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ON A FUNCTION ANALOGOUS TO $\log \eta(\tau)$

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1. Introduction

Let us denote by $\eta(z)$ the classical η-function of Dedekind defined by

$$\eta(z) = e^{\pi i z/12} \prod_{n=1}^{\infty} (1 - e^{2\pi i n z}) , \qquad \mathrm{Im}\,(z) > 0 .$$

If $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, Z)$, then the classical law of transformation of $\log \eta(z)$ asserts that if $\sigma(z) = (az + b)/(cz + d)$, then

$$\begin{aligned} \log \eta(\sigma(z)) &= \log \eta(z) + \frac{\pi i b}{12} \quad (c = 0) \\ &= \log \eta(z) + \frac{1}{2} \log\left(\frac{cz + d}{i}\right) + \pi i \frac{a + d}{12c} - \pi i s(d, c) \quad (c > 0) \end{aligned} \tag{1}$$

where all logarithms are taken with respect to the principal branch and

$$s(d, c) = \sum_{\mu \,(\mathrm{mod}\, c)} \left(\left(\frac{d\mu}{c}\right)\right)\left(\left(\frac{\mu}{c}\right)\right)$$

and where

$$\begin{aligned} ((x)) &= x - [x] - \tfrac{1}{2} \qquad \text{if } x \text{ is not an integer,} \\ &\quad 0 \qquad \text{otherwise.} \end{aligned}$$

The sum $s(d, c)$ is called a Dedekind sum, and appears in many number-theoretic investigations.

In [1], we have introduced a generalization of the function $\eta(z)$ associated to a totally real algebraic number field K. Our generalization arose from a generalization of Kronecker's second limit formula. Moreover, we showed in [1] that a classical conjecture of Hecke concerning class numbers of algebraic number field could be reduced to determining

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how the generalized η-function transforms under the Hilbert modular group—that is, to find a generalization of formula (1). It is the purpose of this paper to find such a generalization.

Throughout this paper, let K be a totally real algebraic number field of degree n and let $\mathfrak{f}, \mathfrak{m}, \mathfrak{n}$ be integral K-ideals, $\mathfrak{d} =$ the (absolute) different of K. Further, let $\Gamma_{\mathfrak{f}}$ denote the group of all totally positive K-units $\equiv 1 \pmod{\mathfrak{f}}$. Finally, let u, v be elements of K having denominators dividing $\mathfrak{d}\mathfrak{f}$. Associated to the this data, we define the following generalized η-function on the product $\boldsymbol{H}^n$ of n complex upper half-planes $\boldsymbol{H}$:

$$\log \eta(z; u, v) = \Lambda(u)N(z) + \sum_{\substack{\beta \in \mathfrak{n} \\ \beta \neq 0 \\ \{\Gamma_{\mathfrak{f}}\}}}^{*} \frac{e^{2\pi i \operatorname{Tr}(v\beta)}}{|N(\beta)|} \sum_{\substack{\mu \in \mathfrak{m}^{-1}\mathfrak{d}^{-1} \\ (\mu+u)\beta \gg 0}} e^{2\pi i \operatorname{Tr}((\mu+u)\beta z)} ,$$

where

$$\Lambda(u) = \frac{N(\mathfrak{m})\,|d_k|^{1/2}}{(2\pi i)^n} \sum_{\substack{\alpha \in \mathfrak{m} \\ \alpha \neq 0 \\ \{\Gamma_{\mathfrak{f}}\}}}^{*} \frac{e^{2\pi i \operatorname{Tr}(u\alpha)}}{|N(\alpha)|^2} ,$$

$z = (z_1, \cdots, z_n) \in \boldsymbol{H}^n$, $N(z) = z_1 \cdots z_n$, $\operatorname{Tr}(\gamma z) = \gamma^{(1)} z_1 + \cdots + \gamma^{(n)} z_n$ $(\gamma \in K)$, and where $\gamma \gg 0$ denotes that $\gamma \in K$ is totally positive and $\sum_{\{\Gamma_{\mathfrak{f}}\}}^{*}$ denotes a sum over a complete set of elements non-associated with respect to $\Gamma_{\mathfrak{f}}$, $d_K =$ the discriminant of K. The function $\Lambda(u)$ should be regarded as a generalized Bernoulli polynomial, since in case $K = \boldsymbol{Q}$, $\Lambda(u)$ is, essentially, the Bernoulli polynomial $B_2(u)$.

In case $u = v = 0$, $K = \boldsymbol{Q}$, the above function is, apart from a constant factor, the function $\log \eta(z)$. In case $K =$ a real quadratic field, $u = v = 0$, the above function was studied by Hecke [3], who computed its transformation formula under the transformation $z_i \mapsto -1/z_i$ $(i = 1, 2)$. In case $K = \boldsymbol{Q}$, u, v arbitrary, the above function is well-known [6] in connection with Kronecker's second limit formula.

Let $G = GL^+(2, K)$ denote the set of all 2×2 matrices over K with totally positive determinant and let G act on $\boldsymbol{H}^n$ via

$$\sigma z = \left(\frac{a^{(1)} z_1 + b^{(1)}}{c^{(1)} z_1 + d^{(1)}}, \cdots, \frac{a^{(n)} z_n + b^{(n)}}{c^{(n)} z_n + d^{(n)}} \right), \quad z = (z_1, \cdots, z_n), \quad \sigma = \begin{pmatrix} u & b \\ c & d \end{pmatrix}.$$

Let $\mathcal{O}$ denote the ring of integers of K and let $SL(2, \mathcal{O})$ denote the subgroup of G consisting of those elements of G with integral coefficients and determinant $+1$. Then $SL(2, \mathcal{O})$, with the above action is just the Hilbert modular group. Let

$$\Gamma(\mathfrak{f}) = \left\{ \sigma \in SL(2, \mathcal{O}) \mid \sigma \equiv (\text{mod } \mathfrak{f}),\ I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\},$$

where congruences are interpreted elementwise.

The main purpose of this paper is to compute $\log \eta(\sigma z; u, v)$ for $\sigma \in \Gamma(\mathfrak{f})$. In case $K = \boldsymbol{Q}$, our formula will generalize (1). Our proof is modelled precisely on the proof of (1) given in [2]. In fact, in so far as is possible, we have modelled our notation on the notation used in that paper in order to facilitate easy comparison.

The formula which we derive in this paper has applications to our theory of relative class number formulas, but in order to keep the length of this paper reasonable, we will reserve presenting the applications for a subsequent paper.

In order to keep the details of our computations as simple as possible, we will make two technical assumptions throughout the paper:

ASSUMPTION 1: $\mathfrak{f}$ and $\mathfrak{d}$ are principal ideals.

ASSUMPTION 2: $\mathfrak{m} = \mathfrak{n} = \mathcal{O}$.

Our derivation of the transformation law for $\log \eta(z; u, v)$ can be best formulated in terms of a general Mellin transform theorem which we present in Section 2. We begin our study of the generalized η-function in Section 3. Section 4 is devoted to the functional equations of certain zeta functions which arise. Section 5 reduces the calculation of the transformation law to the calculation of certain residues. Section 6 computes the residues and completes the derivation of the transformation law.

2. Mellin Transforms

Let V be a locally compact, abelian group under multiplication and let v be a generic element of V. Let V^* denote the Pontrjagin dual of V and let v^* be a typical element of V^*. Further, let Γ be a discrete subgroup of V. Let γ be a typical element of Γ. Our goal in this section is to write down the Fourier inversion formula for the group V/Γ and apply it to a special function on V/Γ.

Let $\Gamma^{\perp} = \{v^* \in V^* \mid v^*(\gamma) = 1 \text{ for all } \gamma \in \Gamma\}$. Then by duality theory, we have natural isomorphisms

$$\Gamma^{\perp} \approx (V/\Gamma)^* ,$$
$$\Gamma^* \approx V^*/\Gamma^{\perp} .$$

Let us fix a Haar measure d_1v on V/Γ and let d_1v^* denote the Haar measure on $(V/\Gamma)^*$ which is dual to d_1v. We will make a specific choice of d_1v later. Let $\mathscr{S}(V/\Gamma)$ denote the space of Schwartz-Bruhat functions on V/Γ. Then, if $f \in \mathscr{S}(V/\Gamma)$, let f^* denote its Fourier transform:

$$f^*(v^*) = \int_{V/\Gamma} f(v)v^*(v)d_1v \qquad (v^* \in (V/\Gamma)^*) ,$$

where we view v^* as a character on V which is trivial on Γ. Moreover, we have the Fourier inversion formula

$$f(v) = \int_{(V/\Gamma)^*} f^*(v^*)v^*(v)^{-1}d_1v^* \qquad (v \in V/\Gamma) .$$

Suppose that $\psi: V \to V_0 \times V_1$ is an isomorphism of locally compact abelian groups such that $\psi(\Gamma) \subseteq V_0 \times \{1\}$. Then let us identify V (resp. Γ) with $V_0 \times V_1$ (resp. $\psi(\Gamma)$) with respect to the isomorphism ψ. Then,

$$V/\Gamma \approx (V_0/\Gamma) \times V_1 ,$$

so that Haar measures d_1v_0 on V_0/Γ and d_1v_1 on V_1 can be chosen so that $d_1v = d_1v_0 \times d_1v_1$ holds. If $d_1v_0^*$ and $d_1v_1^*$ are the respective dual measures to d_1v_0 and d_1v_1, then we have

$$d_1v^* = d_1v_0^* \times d_1v_1^* .$$

Moreover,

$$(V/\Gamma)^* \approx (V_0/\Gamma)^* \times V_1^* .$$

Therefore, the Fourier inversion formula for V/Γ may be written

$$\begin{aligned} f(v) &= \int_{V_1^*}\left(\int_{(V_0/\Gamma)^*} f^*(v_0^*, v_1^*)(v_0^*, v_1^*)((v_0, v_1))^{-1}d_1v_0^*\right)d_1v_1^* \\ &= \int_{V_1^*}\left(\int_{(V_0/\Gamma)^*} f^*(v_0^*, v_1^*)v_0^*(v_0)^{-1}d_1v_0^*\right)v_1^*(v_1)^{-1}d_1v_1^* , \end{aligned} \tag{2}$$

where (v_0, v_1) and (v_0^*, v_1^*) are elements of $V/\Gamma \approx (V_0/\Gamma) \times V_1$ and $(V/\Gamma)^* \approx (V_0/\Gamma)^* \times V_1^*$, respectively.

Let us now apply formula (2) to a specific situation. Let n be a positive integer and let $V = \boldsymbol{R}_+^n$, where $\boldsymbol{R}_+ =$ the group of positive real numbers under multiplication. Further, let Γ be a discrete subgroup of V. For $v = (v_1, \cdots, v_n) \in V$, set $N(v) = v_1 \cdots v_n$ and

$$V_0 = \{v \in V \mid N(v) = 1\} ,$$
$$V_1 = \{v = (v_1, v_1, \cdots, v_1) \in V \mid v_1 \in \boldsymbol{R}_+\} .$$

Then $V \approx V_0 \times V_1$. In fact a natural way of realizing this isomorphism is as follows: If $v \in V$, then v may be written in the form

$$v = \psi_0(v)\psi_1(v) , \quad \psi_0(v) \in V_0 , \quad \psi_1(v) \in V_1 ,$$

where

$$\psi_0(v) = \left(\frac{v_1}{N(v)^{1/n}}, \frac{v_2}{N(v)^{1/n}}, \cdots, \frac{v_n}{N(v)^{1/n}}\right) ,$$
$$\psi_1(v) = (N(v)^{1/n}, \cdots, N(v)^{1/n}) .$$

Let us now restrict ourselves to Γ such that $\Gamma \subseteq V_0$. Then formula (2) shows that for $f \in \mathscr{S}(V/\Gamma)$, we have

$$f(v) = \int_{V_1^*}\left(\int_{(V_0/\Gamma)^*} f^*(v_0^*, v_1^*)v_0^*(\psi_0(v))^{-1}d_1v_0^*\right)v_1^*(\psi_1(v))^{-1}d_1v_1^* , \tag{3}$$

where the measures $d_1v_0^*$ and $d_1v_1^*$ are as chosen above. Let us make the further assumption that Γ is free of rank $n-1$. Then in this case, it is easy to describe a suitable normalization for the measures: For V_0/Γ is then compact and $(V_0/\Gamma)^*$ is discrete. Thus, for d_1v_0, let us choose the measure which gives V_0/Γ the measure 1. For $d_1v_0^*$, let us choose the measure which gives each point of $(V_0/\Gamma)^*$ the measure 1. Then $d_1v_0^*$ and d_1v_0 are dual to one another. In any case, $V_1 \approx \boldsymbol{R}_+$, so that $V_1^* \approx \boldsymbol{R}_+ \approx \boldsymbol{R}$. Let us identify $(v_1, \cdots, v_1) \in V_1$ with $v_1 \in \boldsymbol{R}_+$ and the character $\chi_t(w) = w^{itn}(w \in \boldsymbol{R}_+)$ of V_1 with the real number t. In this way, we explicitly realize the isomorphisms $V_1 \approx \boldsymbol{R}_+$, $V_1^* \approx \boldsymbol{R}$. With respect to this identification, let us write $f^*(v_0^*, t)$ instead of $f^*(v_0^*, \chi_t)$. Moreover, from the classical formulas for the Mellin transform, we see that we may choose

$$d_1v_1 = \frac{dx}{x} , \quad v_1 = (x, \cdots, x) , \quad dx = \text{Lebesgue measure on } \boldsymbol{R},$$

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$$d_1 v_1^* = \frac{1}{2\pi} dt\,, \quad v_1^* = \chi_t\,, \quad dt = \text{Lebesgue measure on } \boldsymbol{R}.$$

Then (3) may be rewritten

$$\begin{aligned} f(v) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{(V_0/\Gamma)^*} f^*(v_0^*, t) v_0^*(\psi_0(v))^{-1} d_1 v_0^*\right)\chi_t(\psi_1(v))^{-1} dt \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\int_{(V_0/\Gamma)^*} f^*(v_0^*, t) v_0^*(\psi_0(v))^{-1} d_1 v_0^*\right) N(v)^{-it} dt\,. \end{aligned}$$

Since we assume that Γ is free of rank $n-1$, $(V_0/\Gamma)^*$ is a discrete group and $(V_0/\Gamma)^* \approx \boldsymbol{Z}^{n-1}$. Let $\chi_1, \cdots, \chi_{n-1}$ be a basis of $(V_0/\Gamma)^*$. Then a typical character of V_0/Γ is of the form

$$\chi_m = \chi_1^{m_1} \cdots \chi_{n-1}^{m_{n-1}}\,, \qquad m = (m_1, \cdots, m_{n-1}) \in \boldsymbol{Z}^{n-1}\,.$$

Therefore, by the way in which we normalized our measures, we have

$$f(v) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{m \in \boldsymbol{Z}^{n-1}} f^*(\chi_m, t)\chi_m(\psi_0(v))^{-1}\right) N(v)^{-it} dt\,, \tag{4}$$

where

$$f^*(\chi_m, t) = \int_0^{\infty}\left(\int_{V_0/\Gamma} f(xv_0)\chi_m(v_0) x^{it}{}_0 d_1 v\right)\frac{dx}{x} \tag{5}$$

for any $f \in \mathscr{S}(V/\Gamma)$.

Let us derive from (4) and (5) the final formulas which will be of use to us. Let $g \in \mathscr{S}(V)$. Then, since Γ is discrete, the function

$$f(v) = \sum_{\gamma \in \Gamma} g(v\gamma) = \int_{\Gamma} g(v\gamma) d\gamma$$

belongs to $\mathscr{S}(V/\Gamma)$. Applying (4) to $f(v)$, we have

$$\sum_{\gamma \in \Gamma} g(v\gamma) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\sum_{m \in \boldsymbol{Z}^{n-1}} \hat{g}(\chi_m, t)\chi_m(\psi_0(v))^{-1}\right) N(v)^{-it} dt\,, \tag{6}$$

where

$$\begin{aligned} \hat{g}(\chi_m, t) &= \int_0^{\infty}\left(\int_{V_0/\Gamma}\left(\sum_{\gamma \in \Gamma} g(xv_0\gamma)\chi_m(v_0)\right) d_1 v_0\right) x^{it}\frac{dx}{x} \\ &= \int_0^{\infty}\left(\int_{V_0/\Gamma}\left(\sum_{\gamma \in \Gamma} g(xv_0\gamma)\chi_m(v_0\gamma)\right) d_1 v_0\right) x^{it}\frac{dx}{x} \\ &= \int_0^{\infty}\left(\int_{V_0} g(xv_0)\chi_m(v_0) dv_0\right) x^{it}\frac{dx}{x}\,, \end{aligned}$$

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where dv_0 is the Haar measure on V_0 defined by $dv_0 = d_1v_0 \times d\gamma$. Therefore, if dv is the Haar measure, $dv_0 \times \dfrac{dx}{x}$ on $V = V_0 \times V_1$, we have

$$\hat{g}(\chi_m, t) = \int_V g(v)\chi_m(\psi_0(v))\chi_t(\psi_1(v))dv \, , \tag{7}$$

where χ_m is regarded as a character of V_0, trivial on Γ. Thus, regarding

$$v \xrightarrow{(\chi_m, t)} \chi_m(\psi_0(v))\chi_t(\psi_1(v))$$

as a character of V, we see that $\hat{g}(\chi_m, t)$ is just the value of the Fourier transform of g (on V) at the character (χ_m, t) of V.

Formulas (6) and (7) are the final formulas which will be of use to us. These formulas were used in a special case by Hecke [3, p. 399], although the general principle explained above was not clear. Note that we could develop similar formulas for Γ of rank $< n-1$, but these will not be required in what follows.

Let us now introduce the class of Γ to which we apply (6) and (7). Let K be a totally real algebraic number field of degree n and let $\mathfrak{f}$ be a K-modulus (not necessarily free of infinite primes). Let us map $K^\times$ into $V = \boldsymbol{R}_+^n$ via the map

$$x \xrightarrow{\Theta} (|x^{(1)}|, \cdots, |x^{(n)}|) \, ,$$

where $x^{(1)}, \cdots, x^{(n)}$ are the conjugates of x over $\boldsymbol{Q}$. Let $\tilde{\Gamma}_{\mathfrak{f}}$ denote the group of all K-units which are $\equiv 1 \pmod{\mathfrak{f}}$ and take for Γ the group $\Theta(\tilde{\Gamma}_{\mathfrak{f}})$. It is clear that $\Gamma \subseteq V_0$ since $\tilde{\Gamma}_{\mathfrak{f}}$ consists of units; and Γ is free of rank $n-1$ by the Dirichlet unit theorem. Thus, Γ satisfies our hypothesis for (6) and (7). Let us write (6) and (7) somewhat more explicitly (if less neatly) for this class of Γ.

By the Dirichlet unit theorem, there exist units $\varepsilon_1, \cdots, \varepsilon_{n-1}$ of K such that

$$\tilde{\Gamma}_{\mathfrak{f}} = \begin{Bmatrix} \langle 1 \rangle \\ \text{or} \\ \langle \pm 1 \rangle \end{Bmatrix} \times \langle \varepsilon_1 \rangle \times \cdots \times \langle \varepsilon_{n-1} \rangle \, ,$$

where $\langle x \rangle$ denotes the multiplicative group generated by $x \in K$. Then

$$\Gamma = \Theta(\tilde{\Gamma}_{\mathfrak{f}}) = \langle \Theta(\varepsilon_1), \cdots, \Theta(\varepsilon_{n-1}) \rangle \, .$$

We may exhibit the characters $\chi_m(m \in \boldsymbol{Z}^{n-1})$ of V_0/Γ as follows: Define the numbers e_j^k $(1 \le j \le n,\ 1 \le k \le n-1)$ by the condition

$$\begin{bmatrix} \frac{1}{n} & \frac{1}{n} & \cdots & \frac{1}{n} \\ e_1^1 & e_2^1 & \cdots & e_n^1 \\ \vdots & & & \\ e_1^{n-1} & e_2^{n-1} & \cdots & e_n^{n-1} \end{bmatrix} = \begin{bmatrix} 1 & \log|\varepsilon_1^{(1)}| & \cdots & \log|\varepsilon_{n-1}^{(1)}| \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ \cdot & \cdot & & \\ 1 & \log|\varepsilon_1^{(n)}| & \cdots & \log|\varepsilon_{n-1}^{(n)}| \end{bmatrix}^{-1}.$$

If $(y_1, \cdots, y_n) \in V_0$ and $\chi \in (V_0/\Gamma)^*$, then χ is a character of V_0 such that χ is trivial on Γ (duality), so that there exist real numbers $\theta_1, \cdots, \theta_n$ such that

$$\chi((y_1, \cdots, y_n)) = y_1^{2\pi i\theta_1} \cdots y_n^{2\pi i\theta_n}$$

and

$$\chi(\gamma) = 1 \qquad (\gamma \in \Gamma)\ .$$

It is easy to see that these last two conditions imply that χ is of the form

$$\chi((y_1, \cdots, y_n)) = \prod_{j=1}^{n} y_j^{2\pi i \sum_{k=1}^{n-1} m_k e_j^k}$$

for $m = (m_1, \cdots, m_{n-1}) \in \boldsymbol{Z}^{n-1}$. Therefore, let us set

$$\alpha_j(m) = 2\pi i \sum_{k=1}^{n-1} m_k e_j^k\ .$$

Then a typical character χ_m $(m \in \boldsymbol{Z}^{n-1})$ of V_0/Γ is of the form

$$\chi_m(y_1, \cdots, y_n) = \prod_{j=1}^{n} y_j^{\alpha_j(m)}\ . \tag{8}$$

Note that χ_m is just a grössencharacter of K defined modulo $\mathfrak{f}$ (See [4].)

We now wish to use (8) to write the formulas (6) and (7) more explicitly. In order to do this, let us replace the measure $d_1v_0 \times \frac{dx}{x}$ on $V_0 \times V_1 = V$ with a measure which is easier to calculate with. Let D be a fundamental domain for V_0/Γ. Then d_1v_0 has been chosen so that

$$\int_D d_1v_0 = 1\ .$$

Moreover, if we set $E = D \times [1, e] \subseteq V_0 \times V_1$, then

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$$\int_E d_1 v_0 \times \frac{dx}{x} = 1 \, .$$

It is clear that $\frac{dv_1}{v_1} \cdots \frac{dv_n}{v_n}$ is a Haar measure on V, so that by the uniqueness of Haar measure, we have

$$\frac{dv_1}{v_1} \cdots \frac{dv_n}{v_n} = C dv_0 \times \frac{dx}{x} \, ,$$

for some $C > 0$. Moreover, it is clear from above that

$$C = \int_E \frac{dv_1}{v_1} \cdots \frac{dv_n}{v_n} \, .$$

Let us evaluate C explicitly: Let us change variables in the integral by setting $w = (v_1 \cdots v_n)^{1/n}$. Then it is easy to see that

$$C = \int_1^e \frac{dw}{w} \int_{D'} \frac{dv_1}{v_1} \cdots \frac{dv_{n-1}}{v_{n-1}} \, ,$$

where

$$D' = \left\{ (v_1, \cdots, v_{n-1}) \in \boldsymbol{R}_+^{n-1} \,|\, v_j = \exp \sum_{i=1}^{n-1} x_i \log |\varepsilon_i^{(j)}| \, , \qquad 0 \le x_i < 1 \right\} .$$

But then a simple calculation shows that

$$C = |R_{\mathfrak{f}}| \, ,$$

where

$$R_{\mathfrak{f}} = \det (\log |\varepsilon_i^{(j)}|)_{1 \le i, j \le n-1} \, .$$

Thus $R_{\mathfrak{f}}$ is just the regulator of the units $\varepsilon_1, \cdots \varepsilon_{n-1}$, and we see that

$$\frac{dv_1}{v_1} \cdots \frac{dv_n}{v_n} = |R_{\mathfrak{f}}| \, d_1 v_0 \times \frac{dx}{x} \, . \tag{9}$$

Thus, by (6), (7) and (9) and the fact that $\chi_{-m} = \chi_m^{-1}$, we finally derive:

THEOREM 2–1: *Let $g \in \mathscr{S}(V)$ and $\Gamma = \Theta(\tilde{\Gamma}_{\mathfrak{f}})$ for some K-modulus $\mathfrak{f}$. Then*

$$\sum_{\gamma \in \Gamma} g(v\gamma) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{m \in Z^{n-1}} \hat{g}_1(\chi_m, t) \chi_m(\psi_0(v)) \right) N(v)^{-it} dt \, ,$$

where

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$$\hat{g}_1(\chi_m, t) = |R_{\mathfrak{f}}|^{-1} \int_0^\infty \cdots \int_0^\infty g(v)\chi_m(\psi_0(v))^{-1}\psi_1(v)^{it} \frac{dv_1}{v_1} \cdots \frac{dv_n}{v_n} .$$

3. The Generalized η-Function

Let

$$\Gamma(\mathfrak{f}) = \{\sigma \in SL(2, \mathcal{O}) \,|\, \sigma \equiv I \ (\mathrm{mod}\ \mathfrak{f})\} ,$$

where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Our main goal, which is finally achieved in Section 6, is to determine the law of transformation of $\log \eta(z; u, v)$ under the transformations of $\Gamma(\mathfrak{f})$. We will assume throughout that assumptions 1 and 2 hold. Let us suppose that $\mathfrak{f} = f\mathcal{O}$, $\mathfrak{d} = \partial\mathcal{O}$, $\mathcal{O} =$ the ring of integers of K. Then, since u, v have denominators dividing $\mathfrak{f}\mathfrak{d}$, we see that $f\partial u$ and $f\partial v$ are K-integers, h and k, respectively. Then we have

$$u = \frac{h}{f\partial} , \qquad v = \frac{k}{f\partial} .$$

Suppose that $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathfrak{f})$. We will treat the cases $c = 0$ and $c \neq 0$ separately.

If $c = 0$, then $\sigma z = \varepsilon^2 z + \lambda$ where ε^2 is a totally positive unit of K such that $\varepsilon \equiv 1 \ (\mathrm{mod}\ \mathfrak{f})$ and $\lambda \equiv 0 \ (\mathrm{mod}\ \mathfrak{f})$. An easy argument shows that

$$\log \eta(\varepsilon^2 z; u, v) = \log \eta(z; u, v) , \tag{10}$$

$$\begin{aligned} \log \eta(z + \lambda; u, v) &= \log \eta(z; u, v) + \Lambda(u)(N(z + \lambda) - N(z)) \\ &= \log \eta(z; u, v) + \Lambda(u) \textstyle\sum z_{i_1} \cdots z_{i_k} \lambda^{(i_{k+1})} \cdots \lambda^{(i_n)} , \end{aligned} \tag{11}$$

where the last sum is over all permutations $\{i_1, \cdots, i_n\}$ of $\{1, \cdots, n\}$, excluding the identity permutation. Equations (10) and (11) completely settle the case $c = 0$.

Henceforth, let us assume that $c \neq 0$. Let us introduce a new variable $w \in \boldsymbol{C}^n$ by setting $w = -i(cz + d)sgn(c)$, where $sgn(c) = c/|c|$. Then $\mathrm{Re}\,(w) > 0$ and

$$z = -\frac{d}{c} + \frac{iw}{|c|} , \qquad \sigma(z) = \frac{a}{c} + \frac{i}{|cw|} . \tag{12}$$

In terms of the variable w, we may write

$$\log \eta(z; u, v) - \Lambda(u) N\left(-\frac{d}{c} + \frac{iw}{|c|}\right)$$

$$= \sum_{\substack{\beta \in \mathfrak{d} - \{0\} \\ \{\Gamma_{\mathfrak{f}}\}}}^{*} \frac{e^{2\pi i \operatorname{Tr}(v\beta)}}{|N(\beta)|} \sum_{\substack{\mu \in \mathfrak{b}^{-1} \\ (\mu + u)\beta \gg 0}} \exp\left\{2\pi i \operatorname{Tr}\left((\mu + u)\beta\left(-\frac{d}{c} + \frac{iw}{c}\right)\right)\right\}$$

$$= \sum_{\substack{\beta \in \mathfrak{d} - \{0\} \\ \{\Gamma_{\mathfrak{f}}\}}}^{*} \frac{e^{2\pi i \operatorname{Tr}(v\beta)}}{|N(\beta)|} \sum_{\substack{\mu \in \mathfrak{f} \\ \partial f(\mu + h)\beta \gg 0}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\beta}{f\partial}(\mu + h)\left(-\frac{d}{c} + \frac{iw}{|c|}\right)\right)\right\} \qquad (13)$$

$$= \sum_{\substack{\beta \in \mathfrak{d} - \{0\} \\ \{\Gamma_{\mathfrak{f}}\}}}^{*} \frac{e^{2\pi i \operatorname{Tr}(v\beta)}}{|N(\beta)|} \sum_{\substack{\mu \equiv h \ (\mathrm{mod}\ \mathfrak{f}) \\ \partial f \mu\beta \gg 0}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\beta\mu}{f\partial}\left(-\frac{d}{c} + \frac{iw}{|c|}\right)\right)\right\}$$

$$= \frac{1}{[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{p \ (\mathrm{mod}\ c\mathfrak{f})} \sum_{\substack{q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd - ck)\right)\right\} G_{p,q}(w) ,$$

where

$$G_{p,q}(w) = \sum_{\substack{\beta \equiv p \ (\mathrm{mod}\ c\mathfrak{f}) \\ \beta \neq 0 \\ \{\Gamma_{c\mathfrak{f}}\}}}^{*} \frac{1}{|N(\beta)|} \sum_{\substack{\mu \equiv q \ (\mathrm{mod}\ c\mathfrak{f}) \\ \mu\beta f \partial \gg 0}} \exp\left\{-2\pi \operatorname{Tr}\left(\frac{\mu\beta w}{|c|\, f\partial}\right)\right\} .$$

Let θ be a typical signature character of K, i.e. a function $\theta: K^{\times} \to \{\pm 1\}$ of the form

$$\theta(x) = \left(\frac{x^{(1)}}{|x^{(1)}|}\right)^{a_1} \cdots \left(\frac{x^{(n)}}{|x^{(n)}|}\right)^{a_n} , \qquad a_j = 0 \quad \text{or } 1.$$

Then for $x \in K^{\times}$, we have

$$\sum_{\theta} \theta(x) = 2^n \quad \text{if} \quad x \gg 0$$
$$0 \quad \text{otherwise,}$$

where the summation ranges over all 2^n signature characters of K. Then from our calculations above, we see that

$$G_{p,q}(w) = 2^{-n} \sum_{\theta} G_{p,q,\theta}(w) , \qquad (14)$$

where

$$G_{p,q,\theta}(w) = \theta(f\partial) \sum_{\substack{\beta \equiv p \ (\mathrm{mod}\ c\mathfrak{f}) \\ \beta \neq 0 \\ \{\Gamma_{c\mathfrak{f}}\}}}^{*} \frac{\theta(\beta)}{|N(\beta)|} \sum_{\substack{\mu \equiv q \ (\mathrm{mod}\ c\mathfrak{f}) \\ \mu \neq 0}} \theta(\mu) \exp\left\{-2\pi \operatorname{Tr}\left(\left|\frac{\mu\beta}{cf\partial}\right| w\right)\right\}$$
$$= \sum_{\varepsilon \in \Gamma_{c\mathfrak{f}}} g_{p,q,\theta}(\varepsilon w) , \qquad (15)$$

where $\varepsilon w = \begin{pmatrix} \varepsilon & 0 \\ 0 & 1 \end{pmatrix} w$ and

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$$g_{p,q,\theta}(w) = \theta(f\partial) \sum^*_{\substack{\beta\equiv p \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \beta\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \frac{\theta(\beta)}{|N(\beta)|} \sum^*_{\substack{\mu\equiv q \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \mu\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \theta(\mu) \exp\left\{-2\pi \operatorname{Tr}\left(\left|\frac{\mu\beta}{cf\partial}\right| w\right)\right\} \tag{16}$$

Let $w = (w_1, \cdots, w_n)$, $w_j > 0$. We will apply Theorem 2–1 to the function

$$g_1(w) = g_{p,q,\theta}(w) N(w)^x \qquad (x > 1) .$$

It is easy to see that the condition $x > 1$ guarantees that $g_1 \in \mathscr{S}(V)$. Then Theorem 2–1 implies that

$$N(w)^x \sum_{\varepsilon\in\Gamma_{\mathfrak{c}\mathfrak{f}}} g_{p,q,\theta}(\varepsilon w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\sum_{m\in \mathbf{Z}^{n-1}} \hat{g}_1(\chi_m, t)\chi_m(\psi_0(w))\right) N(w)^{-it} dt \tag{17}$$

where

$$\begin{aligned}
\hat{g}_1(\chi_m, t) &= |R_{\mathfrak{c}\mathfrak{f}}|^{-1} \int_0^\infty \cdots \int_0^\infty g_1(w)\chi_m(\psi_0(w))^{-1}\psi_1(w)^{it} \frac{dw_1}{w_1} \cdots \frac{dw_n}{w_n} \\
&= |R_{\mathfrak{c}\mathfrak{f}}|^{-1} \int_0^\infty \cdots \int_0^\infty g_{p,q,\theta}(w_1, \cdots, w_n) \prod_{j=1}^n \left(\frac{w_j}{\sqrt[n]{w_1 \cdots w_n}}\right)^{-\alpha_j(m)} N(w)^{it+x} \\
&\qquad \cdot \frac{dw_1}{w_1} \cdots \frac{dw_n}{w_n} \\
&= |R_{\mathfrak{c}\mathfrak{f}}|^{-1} \int_0^\infty \cdots \int_0^\infty g_{p,q,\theta}(w_1, \cdots, w_n) \prod_{j=1}^n w_j^{s-\alpha_j(m)} \frac{dw_1}{w_1} \cdots \frac{dw_n}{w_n} ,
\end{aligned} \tag{18}$$

where $s = x + it$ and where we have used the fact that $\sum_{j=1}^n \alpha_j(m) = 0$. Therefore, by inserting (16) in (18), we see that

$$\begin{aligned}
\hat{g}_1(\chi_m, t) &= |R_{\mathfrak{c}\mathfrak{f}}|^{-1}\theta(f\partial) \sum^*_{\substack{\beta\equiv p \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \beta\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \frac{\theta(\beta)}{|N(\beta)|} \sum_{\substack{\mu\equiv q \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \mu\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \theta(\mu) \\
&\quad \cdot \int_0^\infty \cdots \int_0^\infty \left[\prod_{j=1}^n e^{-2\pi|\mu^{(j)}\beta^{(j)}w_j/c^{(j)}f^{(j)}\partial^{(j)}|} w_j^{s-\alpha_j(m)}\right] \frac{dw_1}{w_1} \cdots \frac{dw_n}{w_n} \\
&= |R_{\mathfrak{c}\mathfrak{f}}|^{-1}\theta(f\partial) \sum^*_{\substack{\beta\equiv p \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \beta\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \frac{\theta(\beta)}{|N(\beta)|} \sum^*_{\substack{\mu\equiv q \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \mu\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \theta(\mu) \\
&\quad \cdot \prod_{j=1}^n \int_0^\infty e^{-2\pi|\mu^{(j)}\beta^{(j)}w_j/c^{(j)}f^{(j)}\partial^{(j)}|} w_j^{s-\alpha_j(m)} \frac{dw_j}{w_j} \\
&= \frac{\theta(f\partial)}{|R_{\mathfrak{c}\mathfrak{f}}|} (2\pi)^{-ns}\Gamma(s, \chi_m)\chi_m(cf\partial)^{-1}|N(cf\partial)|^s \\
&\quad \cdot \sum^*_{\substack{\beta\equiv p \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \beta\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \frac{\theta(\beta)\chi_m(\beta)}{|N(\beta)|^{s+1}} \sum^*_{\substack{\mu\equiv q \ (\mathrm{mod}\ \mathfrak{c}\mathfrak{f}) \\ \mu\neq 0 \\ \{\Gamma_{\mathfrak{c}\mathfrak{f}}\}}} \frac{\theta(\mu)\chi_m(\mu)}{|N(\mu)|^s} ,
\end{aligned} \tag{19}$$

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$$\Gamma(s, \chi_m) = \sum_{j=1}^{n} \Gamma(s - \alpha_j(m)) . \tag{20}$$

Formula (19) suggests that we introduce a new kind of zeta function as follows: Let χ be a character of $K^\times$ which is trivial on $\Gamma_r = \Gamma_{r\mathcal{O}}$ for some $r \in \mathcal{O}$, $r \neq 0$. Further, let $p \in \mathcal{O}$. Then let us define the zeta function

$$\zeta(s, \chi; p, r) = \sum_{\substack{\beta \equiv p \ (\mathrm{mod}\ r) \\ \beta \neq 0 \\ \{\Gamma_r\}}}^{*} \frac{\chi(\beta)}{|N(\beta)|^s} .$$

This Dirichlet series is well-defined since χ is trivial on Γ_r. Moreover, the series converges absolutely and uniformly on compact subsets of the half-plane $\mathrm{Re}(s) > 1$. Usually, it is too much to hope that $\zeta(s, \chi; p, r)$ has an analytic continuation and a functional equation. However, in case $\chi(\beta) = \lambda(\beta)\theta(\beta)$, where λ is a grössencharacter defined modulo (r) and θ is a signature character of K, then $\zeta(s, \chi; p, r)$ can be analytically continued and has a functional equation involving gamma factors of customary type (See Section 4). The functional equation to be proven in the next section will involve the function

$$F_{p,q,\chi,r}(s) = \chi(r\partial)^{-1}\left(\frac{2^n \pi^n}{|N(r\partial)|}\right)^{-s} \Gamma(s, \chi)\zeta(s, \chi; q, r)\zeta(s + 1, \chi; p, r) ,$$

where $\chi = \chi_m \theta$ and $\Gamma(s, \chi) = \Gamma(s, \chi_m)$. A simple calculation shows that

$$\hat{g}_1(\chi_m, t) = \frac{1}{|R_{\mathfrak{cf}}|} F_{p,q,\chi,cf}(s) , \qquad \chi = \chi_m \theta .$$

Since the value of r in the remainder of this paper will always be cf, we will omit the value of r in $\zeta(s, \chi; p, r)$ and $F_{p,q,\chi,r}(s)$, writing instead $\zeta(s, \chi; p)$ and $F_{p,q,\chi}(s)$, respectively.

From equations (15) and (17), we see that for $w \in \boldsymbol{R}_+^n$, $x > 1$,

$$\begin{aligned} G_{p,q,\theta}(w) &= \sum_{\varepsilon \in \Gamma_{cf}} g_{p,q,\theta}(\varepsilon w) \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \sum_{m \in Z^{n-1}} \chi_m(\psi_0(w)) \hat{g}_1(\chi_m, t) N(w)^{-x-it} dt \\ &= \frac{1}{2\pi_i |R_{\mathfrak{cf}}|} \sum_{\substack{\chi_m \\ \chi = \chi_m \theta}} \chi_m(w) \int_{x-i\infty}^{x+i\infty} F_{p,q,\chi}(s) N(w)^{-s} ds . \end{aligned} \tag{21}$$

In concluding this section, let us make a few remarks about formula

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(21). In case $K = \boldsymbol{Q}$, $n = 1$, $\mathfrak{f} = (1)$, there are precisely two signature characters. Moreover, the only character χ_m is the trivial character. Moreover, it is always possible to arrange things so that $c > 0$. (Take $-\sigma$ in place of σ), so that $\theta(c) = 1$ for any signature character θ. The zeta functions in this case are of the form

$$\zeta(s, \theta; p, r) = \sum_{\substack{m \equiv p \ (\mathrm{mod}\ c) \\ m \neq 0 \\ m \in \boldsymbol{Z}}} \frac{\theta(m)}{|m|^s},$$

where $\theta(m) = 1$ or $\theta(m) = \mathrm{sgn}\,(m)$. Moreover,

$$F_{p,q,\theta,c}(s) = \left(\frac{2\pi}{c}\right)^{-s} \Gamma(s)\zeta(s, \theta; q, c)\zeta(s+1, \theta; p, c) ,$$

$$G_{p,q,\theta}(w) = \sum_{\substack{m \equiv p \ (\mathrm{mod}\ c) \\ m \neq 0 \\ m \in \boldsymbol{Z}}} \frac{\theta(m)}{|m|} \sum_{\substack{n \equiv q \ (\mathrm{mod}\ c) \\ n \neq 0}} \theta(n) \exp\left(-2\pi\left|\frac{mn}{c}\right| w\right) \qquad (w = w_1) .$$

Moreover, formula (21) is just the Mellin inversion formula

$$G_{p,q,\theta}(w) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} F_{p,q,\theta}(s) w^{-s} ds .$$

These last three formulas should be compared with equations (6), (5) and (8) of [2] to see how the present proof is a precise generalization of the proof [2] for the 1-dimensional case.

4. Functional Equations

Let us now summarize the basic facts concerning zeta functions of the type $\zeta(s, \chi; p, r)$. Throughout this section, let $\chi = \chi_m\theta$, where χ_m is a grössencharacter of K defined modulo (r) and θ is a signature character of K.

THEOREM 4.1. *The zeta function $\zeta(s, \chi; p, r)$ can be analytically continued to a meromorphic function in the whole s-plane. The only possible singularity is at $s = 1$, which occurs only when $\chi \equiv 1$ and $p \equiv 0$* (mod r). *In case the pole is present, its residue is $2^{n-1}e\,|N(r)N(\partial)^{1/2}|^{-1}$, where R is the regulator of K and e is the index of the group of all totally positive K-units $\equiv 1\,(\mathrm{mod}\ r)$ in the group of all K-units. Moreover, $\zeta(s, \chi; p, r)$ satisfies the functional equation*

$$A^{-s/2}\Gamma^*(s, \chi)\zeta(s, \chi; p, r)$$
$$= \frac{(-i)^a \chi(r\partial)}{\sqrt{|N(r)|}} A^{-(1-s)/2}\Gamma^*(1-s, \bar{\chi}) \sum_{\rho\ (\mathrm{mod}\ r)} e^{2\pi i\, \mathrm{Tr}\,(p\rho/r\partial)} \zeta(1-s, \bar{\chi}; \rho, r) ,$$

where $A = \pi^n/|d_K N(r)|$, d_K = *the discriminant of* K, $a = \sum_{i=1}^{n} a_i$ *and*

$$\Gamma^*(s,\chi) = \prod_{j=1}^{n} \Gamma\left(\frac{s + a_j - \alpha_j(m)}{2}\right).$$

Moreover, the only possible poles of the function $\xi(s,\chi;p,r) = A^{-s/2}\Gamma^*(s,\chi)$ $\zeta(s,\chi;p,r)$ *are at the points* $s = 1$, $s = 0$.

The proof of Theorem 4.1 parallels Hecke's proof for the functional equation for zeta functions with grössencharacters [4] and will be given in detail elsewhere.

Let us now use Theorem 4.1 to derive a functional equation for $F_{p,q,\chi}(s)$. Take the functional equation of Theorem 4.1 at s, q and $s+1$, p and multiply the two. We then derive

$$\begin{aligned} &A^{-\frac{1}{2}-s}\Gamma^*(s,\chi)\Gamma^*(s+1,\chi)\zeta(s,\chi;q,r)\zeta(s+1,\chi;p,r) \\ &\quad = \frac{(-1)^a\chi(r\partial)^2}{|N(r)|} A^{s-\frac{1}{2}}\Gamma^*(-s,\bar\chi)\Gamma^*(1-s,\bar\chi \\ &\qquad \cdot \sum_{\alpha,\beta \pmod r} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\alpha q + \beta p}{r\partial}\right)\right\}\zeta(1-s,\bar\chi;\alpha,r)\zeta(-s,\bar\chi;\beta,r). \end{aligned}$$

By using the duplication formula for the gamma function, we see that

$$\Gamma^*(s,\chi)\Gamma^*(s+1,\chi) = \pi^{n/2}2^{n(1-s)}2^{-a}\Gamma(s,\chi)\cdot\prod_{j=1}^{n}(s-\alpha_j(m))^{a_j}$$

$$\Gamma^*(1-s,\bar\chi)\Gamma^*(-s,\bar\chi) = \pi^{n/2}2^{n(1+s)}2^{-a}\Gamma(-s,\bar\chi)\cdot\prod_{j=1}^{n}(-s+\alpha_j(m))^{a_j}.$$

Therefore, by an elementary computation, we deduce

COROLLARY 4.2.

$$F_{p,q,\chi}(s) = \frac{1}{|N(r)|}\sum_{\alpha,\beta \pmod r}\exp\left\{2\pi i \operatorname{Tr}\left(\frac{\alpha p + \beta q}{r\partial}\right)\right\}F_{\beta,\alpha,\bar\chi}(-s).$$

Theorem 4.1 is a generalization of equation (4) of [2], whereas Corollary 4.2 is a generalization of equation (7) of [2].

Implicit in our proof of Corollary 4.2 is the following useful relationship between $\xi(s,\chi;p,r)$ and $F_{p,q,\chi,r}(s)$:

PROPOSITION 4.3.

$$F_{p,q,\chi,r}(s) = \frac{2^{a-n}}{|N(r\partial)|^{\frac{1}{2}}}\frac{\chi(r\partial)^{-1}}{\prod_{j=1}^{n}(s-\alpha_j(m))^{a_j}}\xi(s,\chi;q,r)\xi(s+1,\chi;p,r).$$

5. Proof of the Transformation Law—First Step

Let us begin the proof of the transformation law for $\log \eta(z; u, v)$ using formula (21). Let us first note that although (21) was proven only for $w = (w_1, \cdots, w_n) \in \boldsymbol{R}_+^n$, we may actually assume that w is any complex number such that $\mathrm{Re}(w) > 0$. For such w, let us define the power w^α ($\alpha \in C$) via the formula $w^\alpha = e^{\alpha \log w}$, where $\log w$ is the principal branch. Then w^α is an analytic function in the region $\mathrm{Re}(w) > 0$. Using this definition of w^α, we can define $\chi_m(w)$ and $N(w)^{-s}$ for any w such that $\mathrm{Re}(w) > 0$. By analytic continuation, the formula (21) remains valid for $w = (w_1, \cdots, w_n) \in \boldsymbol{C}^n$, $\mathrm{Re}(w_j) > 0$ $(1 \le j \le n)$.

Let us apply formula (21) with $x = 3/2$ and let us shift the line of integration to $\mathrm{Re}(s) = -3/2$, adding the residues of the poles of the integrand at points having real parts between $3/2$ and $-3/2$. Among the possible poles are $s = 0, 1, -1$, as well as the poles of $\Gamma(s, \chi)$. The validity of shifting the line of integration is easily checked using straightforward estimates and the Phragmen-Lindelöf Theorem. The result is

$$G_{p,q,\theta}(w) = \frac{1}{2\pi i\,|R_{\mathfrak{cf}}|} \sum_{\chi = \chi_m \theta} \chi_m(w) \int_{-3/2 - i\infty}^{-3/2 + i\infty} F_{p,q,\chi}(s) N(w)^{-s} ds + \frac{1}{|R_{\mathfrak{cf}}|} \sum_{\chi = \chi_m \theta} \chi_m(w) \sum_{-3/2 < \mathrm{Re}(s) < 3/2} \mathrm{Res}\, F_{p,q,\chi}(s) N(w)^{-s} .$$

Therefore, by Corollary 4.2, we see that

$$\begin{aligned} G_{p,q,\theta}(w) &= \frac{1}{2\pi i\,|R_{\mathfrak{cf}}|\,|N(cf)|} \sum_{\chi = \chi_m \theta} \chi_m(w) \sum_{\alpha, \beta \ (\mathrm{mod}\ \mathfrak{cf})} \exp\left\{2\pi i \,\mathrm{Tr}\left(\frac{\alpha p + \beta q}{c f \partial}\right)\right\} \\ &\qquad \cdot \int_{-3/2 - i\infty}^{-3/2 + i\infty} F_{\beta, \alpha, \bar{\chi}}(-s) N(w)^{-s} ds \\ &\quad + \frac{1}{|R_{\mathfrak{cf}}|} \sum_{\chi = \chi_m \theta} \chi_m(w) \sum_{-3/2 < \mathrm{Re}(s) < 3/2} \mathrm{Res}\, F_{p,q,\chi}(s) N(w)^{-s} \\ &= \frac{1}{|N(cf)|} \sum_{\alpha, \beta \ (\mathrm{mod}\ \mathfrak{cf})} \exp\left\{2\pi i \,\mathrm{Tr}\left(\frac{\alpha p + \beta q}{c f \partial}\right)\right\} G_{\beta, \alpha, \theta}(w^{-1}) \\ &\quad + \frac{1}{|R_{\mathfrak{cf}}|} \sum_{\chi = \chi_m \theta} \chi_m(w) \sum_{-3/2 < \mathrm{Re}(s) < 3/2} \mathrm{Res}\, F_{p,q,\chi}(s) N(w)^{-s} . \end{aligned}$$

Therefore, by this last equation and equations (13)–(15), we have

$$\log \eta(z; u, v) - \Lambda(u) N\left(-\frac{d}{c} + \frac{iw}{|c|}\right)$$

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$$= \frac{1}{2^n[\Gamma_{\mathfrak{f}}:\Gamma_{c\mathfrak{f}}]} \sum_{p \pmod{c\mathfrak{f}}} \sum_{\substack{q \pmod{c\mathfrak{f}} \\ q\equiv h \pmod{\mathfrak{f}}}} \sum_{\theta} \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd-ck)\right)\right\} G_{p,q,\theta}(w)$$

$$= \frac{1}{2^n[\Gamma_{\mathfrak{f}}:\Gamma_{c\mathfrak{f}}]\,|N(cf)|} \sum_{\substack{p \pmod{c\mathfrak{f}} \\ q \pmod{c\mathfrak{f}} \\ q\equiv h \pmod{\mathfrak{f}}}} \sum_{\theta} \sum_{\alpha,\beta \pmod{c\mathfrak{f}}} \tag{22}$$

$$\times \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\alpha p+\beta q+kcp-pqd}{cf\partial}\right)\right\} G_{\beta,\alpha,\theta}(w^{-1})$$

$$+ \frac{1}{2^n\,|R_{c\mathfrak{f}}|\,[\Gamma_{\mathfrak{f}}:\Gamma_{c\mathfrak{f}}]} \sum_{\substack{p,q \pmod{c\mathfrak{f}} \\ q\equiv h \pmod{\mathfrak{f}}}} \sum_{\chi=\chi_m\theta} \chi_m(w) \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd-ck)\right)\right\}$$

$$\sum_{-3/2<\operatorname{Re}(s)<3/2} \operatorname{Res} F_{p,q,\chi}(s) N(w)^{-s} .$$

Let us evaluate the first triple sum on the right hand side of equation (22). Since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathfrak{f})$, we have $a \equiv d \equiv 1 \pmod{\mathfrak{f}}$, $b \equiv c \equiv 0 \pmod{\mathfrak{f}}$, $ad - bc = 1$. In particular d and $\mathfrak{f}$ are relatively prime and $ad \equiv 1 \pmod{\mathfrak{f}}$. Next, note that

$$\sum_{p \pmod{c\mathfrak{f}}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\alpha+ck-qd}{cf\partial}p\right)\right\} = |N(cf)|$$

$$\text{if } \alpha+ck-qd \equiv 0 \pmod{cf}, \quad 0 \text{ otherwise.}$$

However, $\alpha + ck - qd \equiv 0 \pmod{c\mathfrak{f}}$ if and only if $q \equiv a(\alpha+ck) \pmod{c\mathfrak{f}}$, since $ad \equiv 1 \pmod{c\mathfrak{f}}$. Therefore,

$$\sum_{\substack{q \pmod{c\mathfrak{f}} \\ q\equiv h \pmod{\mathfrak{f}}}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\beta q}{cf\partial}\right)\right\} \sum_{p \pmod{c\mathfrak{f}}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\alpha+ck-qd}{cf\partial}p\right)\right\}$$

$$= |N(cf)| \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\beta a(\alpha+ck)}{cf\partial}\right)\right\} \qquad \text{if } a(\alpha+ck) \equiv h \pmod{\mathfrak{f}},$$

$$0 \qquad\qquad \text{otherwise.}$$

However, since $(f) = \mathfrak{f}$, $c \equiv 0 \pmod{\mathfrak{f}}$, $d \equiv 1 \pmod{\mathfrak{f}}$, we see that if

$$a(\alpha+ck) \equiv h \pmod{\mathfrak{f}},$$

then $\alpha \equiv h \pmod{\mathfrak{f}}$, so that

$$\sum_{\substack{p,q \pmod{c\mathfrak{f}} \\ q\equiv h \pmod{\mathfrak{f}}}} \sum_{\alpha,\beta \pmod{c\mathfrak{f}}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\alpha p+\beta q+ckp-pqd}{cf\partial}\right)\right\} G_{\beta,\alpha,\theta}(w^{-1})$$

$$= |N(cf)| \sum_{\substack{\alpha,\beta \pmod{c\mathfrak{f}} \\ \alpha\equiv h \pmod{\mathfrak{f}}}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{\beta a(\alpha+ck)}{cf\partial}\right)\right\} G_{\beta,\alpha,\theta}(w^{-1}) \tag{23}$$

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On the other hand, since $a \equiv 1 \pmod{\mathfrak{f}}$, we see that

$$\frac{\beta a k}{f\partial} - \frac{\beta k}{f\partial} = \frac{\beta k}{f\partial}(a-1) \in \beta k \mathfrak{d}^{-1} \subseteq \mathfrak{d}^{-1}\,.$$

Therefore, $\mathrm{Tr}\,(\beta a k/f\partial) - \mathrm{Tr}\,(\beta k/f\partial) \in \boldsymbol{Z}$ and (23) equals

$$|N(cf)| \sum_{\substack{\alpha,\beta \,(\mathrm{mod}\, c\mathfrak{f}) \\ \alpha \equiv h \,(\mathrm{mod}\, \mathfrak{f})}} \exp\left\{2\pi i \,\mathrm{Tr}\left(\frac{\beta}{cf\partial}(a\alpha + ck)\right)\right\} G_{\beta,\alpha,\theta}(w^{-1})\,. \tag{24}$$

Thus, by equation (22), we have

$$\begin{aligned}
&\log \eta(z\,;u,v) - \Lambda(u) N\left(-\frac{d}{c} + \frac{iw}{|c|}\right) \\
&= \frac{1}{2^n[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\theta} \sum_{\substack{\alpha,\beta \,(\mathrm{mod}\, c\mathfrak{f}) \\ \alpha \equiv h \,(\mathrm{mod}\, \mathfrak{f})}} \exp\left\{2\pi i \,\mathrm{Tr}\left(\frac{\beta}{cf\partial}(a\alpha + ck)\right)\right\} G_{\beta,\alpha,\theta}(w^{-1}) \\
&\quad + \frac{1}{2^n |R_{c\mathfrak{f}}| [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\substack{p,q \,(\mathrm{mod}\, c\mathfrak{f}) \\ \theta \\ q \equiv h \,(\mathrm{mod}\, \mathfrak{f})}} \sum_{\chi = \chi_m \theta} \chi_m(w) \\
&\quad \times \exp\left\{-2\pi i \,\mathrm{Tr}\left(\frac{p}{cf\partial}(qd - ck)\right)\right\} \sum_{-3/2 < \mathrm{Re}\,(s) < 3/2} \mathrm{Res}\, F_{p,q,\chi}(s) N(w)^{-s} \\
&= \log \eta\left(\frac{a}{c} + \frac{iw^{-1}}{c}\,;u,v\right) - \Lambda(u) N\left(\frac{a}{c} + \frac{iw^{-1}}{|c|}\right) \\
&\quad + \frac{1}{2^n |R_{c\mathfrak{f}}| [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\substack{p,q \,(\mathrm{mod}\, c\mathfrak{f}) \\ q \equiv h \,(\mathrm{mod}\, \mathfrak{f}) \\ \theta}} \sum_{\chi = \chi_m \theta} \chi_m(w) \\
&\quad \times \exp\left\{-2\pi i \,\mathrm{Tr}\left(\frac{p}{cf\partial}(qd - ck)\right)\right\} \sum_{-3/2 < \mathrm{Re}\,(s) < 3/2} \mathrm{Res}\, F_{p,q,\chi}(s) N(w)^{-s}\,.
\end{aligned}$$

Let us set

$$\begin{aligned}
H(s) = \frac{1}{2^n |R_{c\mathfrak{f}}| [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} &\sum_{\substack{\chi_m,\theta \\ \chi = \chi_m \theta}} \sum_{\substack{p,q \,(\mathrm{mod}\, c\mathfrak{f}) \\ q \equiv h \,(\mathrm{mod}\, \mathfrak{f})}} \chi_m(w) \\
&\times \exp\left\{-2\pi i \,\mathrm{Tr}\left(\frac{p}{cf\partial}(qd - ck)\right)\right\} F_{p,q,\chi}(s)\,.
\end{aligned}$$

Then we may summarize our results so far in

THEOREM 5.1. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathfrak{f})$. *Then*

$$\begin{aligned}
\log \eta\left(\frac{az+b}{cz+d}\,;u,v\right) = \log \eta(z\,;u,v) - \Lambda(u)\left\{N(z) - N\left(\frac{az+b}{cz+d}\right)\right\} \\
- \sum_{-3/2 < \mathrm{Re}\,(s) \leqq 3/2} \mathrm{Res}\, H(s) N(w)^{-s}\,.
\end{aligned}$$

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In essence, we have accomplished the goal of the paper. All that remains to be done is to compute the residues on the right hand side and we will have derived the law of transformation of our generalized η-function. Unfortunately, this last task is exceedingly messy and will be the principal subject of the next section.

6. Proof of the Transformation Law—Second Step

Let us recall the function

$$\xi(s,\chi;p,r) = A^{-s/2}\Gamma^*(s,\chi)\zeta(s,\chi;p,r) ,$$

which is mentioned in Theorem 4.1. As we mentioned, this function is analytic for all s, except possibly for $s = 0, 1$. At $s = 1$, there is a simple pole with residue $\dfrac{2^{n-1} Re(r)}{|N(r)|^{1/2}}$ which occurs if and only if $\chi = \chi_0$, the trivial character where $e(r) = [\Gamma : \Gamma_r]$. At $s = 0$, there is a simple pole with residue $-2^{n-1} Re(r)$ which occurs if and only if $\chi = \chi_0$ and $p \equiv 0 \pmod r$.

Recall from Proposition 4.3 that for $\chi = \chi_m\theta$, we have

$$F_{p,q,\chi,r}(s) = \frac{2^{a-n}}{|N(r\partial)|^{\frac{1}{2}}} \frac{\chi(r\partial)^{-1}}{\prod_{j=1}^{n}(s-\alpha_j(m))^{a_j}}\xi(s,\chi;q,r)\xi(s+1,\chi;p,r) . \quad (25)$$

Therefore, we see that $F_{p,q,\chi,r}(s)$ has possible poles only at the points $s = 0$, 1, -1 and $s = s_j(\chi)$ $(1 \le j \le n)$, where $s_j(\chi) = \alpha_j(m)$ if $\chi = \chi_m\theta$. Thus, $H(s)N(w)^{-s}$ can have poles only at these points.

PROPOSITION 6.1. $\operatorname{Res}_{s=1} H(s)N(w)^{-s} = \dfrac{i^n}{|N(c)|}\Lambda(u)\dfrac{1}{N(w)}$.

Proof. Using the fact that $\xi(s,\chi;p,r)$ is an entire function for $\chi \ne \chi_0$, we see that $F_{p,q,\chi,r}(s)$ is analytic at $s = 1$ for $\chi \ne \chi_0$. Therefore,

$$\begin{aligned} &\operatorname{Res}_{s=1} H(s)N(w)^{-s} \\ &= \frac{1}{2^n |R_{c\mathfrak{f}}| [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\substack{\chi_m,\theta \\ \chi=\chi_m\theta}} \chi_m(w) \sum_{\substack{p,q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q\equiv h\ (\mathrm{mod}\ \mathfrak{f})}} \\ &\qquad \times \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd-ck)\right)\right\} \operatorname{Res}_{s=1} F_{p,q,\chi}(s)N(w)^{-s} \\ &= \frac{1}{2^n |R_{c\mathfrak{f}}| [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\substack{p,q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q\equiv h\ (\mathrm{mod}\ \mathfrak{f})}} \\ &\qquad \times \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd-ck)\right)\right\} \operatorname{Res}_{s=1} F_{p,q,\chi_0}(s)N(w)^{-s} . \end{aligned} \quad (26)$$

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However, by Theorem 4.1,

$$\begin{aligned}
&\operatorname{Res}_{s=1} F_{p,q,\chi_0}(s) N(w)^{-s} \\
&\quad = \operatorname{Res}_{s=1} \left(\frac{2^n \pi^n}{|N(cf\partial)|} \right)^{-s} \Gamma(s)^n \zeta(s, \chi_0; q, cf) \zeta(s+1, \chi_0; p, cf) N(w)^{-s} \\
&\quad = \frac{Re\,(cf)\,|N(\partial)|^{\frac{1}{2}}}{2\pi^n} \zeta(2, \chi_0; p, cf) N(w)^{-1} .
\end{aligned}$$

Therefore,

$$\begin{aligned}
\operatorname{Res}_{s=1} H(s) N(w)^{-s} = {} & \frac{|N(\partial)|^{\frac{1}{2}} N(w)^{-1}}{2(2\pi)^n [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\substack{p,q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \\
& \times \exp \left\{ -2\pi i \operatorname{Tr} \left(\frac{p}{cf\partial} (qd - ck) \right) \right\} \zeta(2, \chi_0; p, cf)
\end{aligned}$$

In the last sum, let us write $q = h + \alpha f$, where α runs modulo c. Then

$$\begin{aligned}
&\operatorname{Res}_{s=1} H(s) N(w)^{-s} \\
&\quad = \frac{|N(\partial)|^{\frac{1}{2}} N(w)^{-1}}{(2\pi)^n [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{p \ (\mathrm{mod}\ c\mathfrak{f})} \exp \left\{ 2\pi i \operatorname{Tr} \left(\frac{pk}{f\partial} \right) \right\} \zeta(2, \chi_0; p, cf) \\
&\qquad \times \exp \left\{ -2\pi i \operatorname{Tr} \left(\frac{phd}{cf\partial} \right) \right\} \sum_{\alpha \ (\mathrm{mod}\ c)} \exp \left\{ -2\pi i \operatorname{Tr} \left(\frac{p\alpha d}{c\partial} \right) \right\} \\
&\quad = \frac{|N(\partial)|^{\frac{1}{2}} |N(c)| N(w)^{-1}}{(2\pi)^n [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\substack{p \ (\mathrm{mod}\ c\mathfrak{f}) \\ p \equiv 0 \ (\mathrm{mod}\ c)}} \\
&\qquad \times e^{2\pi i \operatorname{Tr}(vp)} \exp \left\{ -2\pi i \operatorname{Tr} \left(\frac{phd}{cf\partial} \right) \right\} \zeta(2, \chi_0; p, cf) .
\end{aligned}$$

Since $p \equiv 0 \ (\mathrm{mod}\ c)$, $e^{2\pi i \operatorname{Tr}(vp)} = 1$. Let us set $p = \alpha c$, where α runs modulo $\mathfrak{f}$. Then a simple computation shows that

$$\zeta(2, \chi_0; p, cf) = |N(c)|^{-2} [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}] \zeta(2, \chi_0; \alpha, f) .$$

Therefore,

$$\begin{aligned}
&\operatorname{Res}_{s=1} H(s) N(w)^{-s} \\
&\quad = \frac{|N(\partial)|^{\frac{1}{2}} |N(c)|^{-1} N(w)^{-1}}{(2\pi)^n} \sum_{\alpha \ (\mathrm{mod}\ \mathfrak{f})} \exp \left\{ -2\pi i \operatorname{Tr} \left(\frac{\alpha h d}{f\partial} \right) \right\} \zeta(2, \chi_0; \alpha, f) \\
&\quad = \frac{|N(\partial)|^{\frac{1}{2}} |N(c)|^{-1} N(w)^{-1}}{(2\pi)^n} \sum_{\alpha \ (\mathrm{mod}\ \mathfrak{f})} e^{2\pi i \operatorname{Tr}(\alpha u)} \zeta(2, \chi_0; \alpha, f) .
\end{aligned}$$

Since $d \equiv 1 \ (\mathrm{mod}\ \mathfrak{f})$, $u = h/f\partial$. However, a direct computation shows that

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$$\frac{1}{(2\pi)^n}\sum_{\alpha \,(\mathrm{mod}\, \mathfrak{f})} e^{2\pi i \operatorname{Tr}(\alpha u)}\zeta(2,\chi_0;\alpha,f) = \frac{1}{(2\pi)^n}\sum^{*}_{\substack{\beta\in \mathfrak{o}_K-\{0\}\\ \{\Gamma_{\mathfrak{f}}\}}} \frac{e^{2\pi i \operatorname{Tr}(\beta u)}}{|N(\beta)|^2} = \frac{i^n}{|d_K|^{\frac{1}{2}}}\Lambda(u)\ .$$

Therefore, since $|N(\partial)| = |d_K|$, we have

$$\operatorname{Res}_{s=1} H(s)N(w)^{-s} = \frac{i^n}{|N(c)|}\Lambda(u)\frac{1}{N(w)}\ ,$$

and the proposition is proved.

PROPOSITION 6.2. $\operatorname{Res}_{s=-1} H(s)N(w)^{-s} = -\dfrac{i^n}{|N(c)|}\Lambda(u)N(w)$.

Proof. Similar to the proof of Proposition 6.1, except that the functional equation of Proposition 4.2 is applied to the analogue of (26).

Let us now study the residue at $s = 0$. Let us write

$$\operatorname{Res}_{s=0} H(s)N(w)^{-s} = R_1 + R_2\ , \tag{27}$$

where

$$R_1 = \frac{1}{2^n|R_{c\mathfrak{f}}|\,[\Gamma_{\mathfrak{f}}:\Gamma_{c\mathfrak{f}}]}\sum_{\substack{p,q\ (\mathrm{mod}\ c\mathfrak{f})\\ q\equiv h\ (\mathrm{mod}\ \mathfrak{f})}} \times \exp\left\{2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(-qd+ck)\right)\right\}\operatorname{Res}_{s=0} F_{p,q,\chi_0}(s)N(w)^{-s}\ , \tag{28}$$

$$R_2 = \frac{1}{2^n|R_{c\mathfrak{f}}|\,[\Gamma_{\mathfrak{f}}:\Gamma_{c\mathfrak{f}}]}\sum_{\substack{\chi_m,\theta\\ \chi=\chi_m\theta\neq\chi_0}}\chi_m(w)\sum_{\substack{p,q\ (\mathrm{mod}\ c\mathfrak{f})\\ q\equiv h\ (\mathrm{mod}\ \mathfrak{f})}} \times \exp\left\{2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(-qd+ck)\right)\right\}\operatorname{Res}_{s=0} F_{p,q,\chi}(s)N(w)^{-s}\ . \tag{29}$$

PROPOSITION 6.3. $R_1 = \varepsilon(u)\varepsilon(v)\cdot\dfrac{Re\,(\mathfrak{f})}{2\sqrt{|d_K|}}\log N(w)$, *where* $\varepsilon(\alpha) = 1$ *or* 0 *according as* α *is or is not in* $\mathfrak{d}^{-1}$.

Proof. Since $F_{p,q,\chi_0}(s)$ has at most a pole of the first order at $s = 0$ (it equals $(2^{-n}/|N(cf\partial)|^{\frac{1}{2}})\xi(s,\chi_0;q,r)\xi(s+1,\chi_0;p,r)$) we see that

$$R_1 = \frac{1}{2^{2n}|R_{c\mathfrak{f}}|\,[\Gamma_{\mathfrak{f}}:\Gamma_{c\mathfrak{f}}]\,|N(cf\partial)|^{\frac{1}{2}}}\sum_{\substack{p,q\ (\mathrm{mod}\ c\mathfrak{f})\\ q\equiv h\ (\mathrm{mod}\ \mathfrak{f})}}\exp\left\{-2\pi i\operatorname{Tr}\left(\frac{p}{cf\partial}(qd-ck)\right)\right\} \times \operatorname{Res}_{s=0}(\xi(s,\chi_0;q,cf)\xi(s+1,\chi_0;p,cf)N(w)^{-s}).$$

However, by Theorem 4.1,

$$\frac{1}{|N(cf)|^{\frac{1}{2}}} \sum_{p \,(\mathrm{mod}\, c\mathfrak{f})} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(-qd + ck)\right)\right\} \times \xi(s+1, \chi_0; p, cf) = \xi(-s, \chi_0; -qd + ck, cf) .$$

Therefore, we have

$$R_1 = \frac{1}{2^{2n} |R_{c\mathfrak{f}}| [\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}] |d_K|^{\frac{1}{2}}} \sum_{\substack{q \,(\mathrm{mod}\, c\mathfrak{f}) \\ q \equiv h \,(\mathrm{mod}\, \mathfrak{f})}} \operatorname{Res}_{s=0} (\xi(s, \chi_0; q, cf) \cdot \xi(-s, \chi_0; -qd + ck, cf) N(w)^{-s}) .$$

Since $\xi(s, \chi_0; m, r)$ is regular at $s = 0$, except when $m \equiv 0 \,(\mathrm{mod}\, r)$, we see that the only terms of the sum which can be non-zero are those for which $q \equiv 0 \,(\mathrm{mod}\, c\mathfrak{f})$ and $q \equiv cka \,(\mathrm{mod}\, c\mathfrak{f})$, since $da \equiv 1 \,(\mathrm{mod}\, c\mathfrak{f})$. However, these two condition can hold for a q counted in the sum if and only if $h \equiv 0 \,(\mathrm{mod}\, \mathfrak{f})$. However, since $u = h/f\partial$, $v = k/f\partial$ and $c \equiv 0 \,(\mathrm{mod}\, \mathfrak{f})$, it follows that in order for there to be a contribution to the above sum, we must have $u \in \mathfrak{d}^{-1}$.

Assume, for the moment, that $v \notin \mathfrak{d}^{-1}$. Let us show that in this case, the contribution to the above sum is 0. In this case, the terms corresponding to $q \equiv 0$ and $q \equiv cka \,(\mathrm{mod}\, c\mathfrak{f})$ are distinct and their sum is

$$\operatorname{Res}_{s=0} \xi(s, \chi_0; 0, cf) \xi(-s, \chi_0; ck, cf) N(w)^{-s} + \operatorname{Res}_{s=0} \xi(s, \chi_0; a^*kc, cf) \xi(-s, \chi_0; 0, cf) N(w)^{-s} . \tag{$*$}$$

But $v \notin \mathfrak{d}^{-1}$ implies that $\xi(-s, \chi_0; ck, cf)$ and $\xi(s, \chi_0; a^*kc, cf)$ are regular at $s = 0$. Moreover, $\xi(s, \chi; 0, cf)$ and $\xi(-s, \chi; 0, cf)$ have simple poles at $s = 0$ with residues $-2^{n-1} Re\,(c\mathfrak{f})$ and $2^{n-1} Re\,(c\mathfrak{f})$, respectively. Thus, the sum $(*)$ equals

$$2^{n-1} Re\,(c\mathfrak{f}) \{\xi(0, \chi_0; akc, cf) - \xi(0, \chi_0; ck, cf)\}$$

which equals 0 since $ad \equiv 1 \,(\mathrm{mod}\, c\mathfrak{f})$, so that $ack \equiv ck \,(\mathrm{mod}\, c\mathfrak{f})$. Thus, we see that we may assume that

$$u \in \mathfrak{d}^{-1} , \qquad v \in \mathfrak{d}^{-1} . \tag{30}$$

Let us assume that (30) holds. Then clearly $akc \equiv 0 \,(\mathrm{mod}\, c\mathfrak{f})$, since $c \equiv 0 \,(\mathrm{mod}\, c)$ and $k \equiv 0 \,(\mathrm{mod}\, \mathfrak{f})$. Moreover $h \equiv 0 \,(\mathrm{mod}\, \mathfrak{f})$. Thus, there is a unique non-zero term in the above sum, corresponding to $q = 0$. Therefore, since $-ck \equiv 0 \,(\mathrm{mod}\, c\mathfrak{f})$,

$$R_1 = \frac{1}{2^{2n}|R_{c\mathfrak{f}}|[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]|d_K|^{\frac{1}{2}}} \operatorname{Res}_{s=0} (\xi(s, \chi_0; 0, cf)\xi(-s, \chi_0; 0, cf)N(w)^{-s}) .$$

However, the Laurent expansion of $\xi(s, \chi_0; 0, cf)$ about $s = 0$ begins

$$\xi(s, \chi_0; 0, cf) = -\frac{2^{n-1} Re(c\mathfrak{f})}{s} + A + D(s) , \qquad A \text{ constant.}$$

Therefore, $\xi(s, \chi_0; 0, cf)\xi(-s, \chi_0; 0, cf) - \dfrac{2^{2n-2}R^2e(c\mathfrak{f})^2}{s^2}$ is a regular function at $s = 0$, and therefore

$$R_1 = \frac{Re(\mathfrak{f})}{2^2\sqrt{|d_K|}} \log N(w) ,$$

hence the Proposition.

Since $F_{p,q,\chi}(s)N(w)^{-s}$ is regular at $s = 0$, except possibly when $\chi_m = \chi_0$, we see that

$$\begin{aligned} R_2 &= \frac{1}{2^n|R_{c\mathfrak{f}}|[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\theta \neq \chi_0} \sum_{\substack{p,q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \exp\left\{2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(-qd + ck)\right)\right. \\ &\qquad \cdot \operatorname{Res}_{s=0} F_{p,q,\theta}(s)N(w)^{-s} \\ &= \frac{1}{|R_{c\mathfrak{f}}|[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]|N(cf\partial)|^{\frac{1}{2}}} \sum_{\theta \neq \chi_0} 2^{a-2n}\theta(cf\partial) \sum_{\substack{p,q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \\ &\qquad \times \exp\left\{2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(-qd + ck)\right)\right\} \\ &\qquad \cdot \operatorname{Res}_{s=0}\left(\frac{N(w)^{-s}}{s^a}\xi(s, \theta; q, cf)\xi(s + 1, \theta; p, cf)\right) \\ &= \frac{1}{|R_{c\mathfrak{f}}|[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]|d_K|^{\frac{1}{2}}} \sum_{\theta \neq \chi_0} 2^{a-2n}i^a \sum_{\substack{q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \\ &\qquad \cdot \operatorname{Res}_{s=0}\left(\frac{N(w)^{-s}}{s^a}\xi(s, \theta; q, cf)\xi(-s, \theta; -qd + ck, cf)\right) , \end{aligned}$$

where we have applied the functional equation of Theorem 4.1 (see proof of the preceding Proposition).

Let us define the function

$$S_\theta(\sigma, s) = S_\theta(\sigma, s | u, v, \mathfrak{f}) = \sum_{\substack{q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \xi(-s, \theta; -qd + kc, cf)\xi(s, \theta; q, cf) ,$$

for $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G(\mathfrak{f})$. Although this function is very complicated look-

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ing, it turns out to be a very natural function, closely related to the theory of Dedekind sums in case $K = \boldsymbol{Q}$. In case $\theta \neq \theta_0$, the trivial character, $S_\theta(\sigma, s)$ is an entire function of s, so that

$$R_2 = \frac{1}{|R_{\mathfrak{cf}}|\,[\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]\,|d_K|^{\frac{1}{2}}} \sum_{\theta \neq \theta_0} 2^{a-2n} i^a \operatorname{Res}_{s=0} \frac{N(w)^{-s}}{s^a} S_\theta(\sigma, s)$$

$$= \frac{1}{|R_{\mathfrak{cf}}|\,[\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]\,|d_K|^{\frac{1}{2}}} \sum_{\theta \neq \theta_0} \frac{2^{a-2n} i^a}{(a-1)!} \frac{d^{a-1}}{ds^{a-1}} (N(w)^{-s} S_\theta(\sigma, s))_{s=0}$$

$$= \frac{1}{|R_{\mathfrak{cf}}|\,[\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]\,|d_K|^{\frac{1}{2}}} \sum_{\theta \neq \theta_0} \frac{2^{a-2n} i^a}{(a-1)!} \sum_{j=0}^{a-1} \binom{a-1}{j} (-\log N(w))^{a-1-j} \frac{d^j}{ds^j} S_\theta(\sigma, s)|_{s=0} .$$

Thus, finally we have

PROPOSITION 6.4. *For each signature character θ and each j ($0 \leq j \leq a-1$, $a = a(\theta)$), define $A_j(\theta)$ by*

$$A_j(\theta) = \frac{i^a 2^{a-2n} (-1)^{a-1-j}}{|R_{\mathfrak{cf}}|\,[\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]\,|d_K|^{\frac{1}{2}}} \frac{1}{(a-1)!} \binom{a-1}{j} \frac{d^j}{ds^j} S_\theta(\sigma, s)|_{s=0} .$$

Then

$$\operatorname{Res}_{s=0} H(s) N(w)^{-s} = \frac{\varepsilon(u)\varepsilon(v)\, Re\,(\mathfrak{f})}{2\sqrt{|d_K|}} \log N(w) + \sum_{\theta \neq \theta_0} \sum_{j=0}^{a-1} A_j(\theta) (\log N(w))^{a-1-j} .$$

In particular, the terms in the above sum which do not depend on w are just

$$S(\sigma) = \sum_{\theta \neq \theta_0} A_{a-1}(\theta) .$$

(This last sum is the generalization of the classical Dedekind sum to our setting.)

Propositions 6.1 to 6.4 leave us only the task of computing the residues at the points $s_j(\chi)$ $(1 \leq j \leq n)$. This can be done generally. However, for the sake of simplicity in calculations let us make the following assumption:

ASSUMPTION. Let $m \neq (0, \cdots, 0)$. Then the complex numbers $s_1(\chi_m), \cdots, s_n(\chi_m)$ are all distinct.

This assumption will be automatic in case $n = 1, 2, 3$. In fact, we conjecture that this assumption always holds.* Roughly, we suspect this to be true because the numbers $s_1(\chi_m), \cdots, s_n(\chi_m)$ are essentially elements in one row of a matrix which is the inverse of a regulator matrix. Therefore, one would suspect $n-1$ of them to be algebraically independent. However, this appears to be a difficult problem in transcendental number theory. Throughout the rest of this paper, the above assumption will be in effect.

By using reasoning similar to that used above, we can easily see that for $\chi = \chi_m \neq \chi_0$,

$$\begin{aligned} \operatorname{Res}_{s=s_j(\chi)} H(s)N(w)^{-s} &= \frac{1}{2^n |R_{\mathfrak{cf}}| [\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]} \sum_{\theta} \chi_m(w) \sum_{\substack{p,q \ (\mathrm{mod}\ \mathfrak{cf}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \\ &\times \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd - ck)\right)\right\} \operatorname{Res}_{s=s_j(\chi)} F_{p,q,\chi\theta}(s) N(w)^{-s} . \end{aligned} \tag{31}$$

Therefore, by using the definition of $F_{p,q,\chi\theta}(s)$ and the functional equation of Theorem 4.1, we have for $\chi = \chi_m$, $m \neq (0, \cdots, 0)$,

$$\begin{aligned} &\operatorname{Res}_{s=s_i(\chi)} H(s)N(w)^{-s} \\ &= \frac{\chi(cf\partial)^{-1}}{|R_{\mathfrak{cf}}| [\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}] |N(cf\partial)|^{\frac{1}{2}}} \sum_{\theta} 2^{a-2n} \theta(cf\partial)^{-1} \chi_m(w) \sum_{\substack{p,q \ (\mathrm{mod}\ \mathfrak{cf}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \\ &\quad \cdot \exp\left\{-2\pi i \operatorname{Tr}\left(\frac{p}{cf\partial}(qd - ck)\right)\right\} \operatorname{Res}_{s=s_j(\chi)} \left(\frac{N(w)^{-s}}{\prod_{k=1}^{n} (s - s_k(\chi))^{a_k}}\right. \\ &\quad \left. \cdot\, \xi(s, \chi\theta; q, cf)\xi(s+1, \chi\theta; p, cf)\right) \\ &= \frac{1}{|d_K|^{\frac{1}{2}} |R_{\mathfrak{cf}}| [\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]} \chi_m(w) \sum_{\theta} i^a 2^{a-2n} \sum_{\substack{q \ (\mathrm{mod}\ \mathfrak{cf}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \operatorname{Res}_{s=s_j(\chi)} \\ &\quad \cdot \left(\frac{N(w)^{-s}}{\prod_{k=1}^{n} (s - s_k(\chi))^{a_k}} \xi(s, \chi\theta; q, cf)\xi(-s, \chi^{-1}\theta^{-1}, -qd + ck; cf)\right) \\ &= \frac{1}{|d_K|^{\frac{1}{2}} |R_{\mathfrak{cf}}| [\Gamma_{\mathfrak{f}} : \Gamma_{\mathfrak{cf}}]} \chi_m(w) \sum_{\theta} i^a 2^{a-2n} \operatorname{Res}_{s=s_j(\chi)} \left(\frac{N(w)^{-s}}{\prod_{k=1}^{n} (s - s_j(\chi))^{a_k}}\right. \\ &\quad \left. \cdot\, S_{\chi\theta}(\sigma, s)\right) , \end{aligned}$$

where, consistent with our previous notation, we have set

$$S_{\chi\theta}(\sigma, s) = \sum_{\substack{q \ (\mathrm{mod}\ \mathfrak{cf}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \xi(s, \chi\theta; q, cf)\xi(-s, \chi^{-1}\theta^{-1}; -qd + ck, cf) .$$

* This conjecture is actually due to Hecke [4, p. 228–229].

Since $\chi = \chi_m$, $m \neq (0, \cdots, 0)$, we see that $S_{\chi\theta}(\sigma, s)$ is an entire function of s. Moreover, our hypothesis above guarantees that the pole of the function

$$\frac{N(w)^{-s}}{\prod_{k=1}^{n} (s - s_k(\chi))^{a_k}} S_{\chi\theta}(\sigma, s)$$

at $s = s_j(\chi)$ is simple with residue

$$N(w)^{-s_j(\chi)} S_{\chi\theta}(\sigma, s_j(\chi)) \cdot \frac{a_j}{\prod_{\substack{k=1 \\ k \neq j}}^{n} (s_j(\chi) - s_k(\chi))^{a_k}} .$$

Therefore, we finally have

PROPOSITION 6.5. *For* $1 \leq j \leq n$, *set*

$$\hat{w}_j = \left(w_1, \cdots, w_{j-1}, \frac{w_j}{N(w)}, w_{j+1}, \cdots, w_n\right) .$$

Then for $\chi = \chi_m$, $m \neq (0, \cdots, 0)$, *we have*

$$\mathrm{Res}_{s=s_j(\chi)} H(s) N(w)^{-s} = B_j(\chi) \chi(\hat{w}_j) ,$$

where

$$B_j(\chi) = \frac{2e(c\mathfrak{f})^{-1}}{|d_K|^{\frac{1}{2}} R[\Gamma_{\mathfrak{f}} : \Gamma_{c\mathfrak{f}}]} \sum_{\theta} 2^{a-2n} i^{a} \frac{a_j}{\prod_{\substack{k=1 \\ k \neq j}}^{n} (s_j(\chi) - s_k(\chi))^{a_k}} S_{\chi\theta}(\sigma, s_j(\chi)) .$$

In order to state our final transformation formula for $\log \eta(z; u, v)$, let us introduce some constants suggested by Proposition 6.4. For $1 \leq j \leq n$, set

$$C_j = \begin{cases} \dfrac{\varepsilon(u)\varepsilon(v)\, Re\,(\mathfrak{f})}{2\sqrt{|d_K|}} + \displaystyle\sum_{\substack{\theta \neq \theta_0 \\ a=a(\theta) \geq 2}} A_{a-2}(\theta) & (j = 1) \\ \displaystyle\sum_{\substack{\theta \neq \theta_0 \\ a=a(\theta) \geq j+1}} A_{a-1-j}(\theta) & (j > 1) . \end{cases}$$

Furthermore, let us set $\widehat{\sigma(z)}_j = \widehat{w}_j$. Then, using this notation, Proposition 6.1 to 6.5, together with Theorem 5.1, we can at last state the main result of this paper:

MAIN THEOREM. *Let* $\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(\mathfrak{f})$, $z \in \boldsymbol{H}^n$. *Then*

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$$\log \eta\left(\frac{az+b}{cz+d}; u, v\right) = \log \eta(z; u, v) + \Lambda(u)\left\{N\left(\frac{az+b}{cz+d}\right) - N(z)\right\}$$
$$- \frac{(-1)^n}{N(c)}\Lambda(u)\left\{\frac{1}{N(cz+d)} - N(cz+d)\right\}$$
$$- \sum_{j=1}^{n} C_j \left(\log N\left(sgn(c)\frac{cz+d}{i}\right)\right)^j$$
$$- \sum_{m \in \mathbf{Z}^{n-1}} \sum_{j=1}^{n} B_j(\chi_m)\chi_m(\widehat{\sigma(z)_j})$$
$$- S(\sigma) .$$

7. Examples

Let us give a few illustrations of special cases of our main theorem.

EXAMPLE 1. $K = \mathbf{Q}$.

Here $n = 1$, $R = 1$, $\partial = 1$, $d_K = 1$. Without loss of generality, let us assume that $f > 0$, $c > 0$. All the assumptions made in our discussion are valid in this case since all $\mathbf{Q}$-ideals are principal and since there are no non-trivial grossencharacters of conductor 1. There is only one non-trivial signature character, namely

$$\theta_1(x) = \left(\frac{x}{|x|}\right) \qquad (x \in \mathbf{Q}^\times) .$$

In this case,

$$\log \eta(z; u, v) = \Lambda(u)z + \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{2\pi i m v}}{|m|^2} \sum_{\substack{\ell=-\infty \\ (\ell+u)m>0}}^{\infty} e^{2\pi i(\ell+u)mz} ,$$

where

$$\Lambda(u) = \frac{1}{2\pi i} \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} \frac{e^{2\pi i u m}}{|m|}$$
$$= -2\pi i B_2(u) ,$$

where $B_2(u)$ denotes the second Bernoulli polynomial. In particular,

$$\Lambda(0) = -\frac{\pi i}{6} .$$

Also, for $u = v = 0$, $\log \eta(z; u, v) = -2 \log \eta(z)$, where $\eta(z)$ is the classical Dedekind η-function.

A simple computation shows that

$$C_1 = \varepsilon(u)\varepsilon(v) .$$

Moreover

$$S(\sigma) = A_0(\theta_1) = \frac{\pi i}{2} \sum_{\substack{q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \zeta(0, \theta_1 ; -qd + ck, cf)\zeta(0, \theta_1 ; q, cf)$$

However, we showed [2, p. 297] that

$$\zeta(0, \theta_1 ; t, r) = -2\left(\left(\frac{t}{r}\right)\right) .$$

Therefore,

$$\begin{aligned} S(\sigma) &= 2\pi i \sum_{\substack{q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \left(\left(\frac{-qd + ck}{cf}\right)\right)\left(\left(\frac{q}{cf}\right)\right) \\ &= 2\pi i \sum_{s \ (\mathrm{mod}\ c)} \left(\left(\frac{-sd - ud}{c} + v\right)\right)\left(\left(\frac{s + u}{c}\right)\right) \\ &= -2\pi i \sum_{s \ (\mathrm{mod}\ c)} \left(\left(\frac{sd + ud}{c} - v\right)\right)\left(\left(\frac{s + u}{c}\right)\right) . \end{aligned}$$

In particular, if $u = v = 0$,

$$S(\sigma) = -2\pi i s(d, c) ,$$

and our main theorem is just the law of transformation (1). For general u, v, our main theorem yields

$$\log \eta\left(\frac{az + b}{cz + d} ; u, v\right) = \log \eta(z ; u, v) - \varepsilon(u)\varepsilon(v) \log\left(\frac{cz + d}{i}\right) + \Lambda(u)\frac{a + d}{c} + 2\pi i S(u, v, \sigma)$$

where

$$S(u, v, \sigma) = \sum_{s \ (\mathrm{mod}\ c)} \left(\left(\frac{sd + ud}{c} - v\right)\right)\left(\left(\frac{s + u}{c}\right)\right)$$

is a generalized Dedekind sum. This latter formula is due to Siegel [6, p. 179] and Meyer [5, p. 102].

EXAMPLE 2. Let $K = \boldsymbol{Q}(\sqrt{d_K})$ be a real quadratic field of discriminant d_K and fundamental unit ε_K. Then, $\mathfrak{d}$ is a principal ideal and further, assume that $\mathfrak{f}$ is an arbitrary principal integral ideal. Then all

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assumptions made in the paper are valid. Let ε be a generator for the group of K-units $\equiv 1 \pmod{\mathfrak{f}}$. Note that

$$\begin{pmatrix} 1 & \log \varepsilon^{(1)} \\ 1 & \log \varepsilon^{(2)} \end{pmatrix}^{-1} = \begin{pmatrix} \dfrac{1}{2} & \dfrac{1}{2} \\ \dfrac{1}{2\log\varepsilon} & -\dfrac{1}{2\log\varepsilon} \end{pmatrix},$$

so that for $m \in \boldsymbol{Z}$,

$$2\pi i\alpha_1(m) = \frac{\pi i m}{\log\varepsilon}\,, \qquad 2\pi i\alpha_2(m) = -\frac{\pi i m}{\log\varepsilon}\,,$$

$$\chi_m(y_1, y_2) = y_1^{\pi i m/\log\varepsilon} y_2^{-\pi i m/\log\varepsilon}\,.$$

Notice that for $m \neq 0$, $\alpha_1(m) \neq \alpha_2(m)$, so that the assumption of Section 5 is valid.

In this example, there are four signature characters θ, namely

$$\theta = \theta_0, \theta_1, \theta_1', \theta_2\,,$$

where

$$\theta_0 \equiv 1, \quad \theta_2(x) = \operatorname{sgn}(N(x)) \qquad (x \in K^\times)\,,$$
$$\theta_1(x) = \operatorname{sgn}(x), \quad \theta_1'(x) = \operatorname{sgn}(x') \qquad (x \in K^\times)\,,$$

where $x \mapsto x'$ denotes the non-trivial conjugation map of $K/\boldsymbol{Q}$. Immediate computations show that

$$C_1 = \frac{\varepsilon(\mathfrak{u})\varepsilon(\mathfrak{v})e(\mathfrak{f})\log\varepsilon_K}{2\sqrt{d_K}} + \frac{e(\mathfrak{f})}{2e(c\mathfrak{f})^2\log\varepsilon_K\sqrt{d_K}} S_{\theta_2}(\sigma, 0)\,,$$

$$S(\sigma) = i\frac{e(\mathfrak{f})}{4e(c\mathfrak{f})^2|d_K|^{\frac{1}{2}}\log\varepsilon_K}\left[S_{\theta_1}(\sigma,0) + S_{\theta_1'}(\sigma,0) + 2i\frac{d}{ds}S_{\theta_2}(\sigma,s)|_{s=0}\right].$$

Moreover, for $\theta \neq \theta_0$,

$$S_\theta(\sigma, 0) = \sum_{\substack{q \ (\mathrm{mod}\ c\mathfrak{f}) \\ q \equiv h \ (\mathrm{mod}\ \mathfrak{f})}} \xi(0, \theta; -qd + kc, cf)\xi(0, \theta; q, cf)$$

By applying the functional equation of Theorem 4.1, and letting $s \to 0$ from the left, we see that

$$\xi(0, \theta; p, r) = \frac{(-i)^a\theta(r\partial)\sqrt{d_K}}{\pi^{a/2}} \lim_{s\to 0} \sum_{\substack{\beta \in \mathfrak{o}_K \\ \beta \neq 0 \\ \{\Gamma_r\}}}{}^{*} \frac{\theta(\beta)e^{2\pi i \operatorname{Tr}(\beta p/r\partial)}}{|N(\beta)|^{1+s}}\,,$$

Thus, we see that $\xi(0, \theta; p, r)$ is a generalization of the function (( )), that is, essentially, the first Bernoulli polynomial, to a real quadratic field. Moreover, our above formula for $S(\sigma)$ justifies calling $S(\sigma)$ a Dedekind sum, at least apart from the anomalous term involving

$$\frac{d}{ds} S_{\theta_2}(\sigma, s) \, ,$$

for which no conceptual explanation is available at this time. Fortunately, this term does not contribute to the class number formulas in this case.

Our transformation formula in this example reads

$$\begin{aligned}\log \eta\left(\frac{az+b}{cz+d}; u, v\right) &= \log \eta(z; u, v) + \Lambda(u)\left\{N\left(\frac{az+b}{cz+d}\right) - N(z)\right\} \\ &\quad - \frac{1}{N(c)}\Lambda(u)\left\{\frac{1}{N(cz+d)} - N(cz+d)\right\} \\ &\quad - C_1 \log N\left(sgn(c)\frac{cz+d}{i}\right) \\ &\quad - \sum_{m=-\infty}^{\infty}{}' B_1(\chi_m)\left(sgn(c')\frac{c'z_2+d'}{i}\right)^{-2\pi i m/\log \varepsilon} \\ &\quad - \sum_{m=-\infty}^{\infty}{}' B_2(\chi_m)\left(sgc(c)\frac{cz_1+d}{i}\right)^{+2\pi i m/\log \varepsilon} - S(\sigma) \, ,\end{aligned}$$

where $z = (z_1, z_2)$. The special case of this formula in case $\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ is due to Hecke [3, p. 403].

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