THE MODULAR REPRESENTATION ALGEBRA OF GROUPS WITH SYLOW 2-SUBGROUP $Z_2 \times Z_2$

S. B. CONLON

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Let k be a field of characteristic 2 and let G be a finite group. Let $A(G)$ be the modular representation algebra [1] over the complex numbers C, formed from kG-modules [2]. If the Sylow 2-subgroup of G is isomorphic to $Z_2 \times Z_2$, we show that $A(G)$ is semisimple [1]. We make use of the theorems proved by Green [4], and the results of the author concerning $A(\mathscr{A}_4)$ [2], where $\mathscr{A}_4$ is the alternating group on 4 symbols.

1. Generalities on representation algebras

Let A be any commutative linear algebra over the complex number field C. A *point* of A is a non-zero algebra homomorphism

$$\phi : A \to C.$$

Thus $\phi(A) = C$. A is said to be *semisimple* if for each non-zero element $a \in A$, there exists a point ϕ of A such that $\phi(a) \neq 0$. If $\dim_C A = r$ is finite, A is semisimple if and only if A has r points; A is then the direct sum of r copies of C.

PROPOSITION 1. *If B is an ideal of A such that both A/B and B are semisimple, then A is semisimple.*

PROOF. Take $a \in B$, and let ϕ be a point of B such that $\phi(a) \neq 0$. We extend [3] ϕ to be a point of A by noting that, as $\phi(B) = C$, there exists in B an element b such that $\phi(b) = 1$. For x any element of A, we define $\phi(x) = \phi(xb)$.

Secondly let $a \notin B$. Thus there exists a point ϕ of A/B such that $\phi(a+B) \neq 0$. But ϕ can be regarded as a point of A which is zero on B. Thus $\phi(a) \neq 0$ and so A is semisimple.

Let k be an arbitrary field and G a finite group. Let M be a kG-module

[1] We adopt the definitions and notation of Green in [4].

[2] A kG-module is a finitely generated k-module on which G acts as a group of left operators. kG is the group algebra on G over k.

[3] As in lemma 6 of [4].

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(of finite k-dimension), and write $\{M\}$ for the class of modules kG-isomorphic to M (or simply the "class of M"). As in [4] we form the modular representation algebra $A(G)$ as an algebra over the complex numbers C in which sum corresponds to direct sum of modules and multiplication to tensor product of kG-modules. A basis for $A(G)$ over C is provided by the indecomposable kG-module classes. k_G will denote the trivial kG-module, and $1_G = \{k_G\}$ its class. Then $A(G)$ is a commutative algebra over C with identity 1_G.

Let $\theta : H \to G$ be a homomorphism of groups, L a kH-module and M a kG-module. Then $\theta^* M$ will denote the restricted kH-module, where the operation of a group element $h \in H$ on $m \in M$ is given by

$$h \cdot m = \theta(h)m.$$

$\theta_* L$ will denote the induced kG-module

$$kG \otimes_{kH} L,$$

where kG is regarded as a right kH-module by means of θ. Thus we get induced linear maps:

$$\theta^* : A(G) \to A(H), \quad \theta_* : A(H) \to A(G).$$

θ^* is an algebra homomorphism, while for θ_* we have the identity

(1) [4] $$\theta_* L \otimes M \approx \theta_*(L \otimes \theta^* M).$$

Here '$\otimes$' denotes Tensor (or Kronecker) product of the representation modules. In particular, if H is a subgroup of G, with θ the embedding map, we write $M_H = \theta^* M$, and $L^G = \theta_* L$; also θ^*, θ_* coincide with the maps r_{GH}, t_{HG} respectively of Green [4].

If H is a normal subgroup of G, and L is kH-module, let S denote the set of elements $s \in G$ such that $s \otimes_{kH} L \approx L$ as left kH-modules. Then S is a subgroup of G containing H and is called the *stabilizer* of L in G. If $S = G$, we say that L is *stable* in G. § 2 of [1] contains the following theorem:

(2) If L is indecomposable, then L^G decomposes according to the decomposition of a certain twisted group algebra on S/H into one-sided indecomposable ideals.

(2′) It should be noted that twisted group algebras on cyclic groups are always isomorphic to the group algebras.

PROPOSITION 2. *If $G_1 G_2$ is the direct product of finite groups G_1, G_2 and if $(|G_1|, p) = 1$, where p is the characteristic of k, or if k has characteristic* 0, *then*

[4] For proof of (1), see p. 268 of [3].

$$A(G_1G_2) \approx A(G_1) \otimes_C A(G_2).$$

PROOF. Write $G = G_1G_2$, and let $\sigma_i : G \to G_i$ be the natural homomorphisms $(i = 1, 2)$. Then we have

$$\sigma_i^* : A(G_i) \to A(G),$$

and combining these we get an algebra homomorphism

$$\sigma^* = \sigma_1^* \otimes \sigma_2^* : A(G_1) \otimes_C A(G_2) \to A(G),$$

which we show to be an isomorphism.

By Higman's theorem 1 in [5], every indecomposable kG-module can be considered as a direct summand of L^G, where L is an indecomposable kG_2-module. Now L is stable in G and the twisted group algebra of (2) is the group algebra kG_1. Indeed the endomorphisms $\theta_{\alpha,\beta}$ of L in the analysis of § 2 of [1] may all be taken to be the identity automorphism, and for $g \in G$ we may take

$$D_g = \lambda(\sigma_2(g)) \qquad \text{(Notation as in § 2 of [1])},$$

where λ is a G_2-representation afforded by L. If π is a principal indecomposable G_1-representation, the typical indecomposable G-representation ψ has the form

$$\psi(g) = \pi(\sigma_1(g)) \otimes \lambda(\sigma_2(g)),$$

analogously to proposition 1 of § 2 in [1]. Hence the indecomposable kG-modules have the form [5]

$$P \# L, \tag{3}$$

where P and L are indecomposable kG_1- and kG_2-modules respectively. Then $\sigma^*\{P \otimes_C L\} = \{P \# L\}$, and σ^* is onto.

Moreover, if P, L are indecomposable, and

$$P \# L \approx P' \# L' \qquad \text{(as } kG\text{-modules)}, \tag{4}$$

by restricting to G_1 and G_2 it follows that $P \approx P', L \approx L'$. As $\{P \otimes_C L\}$, $\{P \# L\}$ form free bases over C for $A(G_1) \otimes_C A(G_2)$, $A(G_1G_2)$ respectively, σ^* is $1-1$ and so σ^* is an isomorphism.

Identity (1) has the following consequence when representations are stable.

PROPOSITION 3. *Let H be a normal subgroup of G and suppose all the indecomposable kH-modules are stable in G. Then $A(H)$ isomorphic to an ideal direct summand of $A(G)$.*

PROOF. Let ϕ: $H \to G$ be the inclusion homomorphism and let L be a kH-module. Define

[5] This is the *outer tensor product* as defined on p. 315 of [3].

$$\sigma\{L\} = \frac{1}{m}\phi_*\{L\} \qquad (m = G:H),$$

and then σ induces a homomorphism of $A(H)$ into $A(G)$. For if L, L' are kH-modules, we have

$$\begin{aligned}
\sigma(\{L\}\cdot\{L'\}) &= \sigma\{L \otimes L'\}, \\
&= \frac{1}{m}\phi_*\{L \otimes L'\}, \\
&= \frac{1}{m^2}\phi_*\{L \otimes \phi^*(\phi_*(L'))\} \qquad \text{(as } L' \text{ is stable)}, \qquad (5)\\
&= \frac{1}{m^2}\{\phi_*(L) \otimes \phi_*(L')\} \qquad \text{(by (1))},
\end{aligned}$$

i.e.
$$\sigma(\{L\}\cdot\{L'\}) = \sigma\{L\}\cdot\sigma\{L'\}.$$

Now $I = \sigma(1_H)$ is an idempotent of $A(G)$ from (5), and if M is any kG-module it follows from (1) that

$$I\cdot\{M\} = \sigma\{\phi^*M\}.$$

Again from (5)

$$I\cdot\sigma\{L\} = \sigma\{L\},$$

and so the image of σ is the ideal direct summand $I\cdot A(G)$ of $A(G)$.

Furthermore the restriction ρ of ϕ^* to $I\cdot A(G)$ satisfies the conditions,

$$\rho\sigma = \text{identity homomorphism on } A(H),$$

and

$$\sigma\rho = \text{identity homomorphism on } I\cdot A(G),$$

and so σ is an isomorphism of $A(H)$ onto $I\cdot A(G)$.

Thus we see that in $A(G_1)\otimes_C A(G_2)$ $(\approx A(G_1G_2))$ of proposition 2 we have direct summands isomorphic to $A(G_1)$ and $A(G_2)$.

We will require the following

PROPOSITION 4. *If H is a normal subgroup of G, then the decomposition of $A(G/H)$ as a direct sum of ideals gives rise to a corresponding one for $A(G)$.*

PROOF. Consider the natural map $\theta : G \to G/H$. This induces a monomorphism $\theta^* : A(G/H) \to A(G)$. Moreover $\theta^*(1_{G/H}) = 1_G$. Thus any decomposition of the identity $1_{G/H}$ of $A(G/H)$ into the sum of idempotents is carried over by θ^* into $A(G)$, and similarly for the ideals generated by these idempotents in their respective algebras.

Let P be a subgroup of G, and write $A_P(G)$ for the C-subspace of $A(G)$ spanned by the symbols $\{L\}$ for all P-projective kG-modules L. Write

$A'_P(G)$ for the subspace of $A(G)$ spanned by the symbols $\{L\}$ for all H-projective kG-modules L, where $H \leqq_G P$, $H \neq_G P$ [5']. As in [4], $A_P(G)$ and $A'_P(G)$ are ideals of $A(G)$, with $A'_P(G) \subsetneqq A_P(G)$.

Write $W_P(G) = A_P(G)/A'_P(G)$.

(6) If k has characteristic p, then Green in [4] shows that $A(G)$ is semisimple if, for each p-subgroup P of G, $W_P(N(P))$ is semisimple, where $N(P)$ is the normalizer of P in G.

(7) Proposition 3 in [2] shows that if we take the trivial p-subgroup $P = \{e\}$, then $W_P(N(P)) = A_P(G)$ is the "projective ideal" of $A(G)$, which is an ideal direct summand of $A(G)$ consisting of the direct sum of a finite number of copies of C. Hence for $P = \{e\}$, $W_P(N(P))$ is semisimple. We denote the projective ideal of $A(G)$ by $A_e(G)$..

Finally it should be noted that as far as the question of the semi-simplicity of $A(G)$ is concerned we can assume k to be algebraically closed. For if not let k^* be its algebraic closure and let $A^*(G)$ be the corresponding modular representation algebra. Proposition 1 and (3) of [2] show there is a natural monomorphism

(8) $$A(G) \to A^*(G),$$

and so, if $A^*(G)$ is semisimple, the restrictions of its points to $A(G)$ ensure the semisimplicity of $A(G)$.

2. Representation algebras of $\mathscr{V}_4$ and $\mathscr{A}_4$

Let k be an algebraically closed field of characteristic 2, let $\mathscr{V}_4 = Z_2 \times Z_2$, be the Klein 4-group and let $\mathscr{A}_4$ be the alternating group on 4 symbols. We shall consider $\mathscr{V}_4$ to be identified with the Sylow 2-subgroup of $\mathscr{A}_4$. The following facts are proved in [2].

The indecomposable $k\mathscr{V}_4$-module classes may be written

$$A_0 = B_0,\ A_n,\ B_n,\ C_n(\pi),\ D,$$

where $n > 0$, and $\pi \in k \cup \{\infty\}$. If we write

$$\mathscr{V}_4 = \{x, y | x^2 = y^2 = e,\ xy = yx\},$$

then the vertices of these classes are as follows:

(9i) A_0, A_n $(n > 0)$, B_n $(n > 0)$, $C_n(\pi)(n > 1)$, and $C_1(\pi)$ $(\pi \neq 0, 1, \infty)$ have vertex $\mathscr{V}_4$,

(9ii) $C_1(0)$, $C_1(1)$, $C_1(\infty)$ have vertices $\{y\}$, $\{xy\}$, $\{x\}$ respectively (order 2),

(9iii) and D has vertex $\{e\}$ (order 1).

D is the regular (indecomposable) module class.

[5'] $H \leqq_G P$ means that there exists an element $x \in G$ such that $x^{-1}Hx \leqq P$, etc.

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We require the following products:

(10i) $B_m C_n(\pi) = A_m C_n(\pi) = C_n(\pi) \bmod A_e(\mathscr{V}_4) \quad (n > 0,\ m \geqq 0)$,

(10ii) $C_m(\pi)C_n(\pi') = \left.\begin{array}{l} 0, \text{ if } \pi \neq \pi', \\ 2C_n(\pi), \text{ if } \pi = \pi',\ m \geqq n, \\ \text{except that} \\ C_1(\pi)C_1(\pi) = C_2(\pi) \\ \text{if } \pi \neq 0,\ 1 \text{ or } \infty, \end{array}\right\} \bmod A_e(\mathscr{V}_4).$

The representation algebra $A(\mathscr{V}_4)$ may be written:

(11)
$$A(\mathscr{V}_4) = \left(C\left[X, \frac{1}{X}\right] + \{\bigoplus_{\substack{\pi, n \\ n>0}} CI_{n,\pi}\}\right) \oplus CI_D,$$

where $X^m I_{n,\pi} = I_{n,\pi}$ (all integers m) and where $\{\oplus_{\pi,n>0} CI_{n,\pi}\}$ is the direct sum of ideals isomorphic to C. CI_D is the projective ideal $A_e(\mathscr{V}_4)$. Here we have the following identifications module $A_e(\mathscr{V}_4)$:

(12i) $X^n = A_n \qquad (n \geqq 0)$,

(12ii) $X^{-n} = B_n \qquad (n \geqq 0)$,

$\left.\begin{array}{ll} I_{1,\pi} = \frac{1}{2}C_1(\pi), & \\ I_{n,\pi} = \frac{1}{2}(C_n(\pi) - C_{n-1}(\pi)) & (n > 1), \end{array}\right\}$ when $\pi = 0, 1$ or ∞,

(12iii) $\left.\begin{array}{ll} I_{1,\pi} = \frac{1}{4}(C_2(\pi) - \sqrt{2}\,C_1(\pi)), & \\ I_{2,\pi} = \frac{1}{4}(C_2(\pi) + \sqrt{2}\,C_1(\pi)), & \\ I_{n,\pi} = \frac{1}{2}(C_n(\pi) - C_{n-1}(\pi)) & (n > 2), \end{array}\right\}$ when $\pi \neq 0, 1$ or ∞.

$A(\mathscr{V}_4)$ is semisimple. We may write

(13)
$$W_{\mathscr{V}_4}(\mathscr{V}_4) = C\left[X, \frac{1}{X}\right] + (\{\bigoplus_{\substack{\pi \neq (0,1,\infty) \\ n>0}} CI_{n,\pi}\} \oplus \{\bigoplus_{\substack{\pi = (0,1,\infty) \\ n>1}} CI_{n,\pi}\})$$

and as in the proof of the semisimplicity of $A(\mathscr{V}_4)$ in § 4 of [2], $W_{\mathscr{V}_4}(\mathscr{V}_4)$ is semisimple.

$\mathscr{V}_4 \lhd \mathscr{A}_4$ and so we can consider the stability of $k\mathscr{V}_4$-module classes in $\mathscr{A}_4$. We have that

(14i) $A_0,\ A_n,\ B_n,\ C_n(\omega),\ C_n(\omega^2),\ D$ are stable in $\mathscr{A}_4$,

(14ii) $C_n(\pi)$ $(\pi \neq \omega, \omega^2)$ are not stable in $\mathscr{A}_4$, where ω is a primitive cube root of unity in k.

Say w is an element of order 3 of $\mathscr{A}_4$ with $w^{-2}xw^2 = w^{-1}yw = xy$. Then we have that the $k\mathscr{V}_4$-module class

(15)
$$w \otimes_{k\mathscr{V}_4} C_n(\pi) = C_n(\theta(\pi)),$$

where $\theta(\pi) = (1+\pi)/\pi$, with the obvious interpretation when $\pi = 0$ or ∞. θ gives a permutation on $k \cup \{\infty\}$. We denote the typical class of transitivity by $\mu = (\pi, \theta(\pi), \theta^2(\pi))$, but (ω) and (ω^2) form transitivity classes by themselves. Applying (2) together with Higman's theorem 1 in [5], we see that the indecomposable $k\mathscr{A}_4$-module classes can be written (see [2])

(16i) $$A_0^\alpha,\ A_n^\alpha,\ B_n^\alpha,\ C_n^\alpha(\omega),\ C_n^\alpha(\omega^2),\ D^\alpha$$

(16ii) $$C_n^*(\mu),$$

where $n > 0$ and $\alpha = 0, 1, 2$. Superscripts α will always be taken modulo 3 (0, 1 or 2). Note that

$$(C_n^*(\mu))\mathscr{V}_4 = C_n(\pi)+C_n(\theta(\pi))+C_n(\theta^2(\pi)), \text{ and } (L^\alpha)_{\mathscr{V}_4} = L,$$

where L^α is any one of (16i). The vertices of the above $k\mathscr{A}$-module classes remain the same as the corresponding $k\mathscr{V}_4$-module classes. The representation algebra $A(\mathscr{A}_4)$ may be written

(18)
$$A(\mathscr{A}_4) = \left(C\left[Y_0, \frac{1}{Y_0}\right] + \{\bigoplus_{\substack{n>0\\ \phi=\omega,\omega^2,\mu}} CI_{n0}(\phi)\}\right)$$
$$\oplus\left(\bigoplus_{\beta=1,2}\left[C\left[Y_\beta, \frac{1}{Y_\beta}\right] + \{\bigoplus_{\substack{n>0\\ \phi=\omega,\omega^2}} CI_{n\beta}(\phi)\}\right]\right)$$
$$\oplus (C \oplus C \oplus C),$$

where the last term is the projective ideal $A_e(\mathscr{A}_4)$,

$$Y_\beta^m I_{n\beta}(\omega^\alpha) = u^{-\alpha\beta m} I_{n\beta}(\omega^\alpha) \qquad (\beta = 0, 1, 2;\ \alpha = 1, 2),$$
$$Y_0^m I_{n0}(\mu) = I_{n0}(\mu),$$

with u a primitive cube root of unity in C. We have the following identifications modulo $A_e(\mathscr{A}_4)$:

$$Y_\beta^n = \tfrac{1}{3}(A_n^0+u^\beta A_n^1+u^{2\beta}A_n^2),$$
$$Y_\beta^{-n} = \tfrac{1}{3}(B_n^0+u^\beta B_n^1+u^{2\beta}B_n^2),$$

$I_{n\beta}(\phi)$ = finite linear combination of $C_m^\alpha(\phi)$, for $\alpha, \beta = 0, 1, 2$. $A(\mathscr{A}_4)$ is again semisimple.

3. $A(G)$ for G with Sylow 2-subgroup $Z_2 \times Z_2$

Let k be an algebraically closed field of characteristic 2 and G a finite group with Sylow 2-subgroup isomorphic to $\mathscr{V}_4 = Z_2 \times Z_2$. To see that $A(G)$ is semisimple we use Green's theorem (6) and show that $W_P(N(P))$ is semisimple, where P is a 2-subgroup of G of order 1, 2 or 4.

The case when $|P| = 1$ has been dealt with in (7). When $|P| = 2$, a basis for $W_P(N(P))$ is obtained from the indecomposable direct summands of $(k_P)^{N(P)}$. But these correspond, as in (2), to the principal representations of $k(N(P)/P)$, and it is readily seen that $W_P(N(P))$ is a homomorphic image of the projective ideal[6] of $A(N(P)/P)$. Thus by (7) $W_P(N(P))$ is semisimple.

Now assume that $|P| = 4$, and so $P \approx \mathscr{V}_4$. Write $H = N(P)$. Two cases arise:

(a) the centralizer $C(P)$ of P in H is H itself, and

(b) the centralizer $C(P)$ is not H.

In case (a) it is clear that $H = RP$, the direct product of two groups. Thus by proposition 2

$$A(H) \approx A(R) \otimes_C A(P).$$

Moreover in this correspondence

$$W_P(H) \approx A(R) \otimes_C W_P(P).$$

$A(R)$ is semisimple and of finite dimension over C as $(|R|, 2) = 1$, and $W_P(P)$ is semisimple by (13). Hence $W_P(H)$ is semisimple.

Case (*b*). In this case we show $W_P(H)$ to be semisimple by taking an ideal S of $W_P(H)$ such that both S and $W_P(H)/S$ are semisimple. $W_P(H)$ is itself semisimple by proposition 1.

Structure of H. We can find a complement R to P in H and write $H = RP$. The centralizer $C(P)$ of P in H may be written in the form QP, a direct product of groups, where Q is a normal subgroup of H contained in R. $H/QP \approx R/Q$ has order 3, as elements of R/Q correspond to automorphisms of $\mathscr{V}_4$ whose orders are prime to 2. Take $r \in R$ such that rQ generates R/Q. Then any element $h \in H$ has a unique expression in the form

$$h = r^\beta q p, \tag{20}$$

where $\beta = 0, 1, 2$, $q \in Q$, $p \in P$. Write $\rho_1 : H \to R$ to be the epimorphism $\rho_1(h) = r^\beta q$. We define K to be the extension[7] of P by R/Q, its elements being written in the form $(r^\beta Q)(p)$ or $(r^\beta)(p)$ and satisfying the relation

$$(r^\beta)(p) = (r^\beta p r^{-\beta})(r^\beta).$$

Thus P is its own centralizer in K and $K \approx \mathscr{A}_4$. Further there is an epimorphism $\rho_2 : H \to K$ given by $\rho_2(h) = (r^\beta)(p)$, where h is given by (20). Finally we have a monomorphism,

$$\rho : H \to RK, \tag{21}$$

into the direct product of R and K given by $\rho(h) = \rho_1(h) \cdot \rho_2(h)$.

[6] In fact, $W_P(N(P)) \approx A_0(N(P)/P)$.

[7] $K \approx H/Q$ essentially.

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Indecomposable kH-modules. To obtain the indecomposable kH-modules, we use Higman's theorem 1 in [5] and look at the break-up of kH-modules L^H, where L is an indecomposable kQP-module. As in (3), L has the form $M \# N$, where M is an indecomposable (principal) kQ-module and N is an indecomposable kP-module. By (4), $M \# N$ is stable in H if and only if M is stable in R and N is stable in K. By (2) and (2′), $(M \# N)^H$ is the direct sum of 3 non-isomorphic kH-modules

(22i) $$(M \# N)^\alpha,$$

if $M \# N$ is stable in H, or otherwise

(22ii) $$(M \# N)^H$$

is indecomposable. In the latter case it should be noted that

(22iii) $$(r^\beta \otimes (M \# N))^H \approx ((r^\beta \otimes M) \# (r^\beta \otimes N))^H \approx (M \# N)^H.$$

Moreover the vertex of an indecomposable kH-module so generated is the same as the vertex of N.

Now $W_P(H) = A_P(H)/A'_P(H) = A(H)/A'_P(H)$. Further $A'_P(H) \geqq A_e(H)$, and so in looking at $W_P(H)$ we can work modulo the indecomposable kH-projectives. These last are in $1-1$ correspondence with the indecomposable projectives of kR, for the regular kP-module N is stable in K as in (14i) and if M is any indecomposable kQ-module, $M \# N$ is stable in H if and only if M is stable in R. Hence $(M \# N)^H$ decomposes just as M^R does by (2).

Definition and semisimplicity of S. Consider the subspace S of $W_P(H)$ spanned by classes of indecomposable kH-modules of the form $(M' \# N')^H$ where N' is unstable in K. Then if X is any kH-module such that $X_{(PQ)}$ has form $\oplus (M_\alpha \# N_\alpha)$, then

(23)
$$\begin{aligned} X \otimes (M' \# N')^H &\approx \oplus ((M_\alpha \# N_\alpha) \otimes (M' \# N'))^H \qquad \text{(by (1))}, \\ &\approx \oplus ((M_\alpha \otimes M') \# (N_\alpha \otimes N'))^H. \end{aligned}$$

The unstable classes $\{N\}$ span an ideal of $A(P)$ and so S is an ideal of $W_P(H)$. Furthermore the map

$$(M' \# N')^H \to M' \otimes_C (N')^K$$

is an isomorphism from S onto $A(Q) \otimes_C T$, where T is the subspace of $A_P(K)$ coming from indecomposable kP-modules which are unstable in K. But from (18) T is the direct sum of copies of C and so S is a semisimple ideal of $W_P(H)$.

$W_P(H)/S$. We now consider $W_P(H)/S$. Whereas the basis elements of S came from kP-modules classes which were unstable in K, a basis of

$W_P(H)/S$ will be obtained from kP-module classes which have vertex P, and are stable in K.

The embedding homomorphism ρ of H into the direct product RK as in (21) gives rise to an algebra homomorphism

$$\rho^* : A(RK) \to A(H).$$

By proposition 2, $A(RK) \approx A(R) \otimes_C A(K)$, and so we get a succession of homomorphisms:

$$A(R) \otimes A(K) \approx A(RK) \to A(H) \to A(H)/A'_P(H) = W_P(H) \to W_P(H)/S.$$

Let σ denote the composition of these homomorphisms. We show σ is onto and analyse $W_P(H)/S$ as a quotient of $A(R) \otimes A(K)$.

σ is onto. Let N be an indecomposable kP-module which is stable in K, and let $\nu(p)$ $(p \in P)$ be a representation afforded by this module. As N is stable in K, there exists a matrix R_ν such that

$$\nu(r^{-1}pr) = R_\nu^{-1}\nu(p)R_\nu \qquad (p \in P).$$

Then

$$\nu_\alpha(p) = \nu(p), \quad \nu_\alpha(r) = \omega^\alpha R_\nu, \qquad (\alpha = 0, 1, 2),$$

are 3 inequivalent indecomposable K-representations "contained" in N^K.

Let M be an indecomposable kQ-module. Let $\mu(q)$ $(q \in Q)$ be a representation afforded by this module. Then the kQP-module $M \# N$ is stable in H if and only if M is stable in R.

(i) Say M is unstable in R. Let $\tilde{\mu}$ be the R-representation afforded by the indecomposable kR-module M^R. In the representation ζ afforded by $(M \# N)^H$ (indecomposable) choose a basis according to the direct sum decomposition

$$((M \# N)^H)_{QP} = \oplus\ (r^\beta \otimes M) \# (r^\beta \otimes N),$$

but in the subspace corresponding to $r^\beta \otimes N$ choose the basis such that we have

$$\zeta(qp) = \begin{bmatrix} \mu(q) \otimes \nu(p) & 0 & 0 \\ 0 & \mu(r^{-1}qr) \otimes \nu(p) & 0 \\ 0 & 0 & \mu(r^{-2}qr^2) \otimes \nu(p) \end{bmatrix}.$$

Then $\zeta(r)$ takes the form

$$\zeta(r) = \begin{bmatrix} 0 & 0 & \mu(r^3) \otimes R_\nu \\ I \otimes R_\nu & 0 & 0 \\ 0 & I \otimes R_\nu & 0 \end{bmatrix}.$$

It is now clear that

(24i) $$\zeta(h) = \tilde{\mu}(\rho_1(h)) \otimes \nu_0(\rho_2(h))$$

for all $h \in H$. Thus $\{(M \# N)^H\}$ lies in the image of ρ^*.

(ii) Say M is stable in R. Thus there exists a matrix R_μ such that

$$\mu(r^{-1}qr) = R_\mu^{-1}\mu(q)R_\mu,$$

and

$$\mu_\alpha(q) = \mu(q),\ \mu_\alpha(r) = \omega^\alpha R_\mu, \qquad (\alpha = 0, 1, 2),$$

are 3 inequivalent indecomposable R-representations "contained" in M^R. Now

$$(M \# N)^H \approx \bigoplus_{\alpha=0}^{2} (M \# N)^\alpha,$$

and we can take the representation ζ_α afforded by $(M \# N)^\alpha$ to be in the form

$$\zeta_\alpha(qp) = \mu(P) \otimes \nu(p),\ \zeta_\alpha(r) = \omega^\alpha R_\mu \otimes R_\nu.$$

Thus again we have

(24ii) $$\zeta_\alpha(h) = \mu_\alpha(\rho_1(h)) \otimes \nu_0(\rho_2(h)) \qquad (\alpha = 0, 1, 2).$$

Hence σ is onto $W_P(H)/S$.

Study of ker σ. Elements of the form

$$\{L\} \otimes C_n^*(\mu) \quad (\mu \neq (\omega), (\omega^2)), \quad \{L\} \otimes D^\alpha$$

of $A(R) \otimes A(K)$ (L an kR-module) either have vertex of order less than 4 or map to elements of S. Hence if U is the ideal of $A(R) \otimes A(K)$ generated by the above elements, we can regard σ as a map $\bar{\sigma} : (A(R) \otimes A(K))/U \to W_P(H)/S$. Moreover, from (18) the structure of $(A(R) \otimes A(K))/U$ may be written

(25) $$(A(R) \otimes A(K))/U \approx A(R) \otimes \left[\bigoplus_{\beta=0}^{2}\left(C\left[Y_\beta, \frac{1}{Y_\beta}\right] + \left\{\bigoplus_{\substack{n \geq 1 \\ \phi = \omega, \omega^2}} CI_{n\beta}(\phi)\right\}\right)\right].$$

This is semisimple, as $A(R)$ is the direct sum of a finite number of copies of C and the direct factors on the right are each semisimple as is shown in § 4 of [2].

$\bar{\sigma}$ *and* ρ^*. We next show that $(A(R) \otimes A(K))/U$ is the ideal direct sum of three ideals, two of which are sent to 0 by $\bar{\sigma}$ and the last of which is isomorphic to $W_P(H)/S$ under $\bar{\sigma}$. To this end we look more closely at ρ^*.

If M, N are indecomposable kQ-, kP-modules respectively with N stable in K, then from (24i), (24ii) under ρ^* we obtain

(26i) $M^R \otimes N^\alpha \to (M \# N)^H$ when M is unstable in R, and

(26ii) $M^\beta \otimes N^\alpha \to (M \# N)^{\alpha+\beta}$ when M is stable in R

(superscripts being modulo 3). Clearly the only elements of $(A(R) \otimes A(K))/U$ which can map onto these basis elements are in the subspace generated by $M^R \otimes N^\alpha$ or $M^\beta \otimes N^\alpha$ as the case may be.

(27) Thus in either case we have a subspace of dimension 3 mapping onto a 1-dimensional subspace (if we consider $\alpha+\beta$ fixed (modulo 3) in the second case).

Idempotents of $(A(R) \otimes A(K))/U$. To obtain the ideal direct summands of $(A(R) \otimes A(K))/U$ we proceed to obtain their generating idempotents as follows.

Let E^α, F^α, G^α $(\alpha = 0, 1, 2)$ be the 3 1-dimensional kR-, kK-, kH-modules respectively corresponding to the matrix representations

$$\gamma^\beta \to \omega^{\beta\alpha}.$$

Thus we can write $k_R = E^0$, $k_K = F^0$, $k_H = G^0$. We use the same symbols E^α, F^α, G^α to denote the corresponding module classes. Then under ρ^* we have from (26ii) that

(28) $$E^\alpha \otimes F^\beta \to G^{\alpha+\beta}.$$

Consider the normal subgroup QP of RK. $RK/QP \approx R/Q \cdot K/P$, which is the direct product of two cyclic groups of order 3. We can denote the various $k(R/Q \cdot K/P)$-module classes by $E^\alpha \otimes_C F^\beta$ $(\alpha, \beta = 0, 1, 2)$. Thus we get that $A(R/Q \cdot K/P)$ is the direct sum of 9 copies of C with idempotents

$$I_{\alpha\beta} = \tfrac{1}{3}(E^0+u^\alpha E^1+u^{2\alpha}E^2) \otimes \tfrac{1}{3}(F^0+u^\beta F^1+u^{2\beta}F^2),$$

where α, $\beta = 0, 1, 2$, and u is a primitive cube root of unity in C. By proposition 4 we get a corresponding decomposition of $A(RK) \approx A(R) \otimes A(K)$, and so one induced on the quotient $(A(R) \otimes A(K))/U$. Consider the 3 idempotents

$$J_0 = I_{00}+I_{11}+I_{22} = \tfrac{1}{3}(E^0 \otimes F^0+E^1 \otimes F^2+E^2 \otimes F^1),$$
$$J_1 = I_{10}+I_{21}+I_{02} = \tfrac{1}{3}(E^0 \otimes F^0+uE^1 \otimes F^2+u^2E^2 \otimes F^1),$$
$$J_2 = I_{20}+I_{01}+I_{12} = \tfrac{1}{3}(E^0 \otimes F^0+u^2E^1 \otimes F^2+uE^2 \otimes F^1).$$

Then $\rho^*(J_0) = G^0$, the identity of $A(H)$, and $\rho^*(J_1) = \rho^*(J_2) = 0$. On the other hand none of the following products vanishes:

$$J_\beta \cdot (M^R \otimes N^\alpha) \qquad (M \text{ unstable in } R),$$
$$J_\beta \cdot (M^\beta \otimes N^\alpha) \qquad (M \text{ stable in } R),$$

where $\beta = 0, 1, 2$, and the 3-dimensional subspaces of (27) are the sum of 1-dimensional subspaces one in each of the summands $J_\beta \cdot A(RK)$ $(\beta = 0, 1, 2)$.

Hence restricting $\bar{\sigma}$ to the direct summand $J_0 \cdot (A(R) \otimes A(K))/U$ we have that $\bar{\sigma}$ is one-to-one and onto. Thus

$$W_P(H)/S \approx J_0 \cdot (A(R) \otimes A(K))/U.$$

Hence $W_P(H)/S$ is isomorphic to an ideal (direct summand) of a semisimple algebra (by (25)) and so $W_P(H)/S$ is semisimple.

$W_P(H)$ contains an ideal S such that $W_P(H)/S$ and S are semisimple and so by proposition 1 it is semisimple. This completes the proof of the semisimplicity of $W_P(N(P))$ for P of orders 1, 2 or 4. By Green's theorem (6), $A(G)$ is semisimple. By (8) we can further remove the restriction of k being algebraically closed and so we have the following theorem:

THEOREM. *Let G be a finite group whose Sylow 2-subgroup is isomorphic to* $Z_2 \times Z_2$, *and let k be any field of characteristic* 2. *Then the modular representation algebra* $A(G)$ *formed from kG-modules is semisimple.*

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University of Sydney