhood of A. Thus

$$\begin{aligned}\operatorname{cap}_1(A, B(x,2r)) \le \liminf_{j\to\infty}\int_{\mathbb{R}^n}|\nabla(u_j\eta)|\, d\mathcal{L}^n &= |D\chi_{B(x,r)\cap F}|(\mathbb{R}^n) \\ &\le C\operatorname{Cap}_1(A) + C\varepsilon \quad \text{by (2.20)}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get the second result. □

Definition 2.21. *We say that $A \subset \mathbb{R}^n$ is 1-thin at the point $x \in \mathbb{R}^n$ if*

$$\lim_{r\to 0}\frac{\operatorname{Cap}_1(A \cap B(x,r))}{r^{n-1}} = 0.$$

We also say that a set $U \subset \mathbb{R}^n$ is 1-finely open if $\mathbb{R}^n \setminus U$ is 1-thin at every $x \in U$. Then we define the 1-fine topology as the collection of 1-finely open sets on $\mathbb{R}^n$.

We denote the 1-fine interior of a set $H \subset \mathbb{R}^n$, i.e. the largest 1-finely open set contained in H, by fine-int H. *We denote the 1-fine closure of H, i.e. the smallest 1-finely closed set containing H, by $\overline{H}^1$. The 1-base $b_1 H$ is defined as the set of points where H is not 1-thin.*

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See [17, Section 4] for discussion on definition 2.21, and for a proof of the fact that the 1-fine topology is indeed a topology. In fact, in [17], the criterion

$$\lim_{r\to 0} \frac{\mathrm{cap}_1(A\cap B(x,r), B(x,2r))}{r^{n-1}} = 0$$

for 1-thinness was used, in the context of more general metric measure spaces. By (2.16) and lemma 2.17, this is equivalent with our current definition in the Euclidean setting.

According to [16, corollary 3.5], the 1-fine closure of $A \subset \mathbb{R}^n$ can be characterized as

$$\overline{A}^1 = A \cup b_1 A. \tag{2.22}$$

3. Proof of theorem 1.2

In this section, we prove theorem 1.2. We work in $\mathbb{R}^n$ with $n \geq 2$. First, we give the following simple example demonstrating that K_f is generally not a natural quantity to consider for mappings $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$, let alone mappings of lower regularity.

EXAMPLE 3.1. Let $\{q_j\}_{j=1}^{\infty}$ be an enumeration of points in $\mathbb{R}^n$ with rational coordinates. Let $f \in W^{1,n}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$ be such that the first component function is

$$f_1(x) := \sum_{j=1}^{\infty} 2^{-j} \log \max\{-\log|x-q_j|, 1\}, \quad x \in \mathbb{R}^n.$$

Now clearly $\operatorname{diam} f(B(x,r)) = \infty$ for every $x \in \mathbb{R}^n$ and $r > 0$. Thus

$$K_f(x) = \limsup_{r\to 0} \left(\frac{\operatorname{diam} f(B(x,r))^n}{|f(B(x,r))|}\right)^{1/(n-1)} = \limsup_{r\to 0} \left(\frac{+\infty}{|f(B(x,r))|}\right)^{1/(n-1)}$$

for every $x \in \mathbb{R}^n$, and so regardless of the value of $|f(B(x,r))|$, the quantity K_f is either $+\infty$ or undefined.

The Hardy–Littlewood maximal function of a function $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is defined by

$$\mathcal{M}u(x) := \sup_{r>0} \fint_{B(x,r)} |u|\, d\mathcal{L}^n, \quad x \in \mathbb{R}^n. \tag{3.2}$$

We also define a restricted version $\mathcal{M}_R u(x)$, with $R > 0$, by requiring $0 < r \leq R$ in the supremum.

It is well-known, see e.g. [14, theorem 1.15], that

$$|\{x \in \mathbb{R}^n : \mathcal{M}u(x) > t\}| \leq \frac{C_0}{t} \|u\|_{L^1(\mathbb{R}^n)}, \quad t > 0, \tag{3.3}$$

for a constant C_0 depending only on n.

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The following weak-type estimate is standard, see e.g. [6, theorem 4.18]; in this reference, a slightly different definition for capacity is used, but a small modification of the proof gives the following result.

LEMMA 3.4. *Let $u \in \mathrm{BV}(\mathbb{R}^n)$. Then for some constant C depending only on n, we have*

$$\mathrm{Cap}_1(\{\mathcal{M}u > t\}) \le C\frac{\|u\|_{\mathrm{BV}(\mathbb{R}^n)}}{t} \quad \text{for all } t > 0.$$

We will need the following version of lemma 3.4; recall also the definition of $\mathcal{M}_R u$ from above that lemma.

LEMMA 3.5. *Let $u \in L^1(\mathbb{R}^n)$. Then for some constant C depending only on n, we have*

$$\mathrm{Cap}_1(\{\mathcal{M}_1 u > t\} \cap B(x,1)) \le C\frac{\|u\|_{\mathrm{BV}(B(x,2))}}{t} \quad \text{for all } t > 0.$$

Proof. We can assume that $\|u\|_{\mathrm{BV}(B(x,2))}$ is finite. Denote by Eu an extension of u from $B(x,2)$ to $\mathbb{R}^n$ with $\|Eu\|_{\mathrm{BV}(\mathbb{R}^n)} \le C'\|u\|_{\mathrm{BV}(B(x,2))}$, for some C' depending only on n; see e.g. [1, proposition 3.21]. We estimate

$$\begin{aligned}\mathrm{Cap}_1(\{\mathcal{M}_1 u > t\} \cap B(x,1)) &= \mathrm{Cap}_1(\{\mathcal{M}_1 Eu > t\} \cap B(x,1))\\ &\le \mathrm{Cap}_1(\{\mathcal{M}_1 Eu > t\})\\ &\le C\frac{\|Eu\|_{\mathrm{BV}(\mathbb{R}^n)}}{t} \quad \text{by lemma 3.4}\\ &\le CC'\frac{\|u\|_{\mathrm{BV}(B(x,2))}}{t}.\end{aligned}$$

□

It is known that Sobolev and BV functions are approximately differentiable a.e., in the sense of (3.7) below. In the following theorem, we show a stronger property, namely that these functions are also 1-*finely differentiable* a.e.

Recall the definition of the precise representative from (2.10). Recall also that we denote by ∇f the density of the absolutely continuous part of Df.

THEOREM 3.6. *Let $\Omega \subset \mathbb{R}^n$ be open and let $f \in \mathrm{BV}_{\mathrm{loc}}(\Omega;\mathbb{R}^l)$, with $l \in \mathbb{N}$. Then for a.e. $x \in \Omega$ there exists a 1-finely open set $U \ni x$ such that*

$$\lim_{U\setminus\{x\}\ni y\to x} \frac{|f^*(y) - f^*(x) - \nabla f(x)(y-x)|}{|y-x|} = 0.$$

Proof. Since the issue is local, we can assume that $\Omega = \mathbb{R}^n$. First assume also that $l = 1$. At a.e. $x \in \mathbb{R}^n$, we have

$$\lim_{r\to 0} \fint_{B(x,r)} \frac{|f(y) - f^*(x) - \langle \nabla f(x), y-x\rangle|}{r}\, d\mathcal{L}^n(y) = 0, \tag{3.7}$$

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see [1, theorem 3.83], as well as

$$\lim_{r\to 0} \fint_{B(x,r)} |\nabla f(y)-\nabla f(x)|\,d\mathcal{L}^n(y)=0 \quad\text{and}\quad \lim_{r\to 0}\frac{|D^s f|(B(x,r))}{r^n}=0.$$

Consider such x. Define $L(z):=\langle \nabla u(x),z\rangle$. Thus for the scalings

$$f_{x,r}(z):=\frac{f(x+rz)-f^*(x)}{r},\quad z\in B(0,2), \tag{3.8}$$

we have

$$f_{x,r}(\cdot)\to L(\cdot)\ \text{ in } L^1(B(0,2)) \text{ as } r\to 0 \quad\text{and}\quad \nabla f_{x,r}(z)=\nabla f(x+rz),\ z\in B(0,2).$$

Then

$$\begin{aligned}|D(f_{x,r}-L)|(B(0,2)) &= \int_{B(0,2)} |\nabla f_{x,r}(z)-\nabla f(x)|\,d\mathcal{L}^n(z)+|D^s f_{x,r}|(B(0,2))\\ &= \frac{1}{r^n}\int_{B(x,2r)} |\nabla f(y)-\nabla f(x)|\,d\mathcal{L}^n(y)+\frac{|D^s f|(B(x,2r))}{r^n}\\ &\to 0 \quad \text{as } r\to 0.\end{aligned}$$

In conclusion, we have the norm convergence

$$f_{x,r}\to L \quad \text{in } \mathrm{BV}(B(0,2)). \tag{3.9}$$

Note that $(f^*)_{x,r}=(f_{x,r})^*$ in $B(0,2)$, so we simply use the notation $f^*_{x,r}$. Note also that

$$|f^*_{x,r}-L|=|(f_{x,r}-L)^*|\le |f_{x,r}-L|^*\le \mathcal{M}_1|f_{x,r}-L|,$$

and so for every $j\in\mathbb{N}$ and $t>0$ we get

$$\begin{aligned}&\mathrm{Cap}_1(\{z\in B(0,1)\colon |f^*_{x,2^{-j}}(z)-L(z)|>t\})\\ &\quad\le \mathrm{Cap}_1(\{z\in B(0,1)\colon \mathcal{M}_1|f_{x,2^{-j}}-L|(z)>t\})\\ &\quad\le C\frac{\|f_{x,2^{-j}}-L\|_{\mathrm{BV}(B(0,2))}}{t} \quad \text{by lemma 3.5}\\ &\quad\to 0 \quad \text{as } j\to\infty \quad \text{by (3.9)}.\end{aligned}$$

Thus we can choose numbers $t_j\searrow 0$ such that for the sets

$$D_j:=\{z\in B(0,1)\colon |f^*_{x,2^{-j}}(z)-L(z)|>t_j\},$$

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we get $\mathrm{Cap}_1(D_j) \to 0$ as $j \to \infty$. Define $A_j := D_j \setminus B(0,1/2)$ and $A := \bigcup_{j=1}^{\infty} 2^{-j}A_j + x$. Now for all $k \in \mathbb{N}$, we have

$$\begin{aligned}\mathrm{Cap}_1(A \cap B(x,2^{-k})) &\le \sum_{j=k}^{\infty} \mathrm{Cap}_1(2^{-j}A_j + x)\\ &= \sum_{j=k}^{\infty} 2^{-j(n-1)}\,\mathrm{Cap}_1(A_j)\\ &\le \sum_{j=k}^{\infty} 2^{-j(n-1)}\,\mathrm{Cap}_1(D_j)\\ &\le 2^{-k(n-1)+1}\max_{j\ge k}\mathrm{Cap}_1(D_j).\end{aligned}$$

Since $\mathrm{Cap}_1(D_j) \to 0$, we obtain

$$\frac{\mathrm{Cap}_1(A \cap B(0,2^{-k}))}{2^{-k(n-1)}} \to 0 \quad \text{as } k \to \infty,$$

and so clearly A is 1-thin at x. By (2.22), the 1-finely open set $U := \mathbb{R}^n \setminus \overline{A}^1$ contains x. For any $j \in \mathbb{N}$ and $y \in U \cap B(x,2^{-j}) \setminus B(x,2^{-j-1})$, we have

$$\begin{aligned}\frac{|f^*(y) - f^*(x) - L(y-x)|}{|y-x|} &\le 2\frac{|f^*(y) - f^*(x) - L(y-x)|}{2^{-j}}\\ &= 2|f^*_{x,2^{-j}}((y-x)/2^{-j}) - L((y-x)/2^{-j})|\\ &\le 2t_j \to 0 \quad \text{as } j \to \infty,\end{aligned}$$

and so

$$\lim_{U \ni y \to x} \frac{|f^*(y) - f^*(x) - \langle \nabla f(x), y-x\rangle|}{|y-x|} = 0.$$

Finally, the generalization to the case $l \in \mathbb{N}$ is immediate, since the intersection of a finite number of 1-finely open sets is still 1-finely open. □

Given $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\mathbb{R}^n)$, note that the weak gradient ∇f is a function in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^{n\times n})$ and thus may be understood to be an equivalence class rather than a pointwise defined function. Below, we sometimes consider ∇f at a given point; for this, we can understand ∇f to be well defined everywhere by using the above theorem and by defining ∇f to be zero in the exceptional set.

We restate the following definition already given in §1.

Definition 3.10. *Let $f\colon \mathbb{R}^n \to [-\infty,\infty]^n$ and $U \subset \mathbb{R}^n$. Then we let*

$$K_{f,U}(x,r) := \left(\frac{\operatorname{diam} f(B(x,r)\cap U)^n}{|f(B(x,r))|}\right)^{1/(n-1)} \quad \text{and} \quad K_f^{\mathrm{fine}}(x) := \inf \limsup_{r\to 0} K_{f,U}(x,r),$$

where the infimum is taken over 1-finely open sets $U \ni x$. If $|f(B(x,r))| = 0$, then we interpret $K_{f,U}(x,r) = \infty$.

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Proof of theorem 1.2. Let $f \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n;\mathbb{R}^n)$; since the claim is local, we can assume that in fact $f \in W^{1,1}(\mathbb{R}^n;\mathbb{R}^n)$. Using, e.g., [6, theorem 6.15], we find a Lipschitz mapping $\widehat{f} \in \mathrm{Lip}(\mathbb{R}^n;\mathbb{R}^n)$ such that the complement of the set

$$H := \{z \in \mathbb{R}^n \colon (\widehat{f})^*(z) = f^*(z) \text{ and } \nabla \widehat{f}(z) = \nabla f(z)\}$$

has arbitrarily small Lebesgue measure. By, e.g., [1, lemma 2.74], $\mathcal{L}^n$-almost all of the set $\{z \in \mathbb{R}^n \colon \det \nabla \widehat{f}(z) \neq 0\}$ can be covered by compact sets $\{K_j\}_{j=1}^{\infty}$ such that $\widehat{f}$ is injective in each K_j. Consider a point $x \in \mathbb{R}^n$ for which $\det \nabla f(x) \neq 0$. Since the theorem is formulated as an 'a.e.' result, we can assume that

$$\lim_{r\to 0} \frac{|B(x,r)\cap H|}{|B(x,r)|} = 1 \quad \text{and} \quad \lim_{r\to 0} \frac{|B(x,r)\cap K_j|}{|B(x,r)|} = 1 \quad \text{for some } j,$$

that f is 1-finely differentiable as in theorem 3.6 so that we find a 1-finely open set $U \ni x$ such that

$$\lim_{U\setminus\{x\}\ni y\to x} \frac{|f^*(y) - f^*(x) - \nabla f(x)(y-x)|}{|y-x|} = 0, \tag{3.11}$$

and that x is a Lebesgue point of ∇f:

$$\lim_{r\to 0} \fint_{B(x,r)} |\nabla f(y) - \nabla f(x)|\, d\mathcal{L}^n(y) = 0. \tag{3.12}$$

For the scalings

$$f_r(z) := \frac{f^*(x+rz) - f^*(x)}{r}, \quad z \in B(0,1),$$

we have $\nabla f_r(z) = \nabla f(x+rz)$, with $z \in B(0,1)$, and thus by (3.12),

$$\lim_{r\to 0} \int_{B(0,1)} |\nabla f_r - \nabla f(x)|\, d\mathcal{L}^n = 0. \tag{3.13}$$

Fix $\varepsilon > 0$. Let

$$D^r := \{z \in B(0,1) \colon |\det \nabla f_r(z) - \det \nabla f(x)| < \varepsilon |\det \nabla f(x)|\}.$$

By (3.13), we also have $|B(0,1)\setminus D^r| < \omega_n \varepsilon$ for sufficiently small $r > 0$. Let

$$H_r := r^{-1}(H - x).$$

For sufficiently small $r > 0$, we have in total

$$|B(0,1)\setminus D^r| + |B(0,1)\setminus H_r| + |B(0,1)\setminus (K_j)_r| < \omega_n \varepsilon. \tag{3.14}$$

In the set $D^r \cap H_r \cap (K_j)_r$, we have

$$|\det \nabla \widehat{f}_r| = |\det \nabla f_r| \geq (1-\varepsilon)|\det \nabla f(x)|. \tag{3.15}$$

Now by the area formula, see e.g. [1, theorem 2.71], we get

$$\begin{aligned}|f_r(B(0,1))| &\geq |f_r(D^r\cap H_r\cap (K_j)_r)|\\ &= |\widehat{f}_r(D^r\cap H_r\cap (K_j)_r)|\\ &= \int_{D^r\cap H_r\cap (K_j)_r} |\det \nabla \widehat{f}_r|\, d\mathcal{L}^n\\ &\geq (1-\varepsilon)\int_{D^r\cap H_r\cap (K_j)_r} |\det \nabla f(x)|\, d\mathcal{L}^n \quad \text{by (3.15)}\\ &\geq (1-\varepsilon)^2\omega_n|\det \nabla f(x)| \quad \text{by (3.14)}.\end{aligned}$$

Thus

$$|f^*(B(x,r))| = r^n|f_r(B(0,1))| \geq (1-\varepsilon)^2\omega_n r^n|\det \nabla f(x)|.$$

Thus using also the fine differentiability (3.11), we get

$$\begin{aligned}\limsup_{r\to 0}\frac{\operatorname{diam} f^*(B(x,r)\cap U)^n}{|f^*(B(x,r))|} &\leq \limsup_{r\to 0}\frac{2^n\|\nabla f(x)\|^n r^n}{(1-\varepsilon)^2\omega_n r^n|\det \nabla f(x)|}\\ &= \frac{2^n\|\nabla f(x)\|^n}{(1-\varepsilon)^2\omega_n|\det \nabla f(x)|}.\end{aligned}$$

It follows that

$$(1-\varepsilon)^2 K_{f^*}^{\text{fine}}(x)^{n-1}|\det \nabla f(x)| \leq \frac{2^n}{\omega_n}\|\nabla f(x)\|^n.$$

Letting $\varepsilon\to 0$, we get the result. □

4. Proof of theorem 1.3

In this section, we prove our main theorem 1.3. At first, we work in $\mathbb{R}^n$ with $n\geq 2$, but in our main results we need $n=2$. We start with the following simple lemma.

Lemma 4.1. *Assume $\Omega\subset\mathbb{R}^n$ is open, $f\in W^{1,1}_{\text{loc}}(\Omega;\mathbb{R}^n)$ is continuous, $x\in\Omega$, and suppose $U\ni x$ is a 1-finely open set such that*

$$\lim_{U\setminus\{x\}\ni y\to x}\frac{|f(y)-f(x)-\nabla f(x)(y-x)|}{|y-x|} = 0. \tag{4.2}$$

Then

$$\limsup_{r\to 0}\frac{\operatorname{diam} f(B(x,r)\cap U)^n}{|B(x,r)|} = \frac{2^n}{\omega_n}\|\nabla f(x)\|^n.$$

Proof. By lemma 2.17, we have

$$\lim_{r\to 0}\frac{|B(x,r)\setminus U|}{|B(x,r)|} = 0,$$

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and so for the linear mapping $L(y) := \nabla f(x)(y-x)$, we clearly have

$$\lim_{r\to 0}\frac{\operatorname{diam} L(B(x,r)\cap U)}{r} = 2\|\nabla f(x)\|.$$

Then by the fine differentiability (4.2), we also have

$$\lim_{r\to 0}\frac{\operatorname{diam} f(B(x,r)\cap U)}{r} = 2\|\nabla f(x)\|,$$

and so the claim follows. □

Now we show that the following version of theorem 1.2 holds when f is additionally assumed to be a homeomorphism; recall (2.1).

PROPOSITION 4.3. *Let $\Omega,\Omega' \subset \mathbb{R}^n$ be open and let $f \in W^{1,1}_{\mathrm{loc}}(\Omega;\Omega')$ be a homeomorphism. Then we have*

$$K_f^{\mathrm{fine}}(x)^{n-1}J_f(x) = \frac{2^n}{\omega_n}\|\nabla f(x)\|^n \quad \text{for a.e. } x\in\Omega \text{ for which } K_f^{\mathrm{fine}}(x)<\infty,$$

and $K_f(x) = K_f^{\mathrm{fine}}(x)$ for a.e. $x \in \Omega$ where f is differentiable and $0 < J_f(x) < \infty$.

Proof. Consider $x \in \Omega$ for which $K_f^{\mathrm{fine}}(x) < \infty$. Thus, we find a 1-finely open set $V \ni x$ such that

$$\limsup_{r\to 0}\frac{\operatorname{diam} f(B(x,r)\cap V)^n}{|f(B(x,r))|} < \infty.$$

Excluding a $\mathcal{L}^n$-negligible set, we can also assume that $J_f(x) < \infty$ exists as a limit (recall the discussion after (2.1)), and that we find a 1-finely open set $U \ni x$ with

$$\lim_{U\setminus\{x\}\ni y\to x}\frac{|f(y)-f(x)-\nabla f(x)(y-x)|}{|y-x|} = 0,$$

recall theorem 3.6. To prove one inequality, we estimate

$$\begin{aligned}
K_f^{\mathrm{fine}}(x)^{n-1}J_f(x) &\le \limsup_{r\to 0}\frac{\operatorname{diam} f(B(x,r)\cap V\cap U)^n}{|f(B(x,r))|}\lim_{r\to 0}\frac{|f(B(x,r))|}{|B(x,r)|}\\
&= \limsup_{r\to 0}\frac{\operatorname{diam} f(B(x,r)\cap V\cap U)^n}{|f(B(x,r))|}\frac{|f(B(x,r))|}{|B(x,r)|}\\
&\le \limsup_{r\to 0}\frac{\operatorname{diam} f(B(x,r)\cap U)^n}{|B(x,r)|}\\
&= \frac{2^n}{\omega_n}\|\nabla f(x)\|^n
\end{aligned}$$

by lemma 4.1.

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Then we prove the opposite inequality. Let $\varepsilon > 0$. We can choose the 1-finely open set $V \ni x$ such that

$$K_f^{\mathrm{fine}}(x)^{n-1} > \limsup_{r\to 0} \frac{\operatorname{diam} f(B(x,r)\cap V)^n}{|f(B(x,r))|} - \varepsilon$$
$$\geq \limsup_{r\to 0} \frac{\operatorname{diam} f(B(x,r)\cap V\cap U)^n}{|f(B(x,r))|} - \varepsilon.$$

Then

$$\begin{aligned} K_f^{\mathrm{fine}}(x)^{n-1} J_f(x) &\geq \left(\limsup_{r\to 0} \frac{\operatorname{diam} f(B(x,r)\cap V\cap U)^n}{|f(B(x,r))|} - \varepsilon\right) \lim_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|} \\ &= \limsup_{r\to 0} \frac{\operatorname{diam} f(B(x,r)\cap V\cap U)^n}{|B(x,r)|} - \lim_{r\to 0} \varepsilon \frac{|f(B(x,r))|}{|B(x,r)|} \\ &= \frac{2^n}{\omega_n} \|\nabla f(x)\|^n - \varepsilon J_f(x) \end{aligned}$$

by lemma 4.1. Letting $\varepsilon \to 0$, we get the other inequality.

If f is differentiable at $x \in \Omega$ and $0 < J_f(x) < \infty$, we can again assume that $J_f(x)$ exists as a limit, and then we also have

$$\begin{aligned} K_f(x)^{n-1} J_f(x) &= \limsup_{r\to 0} \frac{\operatorname{diam} f(B(x,r))^n}{|f(B(x,r))|} \lim_{r\to 0} \frac{|f(B(x,r))|}{|B(x,r)|} \\ &= \limsup_{r\to 0} \frac{\operatorname{diam} f(B(x,r))^n}{|B(x,r)|} \\ &= \frac{2^n}{\omega_n} \|\nabla f(x)\|^n, \end{aligned}$$

and so

$$K_f(x)^{n-1} = \frac{2^n}{\omega_n} \frac{\|\nabla f(x)\|^n}{J_f(x)} = K_f^{\mathrm{fine}}(x)^{n-1},$$

where we also used the first part of the proposition, which is applicable since $K_f^{\mathrm{fine}}(x) \leq K_f(x) < \infty$. □

We note that Eq. (1.1) in §1 can be proved similarly to proposition 4.3.

We will use Whitney-type coverings consisting of disks. As with balls so far, a disk is always understood to be open unless otherwise specified.

Lemma 4.4. *Let $A \subset D \subset W$, where $W \subset \mathbb{R}^2$ is an open set and A is dense in D. Given a scale $0 < R < \infty$, there exists a finite or countable Whitney-type covering $\{B_k = B(x_k, r_k)\}_k$ of A in W, with $x_k \in A$, $r_k \leq R$, and the following properties:*

(1) *$B_k \subset W$ and $D \subset \bigcup_k \frac{1}{2} B_k$,*
(2) *If $B_k \cap B_l \neq \emptyset$, then $r_k \leq 2r_l$;*
(3) *The disks B_k can be divided into 6400 collections of pairwise disjoint disks.*

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Proof. For every $x \in A$, let $r_x := \min\{R, \frac{1}{4}\operatorname{dist}(x, \mathbb{R}^n \setminus W)\}$. Consider the covering $\{B(x, \frac{1}{10}r_x)\}_{x\in A}$. Clearly this is also a covering of D. By the 5-covering theorem (see e.g. [6, theorem 1.24]), we can pick an at most countable collection of pairwise disjoint disks $B(x_k, \frac{1}{10}r_k)$ such that the disks $B(x_k, \frac{1}{2}r_k)$ cover D. Denote $B_k = B(x_k, r_k)$. We have established property (1).

Suppose $B_k \cap B_l \neq \emptyset$. If $r_l = \frac{1}{4}\operatorname{dist}(x_l, \mathbb{R}^n \setminus W)$, then

$$4r_l = \operatorname{dist}(x_l, \mathbb{R}^n \setminus W) \geq \operatorname{dist}(x_k, \mathbb{R}^n \setminus W) - r_l - r_k \geq 4r_k - r_l - r_k = 3r_k - r_l,$$

and so we get $2r_l \geq r_k$. If $r_l = R$, then $r_k \leq R = r_l$. Thus we get property (2).

For a given k, denote by I the set of those indices $l \in I$ such that $B_l \cap B_k \neq \emptyset$. For all $l \in I$, by (2), we have $r_k \leq 2r_l$ and $\frac{1}{10}B_l \subset 4B_k$, and so

$$\sum_{l\in I} 400^{-1}\pi r_k^2 \leq \sum_{l\in I} 100^{-1}\pi r_l^2 = \sum_{l\in I} \mathcal{L}^2(\tfrac{1}{10}B_l) \leq \mathcal{L}^2(4B_k) = 16\pi r_k^2,$$

and so the cardinality of I is at most 6400, and we obtain (3). □

Lemma 4.5. *Let $A \subset \mathbb{R}^2$. Then $\mathcal{H}^1_\infty(A) \leq 10\operatorname{Cap}_1(A)$.*

Proof. We can assume that $\operatorname{Cap}_1(A) < \infty$. Let $\varepsilon > 0$. We find a function $u \in W^{1,1}(\mathbb{R}^2)$ such that $u \geq 1$ in a neighbourhood of A and

$$\int_{\mathbb{R}^2} |\nabla u|\, d\mathcal{L}^2 \leq \operatorname{Cap}_1(A) + \varepsilon.$$

Here $u \in W^{1,1}(\mathbb{R}^2) \subset \mathrm{BV}(\mathbb{R}^2)$ with $|Du|(\mathbb{R}^2) = \int_{\mathbb{R}^2} |\nabla u|\, d\mathcal{L}^2$, and then by the coarea formula (2.7) we find a set $E := \{u > t\}$ for some $0 < t < 1$, for which

$$|D\chi_E|(\mathbb{R}^2) \leq |Du|(\mathbb{R}^2) \leq \operatorname{Cap}_1(A) + \varepsilon,$$

and A is contained in the interior of E. Then necessarily $|E| < \infty$, and for every $x \in A$ we find $r_x > 0$ such that

$$\frac{|B(x, r_x) \cap E|}{|B(x, r_x)|} = \frac{1}{2}.$$

From the relative isoperimetric inequality (2.9), we get

$$\frac{\pi r_x^2}{2} = \min\{|B(x, r_x) \cap E|, |B(x, r_x) \setminus E|\} \leq r_x |D\chi_E|(B(x, r_x)).$$

In particular, the radii r_x are uniformly bounded from above by $(2/\pi)|D\chi_E|(\mathbb{R}^2)$. By the 5-covering theorem (see e.g. [11, p. 60]), we can choose a finite or countable collection $\{B(x_j, r_j)\}_j$ of pairwise disjoint disks such that the disks $B(x_j, 5r_j)$ cover A. Then

$$\mathcal{H}^1_\infty(A) \leq \sum_j 10r_j \leq \frac{20}{\pi}\sum_j |D\chi_E|(B(x_j, r_j)) \leq 10|D\chi_E|(\mathbb{R}^2) \leq 10(\operatorname{Cap}_1(A) + \varepsilon).$$

Letting $\varepsilon \to 0$, we get the result. □

For $x \in \mathbb{R}^2$, let $p(x) := |x|$.

LEMMA 4.6. *Let $A \subset \mathbb{R}^2$. Then we have $\mathcal{L}^1(p(A)) \le 10\,\mathrm{Cap}_1(A)$.*

Proof. Note that p is a 1-Lipschitz function. Thus we estimate

$$\mathcal{L}^1(p(A)) = \mathcal{H}^1_\infty(p(A)) \le \mathcal{H}^1_\infty(A) \le 10\,\mathrm{Cap}_1(A)$$

by lemma 4.5. □

The following theorem is a more general version of our main theorem 1.3. Note in particular that the function K_f^{fine} is not generally known to be measurable; in theorem 1.3, measurability is an assumption implicitly contained in the fact that $K_f^{\mathrm{fine}} \in L^{p^*/2}_{\mathrm{loc}}(\mathbb{R}^n)$.

THEOREM 4.7. *Let $\Omega, \Omega' \subset \mathbb{R}^2$ be open sets with $|\Omega'| < \infty$, and let $f\colon \Omega \to \Omega'$ be a homeomorphism. Let $1 \le p \le 2$. Suppose $K_f^{\mathrm{fine}} < \infty$ outside a set $E \subset \Omega$ such that for a.e. line L parallel to a coordinate axis, $E \cap L$ is at most countable. Suppose also that there is $h \ge K_f^{\mathrm{fine}}$ such that $h \in L^{p^*/2}(\Omega)$. Then $f \in D^p(\Omega;\mathbb{R}^2)$, and in the case $p=2$ we obtain that f is quasiconformal with $K_f(x) = K_f^{\mathrm{fine}}(x)$ for a.e. $x \in \Omega$ and*

$$\|\nabla f(x)\|^2 \le \frac{\pi}{4}\|K_f^{\mathrm{fine}}\|_{L^\infty(\Omega)} J_f(x) \quad \text{for a.e. } x \in \Omega. \tag{4.8}$$

Recall that here $D^p(\Omega;\mathbb{R}^2)$ is the Dirichlet space, that is, f is not necessarily in $L^p(\Omega;\mathbb{R}^2)$, only in $L^1_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$.

Proof. We can assume that Ω is nonempty, and at first we also assume that Ω is bounded. The crux of the proof is to show D^1-regularity. For this, we use the fact that $h \in L^{p^*/2}(\Omega) \subset L^1(\Omega)$. First, we assume also that h is lower semicontinuous.

Fix $0 < \varepsilon \le 1$. To every $x \in \Omega \setminus E$, there corresponds a 1-finely open set $U_x \ni x$ for which

$$\lim_{r\to 0} \frac{\mathrm{Cap}_1(B(x,r)\setminus U_x)}{r} = 0$$

and

$$\limsup_{r\to 0} \frac{\mathrm{diam}\, f(B(x,r)\cap U_x)}{|f(B(x,r))|^{1/2}} < K_f^{\mathrm{fine}}(x)^{1/2} + \varepsilon.$$

For each $j \in \mathbb{N}$, let A_j consist of points $x \in \Omega \setminus E$ for which

$$\sup_{0<r\le 1/j} \frac{\mathrm{Cap}_1(B(x,r)\setminus U_x)}{r} < \frac{1}{20} \tag{4.9}$$

and

$$\sup_{0<r\le 1/j} \frac{\mathrm{diam}\, f(B(x,r)\cap U_x)}{|f(B(x,r))|^{1/2}} < K_f^{\mathrm{fine}}(x)^{1/2} + \varepsilon, \tag{4.10}$$

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and also (recall the lower semicontinuity of h)

$$K_f^{\mathrm{fine}}(x) < h(y) + \varepsilon \quad \text{for all } y \in B(x, 1/j) \cap \Omega. \tag{4.11}$$

We have $\Omega = \bigcup_{j=1}^{\infty} A_j \cup E$. For $j = 1, 2 \ldots$, we inductively define

$$D_j := \Omega \cap \overline{A_j \setminus \cup_{l=1}^{j-1} D_l} \setminus \bigcup_{l=1}^{j-1} D_l;$$

note that we do not know the sets A_j to be measurable but the sets D_j are Borel sets. Moreover, the sets D_j are disjoint and $D_j \supset A_j \setminus \bigcup_{l=1}^{j-1} D_l$, so that $\Omega = E \cup \bigcup_{j=1}^{\infty} D_j$. We can pick open sets $W_j \supset D_j$ such that $W_j \subset \Omega$ and

$$\int_{W_j} (h + 2\varepsilon) \, d\mathcal{L}^2 \leq \int_{D_j} (h + 2\varepsilon) \, d\mathcal{L}^2 + 2^{-j} \varepsilon \tag{4.12}$$

and such that

$$|f(W_j)| \leq |f(D_j)| + 2^{-j} \varepsilon. \tag{4.13}$$

Fix $R > 0$. For each $k \in \mathbb{N}$, using lemma 4.4, we take a Whitney-type covering

$$\{B_{j,k} = B(x_{j,k}, r_{j,k})\}_k$$

of $A_j \setminus \bigcup_{l=1}^{j-1} D_l$ in W_j at scale $\min\{R, 1/j\}$. By lemma 4.4(1), we know that $D_j \subset \bigcup_k \frac{1}{2} B_{j,k}$ and so

$$\Omega \setminus E \subset \bigcup_{j,k} \tfrac{1}{2} B_{j,k}. \tag{4.14}$$

For each point $x_{j,k}$, there is the corresponding 1-finely open set $U_{x_{j,k}}$. Denote $U_{j,k} := U_{x_{j,k}} \cap B_{j,k}$. For any $x \in \mathbb{R}^2$ and $r > 0$, denote a circle by $S(x, r)$. From (4.9) and lemma 4.6, we obtain that there exists $s_{j,k} \in (r_{j,k}/2, r_{j,k})$ such that $S(x_{j,k}, s_{j,k}) \subset U_{j,k}$.

Define

$$g := 2 \sum_{j,k} \frac{\operatorname{diam} f(U_{j,k})}{r_{j,k}} \chi_{B_{j,k}}.$$

By assumption, for almost every line L in the direction of a coordinate axis, $L \cap E$ is at most countable. Take a line segment $\gamma \colon [0, \ell] \to L \cap \Omega$ in such a line L, with

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length $\ell > 0$. We denote also the image of γ by the same symbol. If $\ell \geq R$, we have

$$\int_\gamma g\,ds \geq 2 \sum_{j,k,\,\gamma\cap\frac{1}{2}B_{j,k}\neq\emptyset} \int_\gamma \frac{\operatorname{diam} f(U_{j,k})}{r_{j,k}} \chi_{B_{j,k}}\,ds \geq \sum_{j,k,\,\gamma\cap\frac{1}{2}B_{j,k}\neq\emptyset} \operatorname{diam} f(U_{j,k}). \tag{4.15}$$

By (4.14), we have

$$\gamma \setminus E \subset \bigcup_{j,k} \tfrac{1}{2}B_{j,k}.$$

Let $0 < \delta < R$. Since $L \cap E$ is at most countable, using the continuity of f we find a finite or countable collection of disks $\{B_l\}$ intersecting $\gamma \cap E$ such that $\overline{B_l} \subset \Omega$ and

$$\gamma \cap E \subset \bigcup_l B_l \quad \text{and} \quad \sum_l \operatorname{diam} f(\overline{B_l}) < \delta.$$

Since the disks $\frac{1}{2}B_{j,k}$ and B_l are open, there is in fact a finite number of them covering γ. Thus, there are finite index sets I_1 and I_2 such that every disk $\frac{1}{2}B_{j,k}$ with $(j,k) \in I_1$ intersects γ and

$$\gamma \subset \bigcup_{(j,k)\in I_1} \tfrac{1}{2}B_{j,k} \cup \bigcup_{l\in I_2} B_l \subset \bigcup_{(j,k)\in I_1} B(x_{j,k}, s_{j,k}) \cup \bigcup_{l\in I_2} B_l.$$

We find subsets $J_1 \subset I_1$ and $J_2 \subset I_2$ such that among the disks $B(x_{j,k}, s_{j,k})$, $(j,k) \in J_1$, B_l, $l \in J_2$, no disk is fully contained in another disk, and we still have

$$\gamma \subset \bigcup_{(j,k)\in J_1} B(x_{j,k}, s_{j,k}) \cup \bigcup_{l\in J_2} B_l.$$

Relabel the disks $B(x_{j,k}, s_{j,k})$, $(j,k) \in J_1$, and B_l, $l \in J_2$, as $B(y_1, r_1), \ldots, B(y_M, r_M)$. We can assume that γ is in the x_2-coordinate direction. Denote by z_m the x_2-coordinate of the point in $\gamma \cap \overline{B}(y_m, r_m)$ with the smallest x_2-coordinate. We can assume that the disks $B(y_1, r_1), \ldots, B(y_M, r_M)$ are ordered such that $z_1 \leq \cdots \leq z_M$. Since these disks cover γ, for each $m = 1, \ldots, M-1$ we have that $\overline{B}(y_{m+1}, r_{m+1})$ necessarily intersects $\bigcup_{m'=1}^{m} \overline{B}(y_{m'}, r_{m'}) \cap \gamma$, and since none of the disks $\overline{B}(y_{m'}, r_{m'})$, $m' \in \{1, \ldots, M\}$ is contained in another one, in fact $S(y_{m+1}, r_{m+1})$ necessarily intersects $\bigcup_{m'=1}^{m} S(y_{m'}, r_{m'})$. In total, $\bigcup_{m=1}^{M} S(y_m, r_m)$ is a connected set. It follows that

$$\operatorname{diam}\left(\left(\bigcup_{m=1}^{M} f(S(y_m, r_m))\right)\right) \leq \sum_{m=1}^{M} \operatorname{diam} f(S(y_m, r_m)). \tag{4.16}$$

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Denote by ω a modulus of continuity of f at the end points of the line segment γ. Then

$$\begin{aligned}
|f(\gamma(0)) - f(\gamma(\ell))| &\le \operatorname{diam}\left(\left(\bigcup_{m=1}^{M} f(S(y_m, r_m))\right)\right) + 2\omega(R) \\
&\le \sum_{m=1}^{M} \operatorname{diam} f(S(y_m, r_m)) + 2\omega(R) \quad \text{by (4.16)} \\
&\le \sum_{j,k,\, \gamma\cap\frac{1}{2}B_{j,k}\neq\emptyset} \operatorname{diam} f(S(x_{j,k}, s_{j,k})) + \sum_{l} \operatorname{diam} f(\overline{B_l}) + 2\omega(R) \\
&\le \sum_{j,k,\, \gamma\cap\frac{1}{2}B_{j,k}\neq\emptyset} \operatorname{diam} f(U_{j,k}) + \sum_{l} \operatorname{diam} f(\overline{B_l}) + 2\omega(R) \\
&\le \int_\gamma g\, ds + \delta + 2\omega(R) \quad \text{by (4.15)}.
\end{aligned}$$

Letting $\delta \to 0$, we get

$$|f(\gamma(0)) - f(\gamma(\ell))| \le \int_\gamma g\, ds + 2\omega(R). \tag{4.17}$$

By Young's inequality, we have for any $b_1, b_2 \ge 0$ and $0 < \kappa \le 1$ that

$$b_1 b_2 = \kappa^{1/2} b_1 \kappa^{-1/2} b_2 \le \frac{1}{2}\kappa b_1^2 + \frac{1}{2}\kappa^{-1} b_2^2. \tag{4.18}$$

For every $j \in \mathbb{N}$, we estimate

$$\begin{aligned}
2\pi^{-1} \sum_k \frac{\operatorname{diam} f(U_{j,k})}{r_{j,k}} |B_{j,k}| &\le 2 \sum_k \operatorname{diam} f(U_{j,k}) r_{j,k} \\
&\le 2 \sum_k |f(B_{j,k})|^{1/2} \left(K_f^{\text{fine}}(x_{j,k})^{1/2} + \varepsilon\right) r_{j,k} \quad \text{by (4.10)} \\
&\le \kappa \sum_k |f(B_{j,k})| + 2\kappa^{-1} \sum_k (K_f^{\text{fine}}(x_{j,k}) + \varepsilon) r_{j,k}^2 \quad \text{by (4.18)}.
\end{aligned}$$

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Using (4.11), we estimate further

$$
\begin{aligned}
&2\pi^{-1}\sum_k \frac{\operatorname{diam} f(U_{j,k})}{r_{j,k}}|B_{j,k}|\\
&\le \kappa\sum_k |f(B_{j,k})| + 2\kappa^{-1}\sum_k\int_{B_{j,k}}(h+2\varepsilon)\,d\mathcal{L}^2\\
&\le 6400\kappa|f(W_j)| \quad \text{by lemma 4.4(3) and injectivity}\\
&\qquad + 12800\kappa^{-1}\int_{W_j}(h+2\varepsilon)\,d\mathcal{L}^2 \quad \text{by lemma 4.4(3)}\\
&\le 6400\kappa(|f(D_j)|+2^{-j}\varepsilon)+\\
&\qquad + 12800\kappa^{-1}\left(\int_{D_j}(h+2\varepsilon)\,d\mathcal{L}^2+2^{-j}\varepsilon\right) \quad \text{by (4.12) and (4.13).}
\end{aligned}
$$

It follows that for every $j\in\mathbb{N}$,

$$
\begin{aligned}
\pi^{-1}\int_\Omega g\,d\mathcal{L}^2 &= 2\pi^{-1}\sum_{j,k}\frac{\operatorname{diam} f(U_{j,k})}{r_{j,k}}|B_{j,k}|\\
&\le 6400\kappa\sum_{j=1}^{\infty}(|f(D_j)|+2^{-j}\varepsilon)+\\
&\qquad + 12800\kappa^{-1}\sum_{j=1}^{\infty}\left(\int_{D_j}(h+2\varepsilon)\,d\mathcal{L}^2+2^{-j}\varepsilon\right) \qquad (4.19)\\
&\le 6400\kappa(|f(\Omega)|+\varepsilon)+12800\kappa^{-1}\left(\int_\Omega(h+2\varepsilon)\,d\mathcal{L}^2+\varepsilon\right)\\
&\le 6400\kappa|f(\Omega)|+12800\kappa^{-1}\int_\Omega(h+2\varepsilon)\,d\mathcal{L}^2+19200\kappa^{-1}\varepsilon.
\end{aligned}
$$

We can pick functions g as above, with the choices $R=1/i$, to obtain a sequence $\{g_i\}_{i=1}^{\infty}$. Recall the definition of pointwise variation from (2.11), as well as (2.12). By (4.17), for $\mathcal{L}^1$-a.e. $z\in\pi_2(\Omega)$, we get for any line segment $\gamma\colon[0,\ell]\to\Omega$ with $\gamma(s):=(z,t+s)$ for some $t\in\mathbb{R}$ that

$$
|f(\gamma(0))-f(\gamma(\ell))|\le\liminf_{i\to\infty}\left(\int_\gamma g_i\,ds+2\omega(1/i)\right)=\liminf_{i\to\infty}\int_\gamma g_i\,ds,
$$

and so

$$
\operatorname{pV}(f_z,\Omega_z)\le\liminf_{i\to\infty}\int_{\Omega_z} g_i\,ds.
$$

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We estimate

$$\begin{aligned}
&\int_{\pi_2(\Omega)} \mathrm{pV}(f_z, \Omega_z)\, d\mathcal{L}^1(z) \\
&\qquad \le \int_{\pi_2(\Omega)} \liminf_{i\to\infty} \int_{\Omega_z} g_i\, ds\, d\mathcal{L}^1(z) \\
&\qquad \le \liminf_{i\to\infty} \int_{\pi_2(\Omega)} \int_{\Omega_z} g_i\, ds\, d\mathcal{L}^1(z) \quad \text{by Fatou's lemma} \\
&\qquad = \liminf_{i\to\infty} \int_{\Omega} g_i\, d\mathcal{L}^2 \quad \text{by Fubini} \\
&\qquad \le 6400\pi\kappa|f(\Omega)| + 12800\pi\kappa^{-1} \int_{\Omega} (h+2\varepsilon)\, d\mathcal{L}^2 + 19200\pi\kappa^{-1}\varepsilon \quad \text{by (4.19).}
\end{aligned}$$

Recall (2.13). Since we can do the above calculation also in the x_1-coordinate direction, we obtain

$$\mathrm{Var}(f,\Omega) \le 12800\pi\kappa|f(\Omega)| + 25600\pi\kappa^{-1} \int_{\Omega} (h+2\varepsilon)\, d\mathcal{L}^2 + 38400\pi\kappa^{-1}\varepsilon.$$

Letting $\varepsilon \to 0$, we get the estimate

$$\mathrm{Var}(f,\Omega) \le 25600\pi \left(\kappa|f(\Omega)| + \kappa^{-1} \int_{\Omega} h\, d\mathcal{L}^2 \right) < \infty.$$

All of the reasoning so far can be done also in every open subset $W \subset \Omega$, and so we have in fact

$$|Df|(W) \le 25600\pi \left(\kappa|f(W)| + \kappa^{-1} \int_{W} h\, d\mathcal{L}^2 \right).$$

By considering small $\kappa > 0$, we find that $|Df|$ is absolutely continuous with respect to $\mathcal{L}^2$ in Ω. Thus we get $f \in D^1(\Omega;\mathbb{R}^2)$, and choosing $\kappa = 1$, we get the estimate

$$\int_{\Omega} |\nabla f|\, d\mathcal{L}^2 \le 25600\pi \left(|f(\Omega)| + \int_{\Omega} h\, d\mathcal{L}^2 \right).$$

Now we remove the assumption that h is lower semicontinuous. Using the Vitali–Carathéodory theorem (theorem 2.5), we find a sequence $\{h_i\}_{i=1}^{\infty}$ of lower semicontinuous functions in $L^1(\Omega)$ such that $h \le h_{i+1} \le h_i$ for all $i \in \mathbb{N}$, and $h_i \to h$ in $L^1(\Omega)$. Thus, we get

$$\int_{\Omega} |\nabla f|\, d\mathcal{L}^2 \le \liminf_{i\to\infty} 25600\pi \left(|f(\Omega)| + \int_{\Omega} h_i\, d\mathcal{L}^2 \right) \le 25600\pi \left(|f(\Omega)| + \int_{\Omega} h\, d\mathcal{L}^2 \right). \tag{4.20}$$

Now we prove that in fact $f \in D^p(\Omega;\mathbb{R}^2)$. Note that $K_f^{\mathrm{fine}} < \infty$ a.e. in Ω. Since $f \in D^1(\Omega,\Omega') \subset W^{1,1}_{\mathrm{loc}}(\Omega;\Omega')$, by proposition 4.3, we have

$$K_f^{\mathrm{fine}}(x)^{1/2} J_f(x)^{1/2} = \frac{2}{\pi^{1/2}} \|\nabla f(x)\| \quad \text{for a.e. } x \in \Omega. \tag{4.21}$$

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In the case $1 < p < 2$, by Young's inequality, and recalling that $h \geq K_f^{\mathrm{fine}}$, we get

$$\frac{2^p}{\pi^{p/2}} \int_\Omega \|\nabla f\|^p \, d\mathcal{L}^2 \leq \int_\Omega J_f \, d\mathcal{L}^2 + \int_\Omega h^{p/(2-p)} \, d\mathcal{L}^2 < \infty \tag{4.22}$$

by (2.2) and by the assumption $h \in L^{p^*/2}(\Omega)$. Thus $f \in D^p(\Omega; \mathbb{R}^2)$.

In the case $p = 2$, we have $K_f^{\mathrm{fine}} \leq h \in L^\infty(\Omega)$, and then from (4.21) we get

$$\|\nabla f(x)\|^2 \leq \frac{\pi}{4} \|h\|_{L^\infty(\Omega)} J_f(x) \quad \text{for a.e. } x \in \Omega. \tag{4.23}$$

This shows that $f \in D^2(\Omega; \Omega')$. By definition 2.3 and the discussion below it, we then have that in fact f is quasiconformal; note that $\nabla f(x)$ is now a classical gradient for a.e. $x \in \Omega$. Moreover, using, e.g., [10, theorem 9.8], we know that f^{-1} is also quasiconformal and thus absolutely continuous in measure, and so $J_f(x) > 0$ for a.e. $x \in \Omega$. Thus by proposition 4.3, we have $K_f(x) = K_f^{\mathrm{fine}}(x)$ for a.e. $x \in \Omega$, and from (4.23) we obtain that (4.8) holds.

Finally, using the Dirichlet energy estimates (4.20), (4.22), and (4.23), it is easy to generalize to the case where Ω is unbounded. □

Proof of theorem 1.3. For every direction $v \in \partial B(0,1)$, the intersection of E with almost every line L parallel to v is at most countable, see e.g. [21, p. 103]. For every bounded open set $\Omega \subset \mathbb{R}^2$, we have $|f(\Omega)| < \infty$ since f is a homeomorphism, and then by theorem 4.7, we have $f \in D^p(\Omega; \mathbb{R}^2)$, and in the case $p = 2$, moreover $K_f(x) = K_f^{\mathrm{fine}}(x)$ for a.e. $x \in \Omega$ and

$$\|\nabla f(x)\|^2 \leq \frac{\pi}{2} \|K_f^{\mathrm{fine}}\|_{L^\infty(\Omega)} J_f(x) \quad \text{for a.e. } x \in \Omega.$$

Thus $f \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2; \mathbb{R}^2)$, and in the case $p = 2$, we have $K_f(x) = K_f^{\mathrm{fine}}(x)$ for a.e. $x \in \mathbb{R}^2$ and

$$\|\nabla f(x)\|^2 \leq \frac{\pi}{2} \|K_f^{\mathrm{fine}}\|_{L^\infty(\mathbb{R}^n)} J_f(x) \quad \text{for a.e. } x \in \mathbb{R}^2.$$

Thus f is quasiconformal in $\mathbb{R}^2$. □

Proof of corollary 1.4. This follows immediately from theorem 1.3. □

In closing, we note that certain open problems arise naturally from our work. Koskela–Rogovin [15, corollary 1.3] prove that in the definition of K_f, one can replace 'lim sup' by 'lim inf', and the result (analogous to theorem 1.3) still holds. Thus one can ask: does theorem 1.3 still hold if 'lim sup' is replaced by 'lim inf' in the definition of K_f^{fine}?

The assumption $n = 2$ was needed on page 18 to ensure that suitable circles are contained in the relevant 1-finely open sets; these circles could then be seen to intersect each other on page 19. In higher dimensions, it is not necessarily true that similar spheres would be contained in the 1-finely open sets. One can ask: can our results be generalized to $\mathbb{R}^n$ with $n \geq 3$, and further to metric measure spaces? Much of the literature on the topic, discussed in §1, in fact deals with the setting

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of quite general metric measure spaces. We observe that most of the quantities and techniques used in the proof of theorem 1.3 make sense also in metric spaces.

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