REFLECTION OF WATER WAVES IN THE PRESENCE OF SURFACE TENSION BY A NEARLY VERTICAL POROUS WALL

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Abstract

In the present paper the problem of reflection of water waves by a nearly vertical porous wall in the presence of surface tension has been investigated. A perturbational approach for the first-order correction has been employed as compared with the corresponding vertical wall problem. A mixed Fourier transform together with the regularity property of the transformed function along the positive real axis has been used to obtain the potential functions along with the reflection coefficients up to first order. Whilst the problem of water of infinite depth is the subject matter of the present paper, a similar approach is applicable to problems associated with water of finite depth.

1. Introduction

A class of boundary-value problems associated with the two-dimensional Laplace equation appears in the scattering of the linearised theory of water waves involving immersed vertical boundaries. Several methods (see [2, 6, 8]) have been considered to solve such boundary-value problems in the literature. Chakrabarti and Sahoo [2] in their recent work have used a method similar to the one considered by Mandal and Kar [6], Mandal and Chakrabarti [5] and Shaw [11] to handle a boundary-value problem which arises in the study of the reflection of water waves by a nearly vertical porous wall. In the present note we have generalised the result of [2] to include the effect of surface tension also. Because of the effect of surface tension, the solution depends on a parameter λ, known as the edge slope constant (see [10]). By using a perturbational approach which has been discussed by earlier workers, we have converted the problem of concern to two boundary-value problems. We have used a mixed Fourier transform to reduce the Laplace equation to two ordinary differential equations. The solution of one of these two ordinary differential equations is obtained by standard methods whereas the other one is solved by the aid of the Green's function

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technique. The reflection coefficients up to first order have been determined by using the regularity property of the transformed function on the positive real axis. The particular case of this general problem giving rise to the solution of the vertical wall problem has been derived as a check.

2. Formulation of the problem

The problem under consideration is two dimensional in nature. We use a rectangular Cartesian co-ordinate system (x, y), in which the y axis has been chosen vertically downward with $y = 0$, $x > 0$ as the undisturbed mean free surface of water and $x = \epsilon c(y)$ with $y > 0$ representing the nearly vertical porous wall, where $\epsilon > 0$ is a small parameter and $c(y)$ is a bounded function of y with $c(0) = 0$. It is assumed that the fluid in the region $y > 0$ is incompressible, the motion is irrotational and simple harmonic in time and is described by the velocity potential $\Phi(x, y, t) = \text{Re}\,[\phi(x, y)e^{-i\omega t}]$, "Re" denoting the "real part", with ω the angular frequency and g the acceleration due to gravity. We will drop the factor $e^{-i\omega t}$ throughout the rest of this paper. The effect of surface tension on the nearly vertical porous wall will be investigated, by way of determining the velocity potential and the reflection coefficients up to the order of ϵ, where ϕ satisfies the partial differential equation

$$\nabla^2 \phi = 0 \quad \text{in the fluid region,} \quad y > 0, \tag{2.1}$$

with the boundary conditions

$$M\phi_{yyy} + \phi_y + K\phi = 0 \quad \text{on} \quad y = 0,\ x > 0. \tag{2.2}$$

Here the positive constants M and K are as given by the relations $M = T/\rho g$ and $K = \omega^2/g$ (see [7, 8]) with subscripts denoting partial derivatives, and

$$\phi_n + ik\phi = 0 \quad \text{on} \quad x - \epsilon c(y) = 0,\ y > 0, \tag{2.3}$$

where the subscript n denotes the derivative in the inward normal direction to the boundary $x = \epsilon c(y)$, $k = \dfrac{b\rho\omega}{\mu} > 0$, with μ being the coefficient of viscosity, ρ the density of fluid and b being a porosity constant, which has the dimension of length. The derivation of the equation (2.3) has been considered by Chakrabarti and Sahoo [2], using a result of Chwang [3]. Also the function ϕ has to satisfy the requirements

$$\text{(I)} \quad \phi, |\nabla\phi| \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty \tag{2.4}$$

and

$$\text{(II)} \quad \phi \longrightarrow e^{-k_0 y - ik_0 x} + Re^{-k_0 y + ik_0 x} \quad \text{as} \quad x \longrightarrow \infty \tag{2.5}$$

at infinity, where R is an unknown constant representing the "reflection coefficient", to be determined, k_0 is the real positive root of the cubic equation $K = \alpha(1 + M\alpha^2)$ in α. The edge condition to be satisfied by the potential ϕ in this case is given by

$$\phi_{xy} + ik\phi_y = \pi\lambda, \quad \text{at} \quad (x, y) = (0+, 0), \tag{2.6}$$

in which λ is the edge slope constant and is indeterminate in nature (see [10]). The solution of the mathematical problem at hand has ultimately been determined which involves this constant λ.

Using the Taylor series expansion in the relation (2.3) and considering the terms up to order ϵ only and neglecting the higher order terms in ϵ, we obtain

$$\phi_x - \epsilon(c(y)\phi_y)_y = -ik[\phi + \epsilon c(y)\phi_x] \quad \text{on} \quad x = 0. \tag{2.7}$$

Employing a perturbation expansion of the potential ϕ, the reflection coefficient R and the indeterminable edge slope constant λ (as in [2, 5 and 6]) in the forms

$$\phi(x, y, \epsilon) = \phi_0(x, y) + \epsilon\phi_1(x, y) + O(\epsilon^2), \tag{2.8}$$
$$R = R_0 + \epsilon R_1 + O(\epsilon^2), \tag{2.9}$$
$$\lambda = \lambda_0 + \epsilon\lambda_1 + O(\epsilon^2), \tag{2.10}$$

the problem under consideration reduces to two boundary-value problems for ϕ_0 and ϕ_1, as explained below.

Problem for $\phi_0(x, y)$

$$\nabla^2\phi_0 = 0 \quad x > 0, \; y > 0, \tag{2.11}$$

with the boundary conditions

$$M\phi_{0yyy} + \phi_{0y} + K\phi_0 = 0 \quad \text{on} \quad y = 0, \; x > 0, \tag{2.12}$$
$$\phi_{0x} + ik\phi_0 = 0 \quad \text{on} \quad x = 0, \tag{2.13}$$
$$\phi_0, \; |\nabla\phi_0| \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty \tag{2.14}$$
$$\phi_0 \longrightarrow e^{-k_0y - ik_0x} + R_0 e^{-k_0y + ik_0x} \quad \text{as} \quad x \longrightarrow \infty, \tag{2.15}$$

and

$$\phi_{0xy} + ik\phi_{0y} = \pi\lambda_0 \quad \text{at} \quad (x, y) = (0+, 0). \tag{2.16}$$

Problem for $\phi_1(x, y)$

$$\nabla^2 \phi_1 = 0 \quad \text{in the region} \quad x > 0, \quad y > 0, \tag{2.17}$$

with the boundary conditions

$$M\phi_{1yyy} + \phi_{1y} + K\phi_1 = 0 \quad \text{on} \quad y = 0, \ x > 0, \tag{2.18}$$

$$\phi_{1x} + ik\phi_1 = f(y) \quad \text{on} \quad x = 0, \ y > 0, \tag{2.19}$$

where $f(y) = [c(y)\phi_{0y}(0, y)]' - ikc(y)\phi_{0x}(0, y)$,

$$\phi_1, \ |\nabla\phi_1| \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty, \tag{2.20}$$

$$\phi_1 \longrightarrow R_1 e^{-k_0 y + ik_0 x} \quad \text{as} \quad x \longrightarrow \infty, \tag{2.21}$$

and

$$\phi_{1xy} + ik\phi_{1y} = \pi\lambda_1 \quad \text{at} \quad (x, y) = (0+, 0). \tag{2.22}$$

In the next section, we present the method of solution of the above boundary-value problems.

3. The method of solution

Determination of $\phi_0(x, y)$ Set

$$\phi_0(x, y) = \psi_0(x, y) + e^{-k_0 y - ik_0 x} + R_0 e^{-k_0 y + ik_0 x} \tag{3.1}$$

and use an integral transform of $\psi_0(x, y)$ as given by

$$\hat{\psi}_0(\xi, y) = \int_0^\infty \psi_0(x, y)(\xi \cos \xi x - ik \sin \xi x) \, dx. \tag{3.2}$$

Substituting for ϕ_0 in the equation (2.11) with its boundary conditions in the relations (2.12)–(2.16) and transforming the various equations and conditions, we obtain that

$$\hat{\psi}_0'' - \xi^2 \hat{\psi}_0 = A_0 \xi e^{-k_0 y}, \tag{3.3}$$

where a prime denotes an ordinary derivative with respect to y (note that ξ is just a transform parameter), with

$$(1 + M\xi^2)\hat{\psi}_0' + K\hat{\psi}_0 + B_0 M\xi = 0 \quad \text{on} \quad y = 0, \tag{3.4}$$

where $A_0 = ik_0(1 - R_0) - ik(1 + R_0)$, $B_0 = \pi\lambda_0 + ik_0(k - k_0) + R_0 ik_0(k + k_0)$, and

$$\hat{\psi}_0, |\hat{\psi}_0'| \longrightarrow 0, \quad \text{as} \quad y \longrightarrow \infty. \tag{3.5}$$

Then, setting

$$\hat{\psi}_0(\xi, y) = P(\xi, y) - \frac{B_0 M \xi}{K}, \tag{3.6}$$

we obtain from the equations (3.3), (3.4) and (3.5), the ordinary differential equation

$$P'' - \xi^2 P = \xi A_0 e^{-k_0 y} - \frac{\xi^3 B_0 M}{K}, \tag{3.7}$$

with the boundary conditions

$$(1 + M\xi^2)P' + KP = 0 \quad \text{on} \quad y = 0 \tag{3.8}$$

and

$$P' \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty. \tag{3.9}$$

The solution of equation (3.7) satisfying the conditions (3.8) and (3.9) is given by

$$P(\xi, y) = \frac{\pi\lambda_0 \xi M e^{-\xi y}}{\xi(1 + M\xi^2) - K} - \frac{\xi A_0 e^{-k_0 y}}{\xi^2 - k_0^2} + \frac{\xi B_0 M}{K}, \tag{3.10}$$

which gives

$$\hat{\psi}_0(\xi, y) = \frac{\pi\lambda_0 \xi M e^{-\xi y}}{\xi(1 + M\xi^2) - K} - \frac{\xi A_0 e^{-k_0 y}}{\xi^2 - k_0^2}. \tag{3.11}$$

Here we observe that the expression (3.11) for $\hat{\psi}_0(\xi, y)$ involves an unknown constant as well as a singularity at $\xi = k_0$ on the positive real axis. In order to determine this constant, we use the fact that $\hat{\psi}_0$ is a Fourier transform and cannot have a singularity on the positive real axis. This forces the result that

$$\lim_{\xi \to k_0} (\xi - k_0)\hat{\psi}_0(\xi, y) = 0, \tag{3.12}$$

so that

$$A_0 = \frac{2\pi\lambda_0 k_0 M}{1 + 3Mk_0^2}, \tag{3.13}$$

which implies that

$$R_0 = \frac{(k_0 - \frac{b\rho\omega}{\mu})}{(k_0 + \frac{b\rho\omega}{\mu})} + \frac{2iMk_0\pi\lambda_0}{(1 + 3Mk_0^2)(k_0 + \frac{b\rho\omega}{\mu})}. \tag{3.14}$$

Next, using the inverse transform of $\hat{\psi}_0(\xi, y)$, as given by the relation (3.2) (see [12]), we obtain that

$$\psi_0(x, y) = c_0(y)e^{-ikx} + \frac{2}{\pi}\int_0^\infty \frac{\hat{\psi}_0(\xi, y)(\xi\cos\xi x - ik\sin\xi x)}{\xi^2 - k^2}\,d\xi. \tag{3.15}$$

Here and in later occurrences, singular integrals are understood in the sense of their Cauchy principal values and $c_0(y)$ is an arbitrary function of y which has to be determined by using the behaviour of $\psi_0(x, y)$ as $x \to \infty$.

Now writing $(\xi\cos\xi x - ik\sin\xi x) = \frac{1}{2}\{(\xi - k)e^{i\xi x} + (\xi + k)e^{-i\xi x}\}$ and rotating the contour to the positive imaginary axis for the integral involving $e^{i\xi x}$ and to the negative imaginary axis for the integral involving $e^{-i\xi x}$, we can write (3.15) as

$$\psi_0(x, y) = c_0(y)e^{-ikx} + i\hat{\psi}_0(k, y)e^{-ikx}$$
$$+ \frac{2}{\pi}\int_0^\infty e^{-\eta x}A_0(\eta)\{\eta(1 - M\eta^2)\cos\eta y - K\sin\eta y\}\,d\eta, \tag{3.16}$$

where

$$A_0(\eta) = \frac{\pi\lambda_0 M\eta}{(\eta - ik)\{K^2 + \eta^2(1 - M\eta^2)^2\}}. \tag{3.17}$$

Using the behaviour of $\psi_0(x, y)$ at $x = \infty$, we obtain

$$c_0(y) = -i\hat{\psi}_0(k, y). \tag{3.18}$$

Thus

$$\psi_0(x, y) = \frac{2}{\pi}\int_0^\infty e^{-\eta x}A_0(\eta)\{\eta(1 - M\eta^2)\cos\eta y - K\sin\eta y\}\,d\eta, \tag{3.19}$$

where $A_0(\eta)$ is as given above in (3.17).

The function $\phi_0(x, y)$ can finally be determined by (3.1), where $\psi_0(x, y)$ and R_0 are given by (3.19) and (3.14) respectively with $k = b\rho\omega/\mu$. This completes the solution of the problem in this case.

Determination of $\phi_1(x, y)$ We set

$$\phi_1(x, y) = \psi_1(x, y) + R_1 e^{-k_0 y + ik_0 x} \tag{3.20}$$

Then using a transform as defined by (3.2) for $\psi_1(x, y)$, (2.17) with the boundary conditions (2.18) to (2.22) reduce to

$$\hat{\psi}_1{}'' - \xi^2 \hat{\psi}_1 = \xi [A_1 e^{-k_0 y} + f(y)] - \frac{\xi^3 B_1 M}{K}, \tag{3.21}$$

where $A_1 = -i(k+k_0)R_1$ and $B_1 = \pi \lambda_1 + i k_0 R_1 (k+k_0)$, with the boundary conditions

$$(1 + M\xi^2)\hat{\psi}_1{}' + K \hat{\psi}_1 = 0 \quad \text{on} \quad y = 0 \tag{3.22}$$

and

$$\hat{\psi}_1' \longrightarrow 0 \quad \text{as} \quad y \longrightarrow \infty. \tag{3.23}$$

In this case we will use a Green's function technique to solve the ordinary differential equation (3.21) in $\hat{\psi}_1$ satisfying the boundary conditions (3.22) and (3.23). The solution is given by

$$\hat{\psi}_1(\xi, y) = \xi \int_0^\infty G(y, s) \left\{ A_1 e^{-k_0 s} + f(s) - \frac{\xi^2 B_1 M}{K} \right\} ds - \frac{\xi B_1 M}{K}, \tag{3.24}$$

where $G(y, s)$ is given by

$$G(y, s) = \begin{cases} \dfrac{\xi(1 + M\xi^2) \cosh \xi s - K \sinh \xi s}{\{\xi(1 + M\xi^2) - K\}\xi} e^{-\xi y} & \text{for} \quad y \geq s \\[4mm] \dfrac{\xi(1 + M\xi^2) \cosh \xi y - K \sinh \xi y}{\{\xi(1 + M\xi^2) - K\}\xi} e^{-\xi s} & \text{for} \quad y \leq s. \end{cases} \tag{3.25}$$

The expression (3.24) for $\hat{\psi}_1(\xi, y)$ has a singularity at $\xi = k_0$ which suggests that as in the problem for $\hat{\psi}_0$, we must have in this case

$$\lim_{\xi \to k_0} (\xi - k_0)\hat{\psi}_1(\xi, y) = 0. \tag{3.26}$$

Using the result

$$k_0(1 + Mk_0^2) \cosh k_0 y - K \sinh k_0 y = K e^{-k_0 y}$$

we obtain from (3.24) and (3.26) that

$$A_1 = \frac{2[k_0 M B_1 - K a(k_0)]}{(1 + Mk_0^2)}, \tag{3.27}$$

where $a(k_0) = v \int_0^\infty u(s)e^{-k_0 s} ds$.

The relation (3.27) for A_1 implies that

$$R_1 = -\frac{2i\,[Ka(k_0) - \pi k_0 M \lambda_1]}{(k_0 + b\rho\omega/\mu)(1 + 3Mk_0^2)}. \tag{3.28}$$

In this case also by Fourier sine inversion

$$\psi_1(x, y) = c_1(y)e^{-ikx} + \frac{2}{\pi}\int_0^\infty \frac{\hat{\psi}_1(\xi, y)(\xi \cos \xi x - ik \sin \xi x)}{\xi^2 - k^2}\,d\xi, \tag{3.29}$$

with $c_1(y)$ an arbitrary unknown function of y which has to be determined by using the condition on $\hat{\psi}_1$ as $x \to \infty$.

Proceeding as in the previous case we obtain the alternate expression

$$\psi_1(x, y) = c_1(y)e^{-ikx} + i\hat{\psi}_1(k, y)e^{-ikx}$$
$$- \frac{2}{\pi}\int_0^\infty \frac{e^{-\eta x}A_1(\eta)\left\{\eta(1 - M\eta^2)\cos \eta y - iK \sin \eta y\right\}\,d\eta}{(\eta - ik)\{K^2 + \eta^2(1 - M\eta^2)^2\}} \tag{3.30}$$

for $\psi_1(x, y)$, where $A_1(\eta)$ is given by

$$A_1(\eta) = -M\eta\pi\lambda_1 + \int_0^\infty f(s)\left\{\eta(1 - M\eta^2)\cos \eta s - K \sin \eta s\right\}\,ds. \tag{3.31}$$

Using the condition on ψ_1 as x approaches infinity, we obtain that

$$c_1(y) = -i\hat{\psi}_1(k, y). \tag{3.32}$$

Thus, from (3.20), (3.30) and (3.32), we obtain the complete solution

$$\phi_1(x, y) = R_1 e^{-k_0 y + ik_0 x} - \frac{2}{\pi}\int_0^\infty e^{-\eta x}A_1(\eta)\frac{\left\{\eta(1 - M\eta^2)\cos \eta y - K \sin \eta y\right\}}{(\eta - ik)\{K^2 + \eta^2(1 - M\eta^2)^2\}}\,d\eta, \tag{3.33}$$

where R_1 and $A_1(\eta)$ are given by (3.28) and (3.31).

4. Particular cases

Case 1   When surface tension is neglected, we have $M = 0$ and $k_0 = K$. Then

$$\phi_0(x, y) = e^{-Ky - iKx} + R_0 e^{-Ky + iKx}, \tag{4.1}$$

$$\phi_1(x, y) = R_1 e^{-Ky + iKx} + \int_0^\infty e^{-\xi x}A_1(\xi)(\xi \cos \xi y - K \sin \xi y)\,d\xi, \tag{4.2}$$

where

$$A_1(\xi) = \frac{2}{\pi} \frac{\int_0^\infty (\xi \cos \xi s - K \sin \xi s) f(s) \, ds}{(-\xi + i b \rho \omega / \mu)(\xi^2 + K^2)}, \tag{4.3}$$

$$R_0 = \frac{K - b \rho \omega / \mu}{K + b \rho \omega / \mu} \tag{4.4}$$

and

$$R_1 = \frac{-2i K \int_0^\infty e^{-Ks} f(s) \, ds}{K + b \rho \omega / \mu}, \tag{4.5}$$

which agrees with the result obtained by Chakrabarti and Sahoo [2].

Case 2   When porosity is neglected and the wall is vertical we have $b \rho \omega / \mu = 0$ and $\epsilon = 0$. So we get

$$\phi(x, y) = \frac{2\pi i}{1 + 3Mk_0^2} (Ae^{ik_0x} + Be^{-ik_0x})e^{-k_0y} \tag{4.6}$$

$$+ 2M\lambda \int_0^\infty \frac{\left(\xi(1 - M\xi^2) \cos \xi y - K \sin \xi y\right) e^{-\xi x}}{\xi^2 (1 - M\xi^2)^2 + K^2} \, d\xi \tag{4.7}$$

with $A = (1 + 3Mk_0^2)/2\pi i$ and $B = A + M\lambda$. This result agrees with that obtained by Rhodes-Robinson [9] modulo a misprint.

Acknowledgement

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