A THEOREM ON RINGS

I. N. HERSTEIN

In a recent paper, Kaplansky [**2**] proved the following theorem: Let R be a ring with centre Z, and such that $x^{n(x)} \in Z$ for every $x \in R$. If R, in addition, is semi-simple then it is also commutative.

The existence of non-commutative rings in which every element is nilpotent rules out the possibility of extending this result to all rings. One might hope, however, that if R is such that $x^{n(x)} \in Z$ for all $x \in$ R and the nilpotent elements of R are reasonably "well-behaved," then Kaplansky's theorem should be true without the restriction of semi-simplicity.

This is in fact what we obtain in this paper. More precisely, we prove the following two theorems:

THEOREM. *Let R be a ring with centre Z such that $x^{n(x)} \in Z$ for all $x \in R$. Then R is not commutative only if every element in the commutator ideal of R is nilpotent.*

THEOREM. *Let R be a ring with centre Z such that $x^{n(x)} \in Z$ for all $x \in R$. Then if R possesses no non-zero nil-ideals it is commutative.*

Since every nil-ideal of a ring is in the radical of that ring, these results contain that of Kaplansky which we have cited. Any restriction on a ring which will forbid the commutator ideal from being a nil-ideal, in the presence of $x^{n(x)} \in Z$, will force commutativity on the ring in question.

Henceforth every ring R which we consider will have centre Z and the property that $x^{n(x)}$ is in Z for every x in R, where $n(x)$ is a positive integer depending on x. Whenever we use the word ideal we mean a two-sided ideal.

We begin with

THEOREM 1. *Suppose that in R,*

(i) *Z possesses no divisors of zero of R,*

(ii) *there is an $a \in Z$, $a \neq 0$, so that, given any non-zero ideal U of R, then for some integer $m(U)$, $a^{m(u)} \in U$. Under these conditions R is commutative.*

Proof. Consider the set of ordered pairs (r, s) where $r \in R$, $s \neq 0 \in Z$. We define $(r_1, s_1) \sim (r_2, s_2)$ if and only if $r_1 s_2 = r_2 s_1$. Clearly this is a proper equivalence relation. We denote the equivalence class of (r, s) by $[r, s]$. Let R^* be the set of all these equivalence classes. In R^* we define an addition and multiplication by

$$[b, c] + [d, g] = [bg + dc, cg] \tag{1}$$

$$[b, c]\,[d, g] = [bd, cg]. \tag{2}$$

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Since Z is free of divisors of zero of R, these operations are meaningful and independent of the particular representatives of the classes. Moreover R^* forms a ring under these operations. If we denote $[rs, s]$ by $[r, 1]$, then the set $\bar{R} = \{[r, 1] \in R^* \mid r \in R\}$ is a ring isomorphic (in the obvious way) to R. Let Z^* be the centre of R^*. A simple computation shows that $[r, s] \in Z^*$ if and only if $r \in Z$. Moreover Z^* is now a *field*.

We wish to show that R^* is a simple ring. Suppose $U^* \neq (0)$ is an ideal of R^*. Let

$$U = \{x \in R \mid [x, 1] \in U^*\}.$$

U is not merely (0), for if $[b, z] \in U^*$, $b \neq 0$, then $[z, 1]\,[b, z] = [b, 1] \in U^*$. A simple verification shows that U is an ideal of R. Since this is so, by our hypothesis (2), $a^{m(u)} \in U$ for an appropriate integer $m(u)$; that is $[a^{m(u)}, 1] \in U^*$. But since $[a^{m(u)}, 1] \in Z^*$ it has an inverse in R^*, whence $U^* = R^*$. Hence R^* is a simple ring, and so is semi-simple. In R^* we also have $x^{n(x)} \in Z^*$ for each $x \in R^*$, and so, by Kaplansky's theorem, R^* is commutative. Since $R^* \supset \bar{R}$ an isomorphic replica of R we immediately have that R is commutative.

We next proceed to

THEOREM 2. *Suppose in R that $c \neq 0$ is an element of the commutator ideal of R. Then if U is any non-zero ideal of R there exists an integer $m(u)$ so that $c^{m(u)} \in U$.*

Proof. Suppose that there exists an ideal U or R such that

$$U \neq (0), \tag{3}$$

$$c^i \notin U \qquad \text{for all } i = 1, 2, \ldots. \tag{4}$$

By Zorn's lemma there exists an ideal V of R possessing the properties (3) and (4) of U and such that if W is any ideal of R with $W \supset V$ and $W \neq V$ then for some integer K, $c^K \in W$. Consider $\bar{R} = R/V$. In $\bar{R}$, from the choice of V, we have

$$\bar{x}^{n(\bar{x})} \in \bar{Z} \qquad \text{for each } \bar{x} \in \bar{R};$$

and if $\bar{a} = \bar{c}^{n(c)} \neq 0 \in \bar{Z}$, then for any non-zero ideal $\bar{T}$ of $\bar{R}$ some power of $\bar{a}$ is in $\bar{T}$.

We claim that there are no divisors of zero of $\bar{R}$ in $\bar{Z}$. For suppose $\bar{z} \in \bar{Z}$ and that $\bar{z}\bar{x} = 0$, $\bar{z} \neq 0 \neq \bar{x}$. Let

$$A(\bar{z}) = \{\bar{x} \in \bar{R} \mid \bar{z}\bar{x} = 0\}.$$

Clearly $A(\bar{z})$ is an ideal of R and is not the zero ideal, hence $\bar{a}^i \in A(\bar{z})$ for an appropriate i. Thus $za^i \in V$ where $\bar{z} = z + V$, $\bar{a} = a + V$. Without loss of generality $i = n(a)^t$ for large enough t. Let

$$T = \{y \in R \mid ya^i \in V\}.$$

Since $a^i \in Z$, T is an ideal of R, and clearly $T \supset V$. If $T \neq V$ then $c^j \in T$ for appropriate j, whence a^K is in T for appropriate K; thus $a^{i+K} \in V$ and so

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some power of c is in V, a contradiction. Hence $\bar{Z}$ has no zero-divisors of $\bar{R}$. But then all the conditions of Theorem 1 are fulfilled, so $\bar{R} = R/V$ is commutative. Hence $V \supset$ commutator ideal $\supset c$, a contradiction, and Theorem 2 is established.

We are now led to

THEOREM 3. *Suppose $0 \neq c$ is in the commutator ideal of R and that c is not nilpotent. Then c is not a divisor of zero.*

Proof. Let $a = c^{n(c)} \neq 0$; $a \in Z$, and suppose that $cx = 0$, $x \neq 0$. Then certainly $ax = 0$. Let

$$A(a) = \{x \in R \mid ax = 0\}.$$

Since $a \in Z, A(a)$ is an ideal of R and $A(a) \neq (0)$. So by Theorem 2, $c^t \in A(a)$ for a suitable t, and so $a^j \in A(a)$ for some j; then $a^{j+1} = 0$, forcing a, and so c, to be nilpotent, contradicting our hypothesis. Thus Theorem 3 is established.

We are now in a position where we can prove the main theorem of the paper, namely,

THEOREM 4. *Suppose R is a ring with centre Z and such that $x^{n(x)} \in Z$ for all $x \in R$. Then if R is not commutative, the commutator ideal of R is a non-zero nil-ideal.*

Proof. Suppose that R is not commutative, and that $c \neq 0$ is in the commutator ideal of R and is not nilpotent. By Theorem 3, c is not a divisor of zero. Suppose there is a $z \neq 0$ in Z which is a divisor of zero, say $zx = 0$, $x \neq 0$. Let $A(z) = \{x \in R \mid zx = 0\}$. $A(z)$ is an ideal of R and is not (0). So by Theorem 2, $c^i \in A(z)$ for some i. But then $c^i z = 0$, whence $z = 0$ since c is not a zero-divisor. So no $0 \neq z \in Z$ is a divisor of zero of R. Let us list the properties of R:

(a) $x^{n(x)} \in Z$ for all $x \in R$.

(b) Z has no divisors of zero of R.

(c) There exists a $b \in Z$ which is not nilpotent such that given any non-zero ideal U of R then $b^{m(u)} \in U$ (by the above remarks, $b = c^{n(c)}$ will do if c is any non-nilpotent element of the commutator ideal).

Thus all the conditions of Theorem 1 are satisfied, and so R is commutative, contrary to our assumption. Hence we are forced to conclude that every element in the commutator ideal is nilpotent, proving Theorem 4.

THEOREM 5. *Suppose the ring R is such that $x^{n(x)} \in Z$, the centre of R, for all $x \in R$. Then if R has no non-zero nil-ideals, it must be commutative.*

Theorem 5 is an immediate consequence of Theorem 4, but actually the two results are equivalent. For suppose R is a ring with non-zero nil-ideals, then by a result of Köthe [**3**] the sum of all nil-ideals of R is a nil-ideal T, R/T possesses no non-zero nil-ideals, and in R/T, $x^{n(x)}$ is in the centre. So by Theorem 5, R/T

is commutative; hence $T \supset$ commutator ideal, which thus must be a nil-ideal; consequently Theorem 5 implies Theorem 4.

It might be pointed out that Theorem 5 cannot be appreciably weakened. The only plausible weakening would be to change "no nil-ideals" to "no nil-potent ideals" in the statement of Theorem 5, but there is an example, due to Baer [**1**], of a nil-ring with no nilpotent ideals; this rules out the possibility of the stronger result.

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