REPRESENTATIONS OF RANK ONE ALGEBRAIC MONOIDS

by LEX E. RENNER

(Received 4 February, 1987)

0. Introduction. One of the fundamental results of representation theory is the identification of the irreducible representations of a semisimple group by their dominant weights [**3**]. The purpose of this paper is to establish similar results for a class of reductive algebraic monoids.

Let k be an algebraically closed field. An *algebraic monoid* is an affine algebraic variety M defined over k, together with an associative morphism $m: M \times M \to M$ and a two-sided unit $1 \in M$ for m.

In [**5**] the set of normal, algebraic monoids with unit group $G = Sl_2(k) \times k^*$, $Gl_2(k)$ or $PGl_2(k) \times k^*$ is determined numerically. That construction, however, does not yield directly any irreducible representations of these monoids. In this paper I produce a complete list (see 3.7 and 3.11). This list is fundamental for studying the relationship (in general) of irreducible representations to the system of idempotents of closely related monoids [**4**].

1. Preliminaries. Let M be an algebraic monoid, and let

$$G = G(M) = \{x \in M \mid x^{-1} \in M\}.$$

Then $G \subseteq M$ is an affine, open, algebraic subgroup. If M is irreducible then $\bar{G} = M$ (Zariski closure). If $T \subseteq G$ is a maximal torus, then $\bar{T} \subseteq M$ is a maximal, irreducible, closed D-submonoid. $\bar{T}$ is determined to within an isomorphism by the commutative monoid

$$X(\bar{T}) = \{\chi \in X(T) \mid \chi \text{ extends to } \bar{\chi} : \bar{T} \to k\}.$$

A rational representation of $\bar{T}$ is simply an $X(\bar{T})$-graded vector space over k. See [**5**] for details and references.

An irreducible, algebraic monoid M is *reductive* if $G(M)$ is a reductive group.

2. Representations of $Sl_2(k) \times k^*$. This section is a summary of some of the basic properties of irreducible representations of $Sl_2(k) \times k^*$.

Let $k, l \in \mathbb{N}$. Then there exists a representation

$$\tau_l : Sl_2(k) \to Gl_{l+1}(k).$$

τ_l is the l-th symmetric power of the canonical representation $Sl_2(k) \subseteq Gl_2(k)$. Note that τ_l is not in general irreducible. We thus define for each $l \in \mathbb{N}$, $k \in \mathbb{Z}$,

$$\rho_{k,l} : Sl_2(k) \times k^* \to Gl_{l+1}(k) = Gl(V(k, l))$$

by

$$\rho_{k,l}(x, t) = (\text{diag}[t^k, \ldots, t^k])\tau_l(x).$$

Glasgow Math. J. **30** (1988) 237–241.

Let $T = k^* \times k^* \subseteq Sl_2(k) \times k^*$ be a maximal torus and let

$$x = p_2 : T \to k^*,$$
$$y = p_1 : T \to k^*.$$

With respect to the basis $\{x, y\} \subseteq X(T)$, we obtain the following weight decomposition of $V = V(k, l)$:

$$V = V_{kx+ly} \oplus V_{kx+(l-2)y} \oplus \ldots \oplus V_{kx-ly}.$$

Thus, if $\overline{\rho_{k,l}(T)} = Z$ is the Zariski closure of $\rho_{k,l}(T)$ in $\mathrm{End}(V)$, then

$$X(Z) = \langle x^k y^l, x^k y^{l-2}, \ldots, x^k y^{-l} \rangle,$$

the submonoid of $X(\rho_{k,l}(T))$ generated by $\{x^k y^l, \ldots, x^k y^{-l}\}$. (We are using both additive and multiplicative notation for characters.) Let

$$W(k, l) = \{kx + ly, \ldots, kx - ly\}. \tag{1}$$

2.1. Remark. We have canonical morphisms

$$m : Sl_2(k) \times k^* \to Gl_2(k) \quad \text{and} \quad c : Sl_2(k) \times k^* \to PGl_2(k) \times k^*,$$

both of degree two.

(a) $\rho_{k,l}$ factors through m if and only if $k + l$ is even.
(b) $\rho_{k,l}$ factors through c if and only if l is even.

3. Irreducible representations of monoids. This section contains our main result: the enumeration of all irreducible representations of any normal, irreducible, algebraic monoid M with unit group $Sl_2(k) \times k^*$, $Gl_2(k^*)$ or $PGl_2(k) \times k^*$. Let us first recall the way in which these monoids are classified.

3.1. Theorem [5]. *Let G be one of the above groups and let*

$$\mathscr{E}(G) = \{M \mid G(M) \cong G, M \text{ is normal and } 0 \in M\}.$$

There is a canonical one-to-one correspondence

$$\mathscr{E}(G) \cong \mathbb{Q}^+,$$

where $\mathbb{Q}^+$ denotes the set of positive rational numbers.

For the purposes of this paper we shall need a recipe for the set of characters of the closure $\bar{T}(r)$ in $M_r (r \in \mathbb{Q}^+$, as in 3.1) of a maximal torus T of G. The computation may be found in section 4.5 of [5]. Here, $\langle \ldots \rangle$ denotes "submonoid generated by".

3.2. $Sl_2(k) \times k^*$.

$$X(\bar{T}(r)) = \{\chi \in X(T) \mid k\chi \in \langle ma + nb, ma - nb \rangle \quad \text{some} \quad k > 0\}.$$

where $r = n/m$, $(n, m) = 1$.

3.3. $Gl_2(k)$.

$$X(\bar{T}(r)) = \{\chi \in X(T) \mid k\chi \in \langle (m+n)u + (m-n)v, (m+n)v + (m-n)u \rangle \quad \text{some} \quad k > 0\},$$

where $r = n/m$, $(m, n) = 1$.

3.4. $PGl_2(k) \times k^*$.

$$X(\bar{T}(r)) = \{\chi \in X(T) \mid k\chi \in \langle mx + ny, mx - ny \rangle \quad \text{some} \quad k > 0\},$$

where $r = n/m$, $(m, n) = 1$.

3.5. Lemma. (a) *Let $M_r \in \mathscr{E}(G)$, $G = Sl_2(k) \times k^*$. Then*

$$W(k, l) \subseteq X(\bar{T}(r)) \quad \text{iff} \quad l/k \leq r.$$

(b) *Let $M_r \in \mathscr{E}(G)$, $G = Gl_2(k)$. Then $W(k, l) \subseteq X(\bar{T}(r))$ iff $2 \mid k + l$ and $l/k \leq r$.*
(c) *Let $M_r \in \mathscr{E}(C)$, $G = PGl_2(k) \times k^*$. Then $W(k, l) \subseteq X(\bar{T}(r))$ iff $2 \mid l$ and $l/k \leq 2r$.*

Proof. It follows from 3.2 that

$$X(\bar{T}(r)) = \{(ka, lb) \in X(T) \mid -r \leq l/k \leq r\} \cup \{(0, 0)\}.$$

Thus, the conclusion follows from (1) above. For (b) and (c), apply (a) using 2.1.

3.6. Proposition [**5**]. *Let $M \in \mathscr{E}(G)$ and suppose we have morphisms $\alpha: G \to M'$ and $\beta: \bar{T} \to M'$ such that $\alpha|_T = \beta|_T$. Then there exists a unique morphism $\rho: M \to M'$ such that $\rho|_G = \alpha$ and $\rho|_{\bar{T}} = \beta$.*

Proof (*sketch*). Let $e \in \bar{T}$ be a maximal idempotent, and let $T(e) \subseteq \bar{T}$ be the unique open submonoid of $\bar{T}$ such that $E(T(e)) = \{1, e\}$. Let B and B^- be the Borel subgroups of G that contain T. Then $m: B_u \times T(e) \times B_u^- \to M$, $m(x, y, z) = xyz$, is an open embedding with image, say U (the "big cell"). Notice that $U \cap (M \backslash G) \neq \varnothing$. Thus, $\operatorname{codim}(M \backslash (G \cup U), M) \geq 2$ since $M \backslash G$ is irreducible. Define $\rho': U \to M'$ by

$$\rho'(x, y, z) = \alpha(x)\beta(y)\alpha(z).$$

Then $\rho'|_{G \cap U} = \alpha|_{G \cap U}$. Thus, ρ' can be extended to $G \cup U$. Hence, by Lemma 5.1 of [**2**], ρ' can be extended to $\rho: M \to M'$.

Let G be as above and let $M \in \mathscr{E}(M)$. Then we let

$$\operatorname{Rep}(M) = \{\rho_{k,l} \mid \rho_{k,l} \text{ extends to } M\},$$

since any such extension is unique. Recall that these representations are not in general irreducible.

3.7. Theorem. (a) *Let $G = Sl_2(k) \times k^*$, $M = M_r$.*

$$\operatorname{Rep}(M) = \{\rho_{k,l} \mid l/k \leq r\}.$$

(b) *Let $G = Gl_2(k)$, $M = M_r$.*

$$\operatorname{Rep}(M) = \{\rho_{k,l} \mid l/k \leq r, 2 \mid k + l\}.$$

(c) *Let* $G = PGl_2(k) \times k^*$, $M = M_r$.

$$\mathrm{Rep}(M) = \{\rho_{k,l} \mid l/k \le 2r,\ 2 \mid l\}.$$

Proof. It will suffice to prove (a). (b) and (c) are similar. If $\rho_{k,l}: G \to Gl_{l+1}(k)$ extends to M_r then $W(k, l) \subseteq X(\bar{T}(r))$. So by 3.5, $l/k \le r$.

Conversely, if $l/k \le r$ then again by 3.5, $\rho_{k,l}|_T$ extends to $\bar{T}(r)$ since $W(k, l) \subseteq X(\bar{T}(r))$ (see Lemma 4.1 of [**5**]). But then 3.6 applies to yield the desired extension $\rho_{k,l}: M \to \mathrm{End}_k(k^{l+1})$.

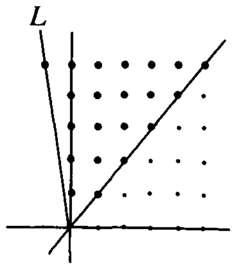
3.8. Scholium. In each case $\mathrm{Rep}(M_r)$ can be identified with a subset of $X(\bar{T}(r))$:

(a) $\eta: \mathrm{Rep}(M_r) \to X(\bar{T}(r))$, $\rho_{k,l} \to (k, l)$,

(b) $\eta: \mathrm{Rep}(M_r) \to X(\bar{T}(r))$, $\rho_{k,l} \to ((k-l)/2, (k+l)/2)$,

(c) $\eta: \mathrm{Rep}(M_r) \to X(\bar{T}(r))$, $\rho_{k,l} \to (k, l/2)$.

3.9. Example. Let $G = Gl_2(k)$, $M = M(3/2) \in \mathscr{E}(G)$.



3.10. Remarks. (1) In each case of 3.8, $\mathrm{Rep}(M_r)$ contains exactly one element of each $\mathbb{Z}_2 = N_G(T)/T$ orbit of $X(\bar{T}(r))$.

(2) The character $\eta(\rho_{k,l})$ of 3.8 is the element of $W(k, l)$ with "highest slope".

3.11. Theorem. *Let* $\mathrm{Rep}(M_r)$ *be as above. Then there is a canonical one-to-one correspondence*

$$\{(\rho, V) \mid \rho \text{ is an irreducible representation of } M_r\} \cong \mathrm{Rep}(M_r).$$

In particular, the remarks of 3.10 *can be applied to the* irreducible *representations of* M_r.

Proof. If $\mathrm{char}(k) = 0$ then each $\rho_{k,l}$ is irreducible. So assume char $k = p > 0$.

Let S^l denote the l-th symmetric power of k^2. Given $\rho_{k,l}: M_r \to \mathrm{End}_k(S^l)$, let (ρ, V) be the irreducible representation of G with highest weight l. Assume here, that $G = Sl_2(k) \times k^*$. By [**1**, A §7.5]

$$V \cong \pi_0 \otimes \pi_1^F \otimes \ldots \otimes \pi_s^{F^s} = \pi$$

where each π_i is infinitesimally irreducible and $\pi_i^{F^i}$ is obtained from π_i by composition with the ith power of the Frobenius morphism. Furthermore, $\pi_i \cong S^{l_i}$ as $Sl_2(k)$-modules for some $0 \leq l_i < p$, and the highest weight of V is $l = \sum_{i=0} l_i p^i$. Hence, we obtain

$$\pi_0 \otimes \ldots \otimes \pi_s^{F^s} \xrightarrow[\otimes j_i]{} S^{l_0} \otimes \ldots \otimes S^{p^s l_s} \xrightarrow[\mu]{} S^l$$

where j_i is the inclusion and μ is the multiplication map, $\mu(v_0 \otimes \ldots \otimes v_s) = v_0 \cdot \ldots \cdot v_s$. Since $\mu \circ (\otimes j_i)$ is nonzero, it is an embedding of $Sl_2(k)$-modules. Thus, π is actually an M_r-submodule of S^l since it is $G(M_r)$-stable and $G(M_r)$ is dense in M_r.

Conversely, if $\rho : M_r \to \mathrm{End}(V)$ is irreducible, then

(i) $\rho|_{k^*} = \chi_k$, $\chi_k(t) = t^k$, some k.
(ii) $\rho|_{Sl_2(k)} \cong \pi$ (as representations) for some π as above.

But then $\rho|_G$ is the restriction of $\rho_{k,l}|_G$ to $\pi \subseteq S^l$.

Thus we have

$$W(k, l) \subseteq X(T)$$

and

$$mW(k, l) \subseteq X(\bar{T}), \quad \text{some} \quad m > 0,$$

where $T \subseteq G$ is a maximal torus (i.e. some multiple of each weight of T on S^l is contained in the semigroup of weights of T on π). Hence, since $\bar{T}$ is normal [5], $W(k, l) \subseteq X(\bar{T})$. But then 3.5 and 3.7 combine to prove that $\rho_{k,l}$ extends from G to M_r.

REFERENCES

1. A. Borel, R. Carter, C. W. Curtis, N. Iwahori, T. A. Springer and R. Steinberg, *Seminar on algebraic groups and related finite groups*, Lecture Notes in Mathematics 131 (Springer, 1970).
2. F. Grosshans, Observable groups and Hilbert's fourteenth problem. *Amer. J. Math.* **95** (1973), 229–253.
3. J. Humphreys, *Linear algebraic groups*, Graduate Texts in Mathematics (Springer, 1981).
4. M. Putcha and L. Renner, The lattice of J-classes and the system of idempotents of reductive algebraic monoids, *J. Algebra*, to appear.
5. L. Renner, Classification of semisimple rank one monoids, *Trans. Amer. Math. Soc.* **287** (1985), 457–473.

UNIVERSITY OF WESTERN ONTARIO
LONDON, CANADA N6A 5B7