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A PROPERTY OF CLOSED FINITE TYPE CURVES

MIROSLAVA PETROVIĆ-TORGAŠEV

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Abstract

In the paper we prove that any closed finite type curve in the Euclidean space E^n $(n \geq 2)$ lies in a null-space of a non-trivial polynomial $P = P(x_1, \ldots, x_n)$ of variables $x_1, \ldots, x_n$, and thus lies on a surface of finite degree.

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We shall first recall the notion of a closed finite type curve in the Euclidean space E^n.

According to a well-known result of Chen [1, 2], a closed curve $\gamma(s)$ of length $2\pi r$ is of finite l-type $(l \geq 1)$ if and only if it can be written in the form

$$\gamma(s) = A_0 + \sum_{\nu=1}^{l} \left(A_{p_\nu} \cos \frac{p_\nu s}{r} + B_{p_\nu} \sin \frac{p_\nu s}{r} \right), \qquad (*)$$

where $A_0 \in E^n$, $p_1 < p_2 < \cdots < p_l$ are non-zero natural numbers and

$$A_{p_1}, A_{p_2}, \ldots, A_{p_l}, B_{p_1}, B_{p_2}, \ldots, B_{p_l}$$

are vectors in E^n such that, for each $\nu \in \{1, 2, \ldots, l\}$, A_{p_ν} and B_{p_ν} are not simultaneously zero. In that case the numbers $p_1, p_2, \ldots, p_l$ are called the *frequency numbers* of the curve γ.

Substituting parameter s into t by $(s/r) = t$ $(0 \leq t \leq 2\pi)$ in the equation $(*)$ and extending the sum in $(*)$ with $A_i = B_i = 0$ $(i \leq m, i \neq p_1, \ldots, p_l)$, it follows that the curve $\gamma(s)$ can also be written in the following form:

$$\gamma(t) = A_0 + \sum_{k=1}^{m} (A_k \cos kt + B_k \sin kt) \quad (0 \leq t \leq 2\pi)$$

where $m = p_l$ and $\|A_i\|^2 + \|B_i\|^2 \neq 0$ $(i = p_1, \ldots, p_l = m)$. In particular, we have that $\|A_m\|^2 + \|B_m\|^2 \neq 0$.

In several papers (see, for example, [3–5]) we have characterized the closed finite type curves in the space E^n which lie on quadrics in that space.

Next, we shall call a surface in the space E^n a 'surface of finite degree' (more exactly 'of degree d') if it is defined as a null-space $\mathcal{N}(P)$ of a non-trivial polynomial $P = P(x_1, \ldots, x_n)$ of variables $x_1, \ldots, x_n$ of some degree d ($d \geq 1$).

Note that surfaces of degree one are the simplest in this class since they have equations of the form

$$a_1 x_1 + \cdots + a_n x_n + b = 0 \quad (a_1^2 + \cdots + a_n^2 \neq 0).$$

Thus, they are exactly hyperplanes in the space E^n.

Surfaces of degree two are in fact quadrics in the space E^n, surfaces of degree three are cubics in the space E^n, and so forth.

In the present paper we shall consider the closed finite type curves and the surfaces of finite degree in a Euclidean space, and the following problem.

> *Do any closed finite type curves $\gamma(s)$ in the Euclidean space E^n lie on a surface of finite degree?*

This statement is obviously not true for $n = 1$ since the null-space of any polynomial $P(x)$ of one variable is finite, and finite type curves are continuous, differentiable, and so on. However, in what follows we positively solve this question for every $n \geq 2$.

The key of the proof is the next lemma.

LEMMA. *For any two trigonometrical polynomials*

$$x = \sum_{k=0}^{m} (a_k \cos kt + b_k \sin kt), \qquad y = \sum_{k=0}^{m} (c_k \cos kt + d_k \sin kt) \quad (0 \leq t \leq 2\pi),$$

there is at least one polynomial $P_s(x, y)$ of two variables x, y, whose degree is s $(0 < s \leq 2m + 1)$ such that $P_s(x(t), y(t)) \equiv 0$, that is the curve $\gamma(t) = (x(t), y(t)) \in R^2$ lies in the null-space $\mathcal{N}(P)$ of that polynomial.

If at least one of the coefficients a_m, b_m, c_m, $d_m \neq 0$, then this polynomial is non-trivial.

PROOF. The basic idea is to construct at least one resultant of two trigonometric polynomials as was done for two algebraic polynomials.

Consider the equations of the curve $\gamma(t)$ in the form

$$a_0 - x + \sum_{k=1}^{m} (a_k \cos kt + b_k \sin kt) = 0, \tag{1}$$

$$c_0 - y + \sum_{k=1}^{m} (c_k \cos kt + d_k \sin kt) = 0. \tag{2}$$

Multiplying equation (1) respectively by 1, $\cos t, \cos 2t, \ldots, \cos mt, \sin t$, $\sin 2t, \ldots, \sin mt$, then equation (2) respectively by $\cos t, \cos 2t, \ldots, \cos mt, \sin t$, $\sin 2t, \ldots, \sin mt$, we obtain the following equations:

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$$a_0 - x + \sum_{k=1}^{m}(a_k \cos kt + b_k \sin kt) = 0, \tag{3}$$

$$(a_0 - x)\cos pt + \sum_{k=1}^{m}(a_k \cos kt \cos pt + b_k \sin kt \cos pt) = 0 \quad (p = 1, \ldots, m), \tag{3p}$$

$$(a_0 - x)\sin pt + \sum_{k=1}^{m}(a_k \cos kt \sin pt + b_k \sin kt \sin pt) = 0 \quad (p = 1, \ldots, m), \tag{4p}$$

$$(c_0 - y)\cos pt + \sum_{k=1}^{m}(c_k \cos kt \cos pt + d_k \sin kt \cos pt) = 0 \quad (p = 1, \ldots, m), \tag{5p}$$

$$(c_0 - y)\sin pt + \sum_{k=1}^{m}(c_k \cos kt \sin pt + d_k \sin kt \sin pt) = 0 \quad (p = 1, \ldots, m). \tag{6p}$$

Next, using the well-known additional theorems for products of trigonometric functions $\cos x$, $\sin y$, one can transform each of the above $4m + 1$ equations into an equation which shows that a linear combination of the functions $1; \cos t, \ldots, \cos 2mt; \sin t, \ldots, \sin 2mt$ equals zero.

Hence, we obtain a linear system of homogenous equations of order $4m + 1$ with unknowns $1, \cos t, \ldots, \cos 2mt; \sin t, \ldots, \sin 2mt$. Since for each t it has a non-trivial solution, the corresponding determinant must be zero.

Note that this determinant $\det(A_{ij})$ $(i, j \le 4m + 1)$ has for almost all entries A_{ij} $(i, j \le 4m + 1)$ some real numbers depending on the coefficients $a_0, c_0, a_p, b_p, c_p, d_p$ $(p = 1, \ldots, m)$, and the only entries involving x and y are the following:

$$\begin{gathered}
A_{11} = 2X, \quad A_{1+p,1+p} = 2X + \epsilon_p a_{2p}, \\
A_{m+1+p,2m+1+p} = 2X - \epsilon_p a_{2p}, \\
A_{2m+1+p,1+p} = 2Y + \epsilon_p a_{2p}, \\
A_{3m+1+p,2m+1+p} = 2Y - \epsilon_p a_{2p},
\end{gathered}$$

for any $p = 1, 2, \ldots, m$, where $X = a_0 - x$, $Y = c_0 - y$, $\epsilon_p = 1$ $(p \le m/2)$ and $\epsilon_p = 0$ $(p > m/2)$, $p = 1, 2, \ldots, m$.

It is clear that the development of this determinant by the Laplace formula gives a polynomial $P(x, y)$ of two variables x, y, and its degree is at most $2m + 1$.

It is also obvious that the above polynomial is not unique, and there are many such polynomials with a similar property, which can be constructed in a similar way.

Moreover, we notice the following fact. Observe that all entries of the last column of the determinant by which the polynomial $P(x, y)$ is defined are equal to a_m, b_m, c_m, d_m or zero. Hence the above polynomial is obviously identically zero if $a_m = b_m = c_m = c_m = 0$.

Now we shall prove that this is the only case when this polynomial is trivial. Thus, we assume that $P(x, y) \equiv 0$, and therefore that all its coefficients are zero, and we prove that then $a_m = b_m = c_m = d_m = 0$. Observe that the coefficient at X^{2m+1} is also zero, and up to the multiple $\pm 2^{2m}$ this coefficient reads

$$A_{2m}(c_1, \ldots, c_m; d_1, \ldots, d_m) = \begin{vmatrix} C_m & D_m \\ -D_m & C_m \end{vmatrix},$$

where

$$C_m = \begin{bmatrix} c_m & 0 & \ldots & 0 \\ c_{m-1} & c_m & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & \ldots & c_m \end{bmatrix}, \quad D_m = \begin{bmatrix} d_m & 0 & \ldots & 0 \\ d_{m-1} & d_m & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ d_1 & d_2 & \ldots & d_m \end{bmatrix}.$$

But, since this coefficient satisfies the relation

$$A_{2m}(c_1, \ldots, c_m; d_1, \ldots, d_m) = (c_m^2 + d_m^2) A_{2m-2}(c_2, \ldots, c_m; d_2, \ldots, d_m),$$

we easily find that $A_{2m}(c_1, \ldots, c_m; d_1, \ldots, d_m) = (c_m^2 + d_m^2)^m$. From $A_{2m} = 0$ we obtain that $c_m = d_m = 0$.

Similarly, assuming that $P(x, y) \equiv 0$, and considering the coefficient at XY^{2m}, we find that $A_{2m}(a_1, \ldots, a_m; b_1, \ldots, b_m) = 0$.

This immediately gives that $a_m = b_m = 0$. Hence we finally get that $a_m = b_m = c_m = d_m = 0$. So we conclude that the above polynomial $P(x, y)$ is identically zero if and only if $a_m = b_m = c_m = d_m = 0$. □

In addition, we give the full form of the above polynomial in the simplest case $m = 1$. Then the polynomial from the above lemma has degree three and it reads

$$\begin{aligned} P(x, y) &= \begin{vmatrix} a_0 - x & a_1 & 0 & b_1 & 0 \\ a_1 & 2(a_0 - x) & a_1 & 0 & b_1 \\ b_1 & 0 & -b_1 & 2(a_0 - x) & a_1 \\ c_1 & 2(c_0 - y) & c_1 & 0 & d_1 \\ d_1 & 0 & -d_1 & 2(c_0 - y) & c_1 \end{vmatrix} \\ &= -4(c_1^2 + d_1^2)X^3 + 8(a_1 c_1 + b_1 d_1)X^2 Y - 4(a_1^2 + b_1^2)XY^2 \\ &\quad + 4(a_1^2 d_1^2 - b_1^2 c_1^2)X + 4(b_1^2 c_1^2 - a_1^2 d_1^2)Y, \end{aligned}$$

where $X = a_0 - x$ and $Y = c_0 - y$.

By the above lemma, we immediately get the main theorem.

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THEOREM. *If $\gamma(s)$ is any closed finite type curve in the Euclidean space E^n $(n \geq 2)$ whose frequency numbers are $p_1 < \cdots < p_k = m$, then there is an index $i = i_0 \leq n$ such that for any index $j \leq n$, $j \neq i_0$ there is a non-trivial polynomial $P^{i_0,j}_{2m+1}(x_{i_0}, x_j)$ of degree $2m+1$ such that $P^{i_0,j}_{2m+1}(x_{i_0}(s), x_j(s)) \equiv 0$, that is, the projected curve $(x_{i_0}(s), x_j(s)) \subseteq \mathcal{N}(P^{i_0,j}_{2m+1})$.*

PROOF. Since $\gamma(t)$ is a closed finite type curve in the space E^n, its equation has the form

$$x_i(t) = a_{i0} + \sum_{k=1}^{m}(a_{ik}\cos kt + b_{ik}\sin kt),$$

$(i = 1, \ldots, n)$. Then $A_m = (a_{1m}, \ldots, a_{nm})^\top$, $B_m = (b_{1m}, \ldots, b_{nm})^\top$ and $\|A_m\|^2 + \|B_m\|^2 \neq 0$. The last relation gives that $a_{im} \neq 0$ or $b_{im} \neq 0$ for some $i = i_0 \leq n$. Assuming that for instance $a_{i_0 m} \neq 0$, the previous lemma provides that, for any index $j \neq i_0$ $(j \leq n)$, there is a non-trivial polynomial $P^{i_0,j}_{2m+1}(x_{i_0}, x_j)$ whose degree is $2m+1$, which is not identically zero, such that $\gamma(t) \subseteq \mathcal{N}(P^{i_0,j}_{2m+1})$. Hence

$$\gamma(t) \subseteq \bigcap_{j \neq i_0} \mathcal{N}(P^{i_0,j}_{2m+1}).$$

This completes the proof. □

Note that the null-spaces $\mathcal{N}(P)$ are for $n \geq 3$ some cylindrical surfaces in the space E^n, and every polynomial P of two variables x_i, x_j $(i < j)$ is also a polynomial of n variables $x_1, x_2, \ldots, x_n$. Hence, any closed finite type curve $\gamma(s)$ lies in a null-space $\mathcal{N}(Q)$ of a polynomial Q of n variables $x_1, \ldots, x_n$, and moreover it lies in the intersection of such $n-1$ cylindrical surfaces.

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University of Kragujevac
Faculty of Science
Radoja Domanovića 12
34000 Kragujevac
Serbia
e-mail: mirapt@kg.ac.yu

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