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ADAPTIVE ELASTICITY*

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Certain natural solid materials adjust to their ambient environmentally applied mechanical loads by slowly changing their overall shape and their local density or microstructure. These materials include living bone, living wood and certain saturated porous geological materials. The theory of adaptive elasticity attempts to model these complex stress adaptation processes with a simple continuum model. The model is composed of a porous anisotropic linear elastic solid perfused with, and surrounded by, a fluid. The chemical reactions of the stress adaptation process are modeled by the transfer of mass from the fluid to the porous solid matrix, and *vice versa*. As a result of the chemical reactions mass is transferred to (from) the solid matrix so that it either increases (decreases) the overall size of the body or increases (decreases) the density of the body. The rates of these chemical reactions are very slow compared to the characteristic time of inertia effects. The rate of change of the overall size and shape of the body is controlled by surface strain, and the rate of change of density at a point is controlled by the local matrix strain.

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The details of this model, as well as its physical motivation, are described and applied to several problems of interest. The problems described include the devolution of initial density inhomogeneities and the shape changes in a hollow circular cylinder due to changes in the axial load. The application of these results to living animals and trees is discussed.

1. Introduction

There are three major portions of this paper. In the first portion the observations and experiments of stress adaptation of overall shape and of local microstructure in living bones, living trees and certain saturated porous geological materials are reviewed. This material is presented in the first three sections following the introduction. The second portion of the paper describes the development of a continuum model for the stress adaptation process. This material is combined in sections five, six and seven. The third and final portion of the paper concerns applications of the theory to some elementary problems in bone mechanics. These applications are described in the latter part of section six and in section eight.

2. Stress adaptation in bone

Functional adaptation is the term used to describe the ability of organisms to increase their capacity to accomplish their function with an increased demand and to decrease their capacity with lesser demand. Living bone is continually undergoing processes of growth, reinforcement and resorption which are collectively termed "remodeling". The remodeling processes in living bone are the mechanisms by which the bone adapts its overall structure to changes in its load inironment. The time scale of the remodeling processes is on the order of months or years. Changes in life style which change the loading environment, for example taking up jogging, have remodeling times on the order of many months. Bone remodeling associated with trauma has a shorter remodeling time, on the order of weeks in humans. The time scales of these remodeling processes should be distinguished from developments in bone due to growth, which have a time scale on the order of decades in humans, and the developments due to natural selection which have a time scale of many lifetimes.

It is necessary to describe the nature of bone as a material before

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describing the remodeling processes that occur in bone. Experiments have shown that bone can be modeled as an inhomogeneous transversely isotropic or orthotropic elastic material, with the degree of anisotropy varying inhomogeneously also.

There are two major classes of bone tissue which significantly contribute to the structural strength of the skeletal system. They are called cancellous and cortical bone. Cortical bone is the hard tissue on the outer surface or cortex of the femur (that is, thigh bone). It is dense, it contains no marrow and its blood vessels are microscopically small. Cancellous bone occurs in the interior of the femur. It consists of a network of hard, interconnected filaments called "trabeculae" interspersed with marrow and a large number of small blood vessels. Cancellous bone is also called trabecular bone or spongy bone. Generally cancellous bone is structurally predominant in the neighborhood of the joints and cortical bone is structurally predominant in the central sections of a femur away from the joints. Bone tissue contains an abundance of extracellular material or matrix. The volume fraction of the matrix is orders of magnitude larger than the volume fraction of bone cells. The matrix accounts for virtually all the structural strength of bone.

The concept of stress or strain induced bone remodeling was first publicized by the German anatomist Wolff [*32*], and is often called Wolff's law. Bassett and Becker [*2*], Shamos, Lavine and Shamos [*25*], Justus and Luft [*17*], and Somjen, Binderman, Berger and Harell [*27*], have proposed various mechanisms for bone remodeling in terms of certain electrical and chemical properties of bone. The distinction made by Frost [*12*] between surface and internal remodeling is followed here. *Surface* remodeling refers to the resorption or deposition of the bone material on the external surface of the bone. The details of the process of deposition of new lamina at the surface of a bone are described by Currey [*9*]. *Internal* remodeling refers to the resorption or reinforcement of the bone tissue internally by changing the bulk density of the tissue. The study of Kazarian and Von Gierke [*18*] very graphically illustrated internal remodeling in cancellous bone. In this study 16 male Rhesus monkeys were immobilized in full body casts for a period of sixty days.* Another set of

* The Editor records his horror that such experiments are performed in the name of science.

16 male Rhesus monkeys were used as control and allowed the freedom of movement possible in a cage. A subsequent comparison of the bone tissue of the immobilized monkeys with the tissue of the control monkeys showed considerable resorption of the bone tissue of the immobilized monkeys. Qualitative radiographic techniques demonstrated increased bone resorption in the metaphysis of the axially-loaded long bones, as well as the loss of cortical bone. Mechanical testing of the bone tissue also reflected the remodeling loss in the immobilized monkeys.

The effect of an increased loading environment on the remodeling of bone tissue is illustrated in a Latvian study reported by Shumskii, Merten and Dzenis [26]. In this study the acoustic velocity in tibia of nine groups of individuals was determined ultrasonically. The nine groups were highly trained athletes (masters of sports, candidates for master of sports, and first degree athletes), swimmers, biathlon athletes, middle distance runners, high jumpers, hurdlers, track and field athletes, second and third degree middle distance runners, and some non-athletic individuals. The data from this study is presented in Table I. It is easy to see that the acoustic velocity in the tibia increases with the group's athletic expertise. Acoustic velocity is proportional to the square root of the Young's modulus and inversely proportional to the square root of the bulk density; thus there is an implication of greater modulus which implies bone deposition and increased density of the bone tissue with increasing athletic expertise.

Surface remodeling can be induced in the leg bones of animals by superposing axial and/or bending loads. Woo *et al* [33] has shown that increased physical activity (jogging) in pigs can cause the periosteal surface of the leg bone to move out and the endosteal surface to move in. Meade *et al* [22] superposed a constant compressive force along the axis of the canine femur by an implanted spring system. This study showed a quantifiable increase in cross section area with increasing magnitude of the superposed compressive force. Lišková and Heřt [20] have shown that intermittent bending applied to the rabbit tibia can cause the periosteal surface to move out. Surface remodeling can also be induced in the leg bones of animals by reducing the loads on the limb. In two studies Uhthoff and Jaworski [31] and Jaworski *et al* [16] immobilized one of the forelimbs of beagles. In the study of Uhthoff and Jaworski [31], young beagles were

TABLE I

The acoustic velocity in the human tibia. This data is from Shumskii, Merten and Dzenis [*26*].

Group No.	Group Characteristics	Acoustic Velocity in m/s	
		Right Leg	Left Leg
1	Individuals not participating in sports	1257	1270
2	Third-degree middle-distance runners	1315	-
3	Second-degree middle-distance runners	1430	-
4	First-degree middle-distance runners	1656	1710
5	Swimmers, 2 masters of sports, 2 candidates for master of sports, and 6 first-degree athletes	1346	1365
6	Biathlon athletes, candidates for master of sports	1502	1490
7	High jumpers, 4 candidates for master of sports, 6 first-degree athletes	1775	1860
8	First-degree hurdlers	1702	1576
9	First-degree track-and-field athletes	1876	1820

used and it was found that the endosteal surface showed little movement while there was much resorption on the periosteal surface. However, in the study with older beagles (Jaworski *et al* [*16*]), it was observed that the periosteal surface showed little movement while on the endosteal surface there was much resorption.

The feedback mechanism by which the bone tissue senses the changes in load environment and initiates the deposition or resorption of bone tissue is not well understood. The two candidates are a piezo-electric effect that occurs in bone and the calcium ion concentration. These mechanisms are described in further detail by Cowin and Hegedus [5]. A fairly comprehensive survey of the electromechanical properties of bone was accomplished recently by Guzelsu and Demiray [*14*].

3. Stress adaptation in trees

Functional adaptation also occurs in trees and plants. Trees adapt their shape and structure to their environmental loading in a manner qualitatively similar to that observed in bones. The exact mechanism of stress adaptation in trees is unknown, but a combination of mechanical stress and the hormone auxin have been suggested by a number of studies [35], [36].

It has been observed that threes growing in dense forest strands have smaller trunk diameter than trees growing at the edge of the strand and are more inclined to be blown over than those at the edge. It has also been observed that nursery trees grown close together in containers are tall and spindly while those placed further apart are greater in trunk diameter. An interesting experiment which quantified this phenomenon was reported by Neel and Harris [23], [1]. These environmental horticulturists obtained eight matching pairs of young sweet-gum trees (Liquidambar). The trees were placed in four gallon cans in a greenhouse. Each morning at 8.30 for 27 mornings, one tree in each pair was shaken for 30 seconds. At the end of the 27 day period the shaken trees had reached a height which was only 20% of the height of the unshaken trees. However, the trunks of the shaken trees were larger than those of the unshaken trees. At a distance of 5 cm from the ground the diameter of the shaken trees had increased by 8.3 mm while those of the unshaken had increased by only 6.8 mm. The wood fiber length and the vessel member length were significantly shorter in the shaken trees. Although the authors did not measure the elastic moduli, the changes in the wood tissue microstructure suggests that the elastic moduli are different in the shaken and unshaken trees. One would reasonably expect the moduli to be higher in the unshaken trees.

The wood tissue that is deposited on an external surface of a tree trunk in response to a superposed bending moment in a particular direction is called reaction wood. Reaction wood is visible when a transverse section of the tree trunk is viewed because it distorts the growth rings. In a tree that has been bent in a particular direction the growth rings will not be circular, but they will be distorted ovals in the particular direction associated with the bending. The additional wood tissue deposited will increase the area moment of inertia of the trunk cross section and decrease the stress experienced by the wood tissue.

The elastic properties of wood are best modeled by orthotropic linear elasticity theory. The Young's modulus in the direction of the wood grain is the largest Young's modulus. The Young's moduli are the largest at the base of the tree and they decrease in magnitude for wood located at a distance up the tree away from the ground, generally increasing with limb diameter.

4. Stress adaptation in certain porous solid geological materials

The effect known as "pressure solution" in the geophysical literature suggests that functional adaptation to stress is not restricted to organic materials. Pressure solution is described as the increasing solubility of a saturated porous solid matrix with increasing solid matrix strain. Three recent reports by Sprunt and Nur [*28*, *29*, *30*] attempt to quantify this effect. The materials studied were all porous and saturated and included sandstone, limestone, dolomitic limestone, marble, quartzite and novaculite. These materials all adapted their microstructure by surface resorption or by changing their porosity. These microstructural changes were found to be proportional to the strain in the solid matrix and not the effect of a transport mechanism.

Although Sprunt and Nur report on experimental situations in which there was resorption of the solid matrix material, they indicate that they believe that this result is due to the fact that they employed an open system. They state "If our system were closed, we would expect solution only in the regions of large compressive stress and deposition in the regions of tension or small compressive stress. Examples of pressure solution in both open and closed systems are well known in nature. Large net reductions in rock mass, where material is dissolved along stylolites and removed from the rock formation (for example, [*13*], [*21*]). Pressure shadows which form by solution of material at points of high stress around a stiff inclusion accompanied by transfer and recrystallization of the same material at points of lower stress around the same inclusion [*3*], [*10*], are examples of natural closed systems". The 1935 experiments of Russell [*24*] support the deposition aspect of the pressure solution effect. Russell used a smooth-surfaced crystal of ammonium alum in a saturated alum solution at constant temperature to demonstrate that material dissolved from one part of a crystal because of local stress may be redeposited on

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another part of the same crystal where the strain is relatively less.

The sandstone experiments reported by Sprunt and Nur [28, 29] employed a different geometry from their experiments on the other geological materials reported in [30]. The sandstone experiments were performed on a hollow circular cylinder where the matrix stress was induced by mechanical pressure applied to the external cylindrical surface. The experiments on the other materials were performed on specimens in the shape of rectangular parallelopipeds with a cylindrical hole. The matrix stress was induced by a compressive load in one direction. The experimental observations reported in the sandstone experiments were porosity changes while the observations reported in the experiments with other materials were surface resorption. It would then appear that there is a difference between surface and internal pressure solution effects. Most likely the difference in these effects is not in the basic mechanics of the reaction, but in the rate at which the reaction occurs. One would suspect that the reaction can proceed at a more rapid rate on the free surface because of the greater mobility of the solvent.

5. Modeling of the stress adaptation process

At this early stage of development, the stress adaptation processes in bone, trees and saturated porous solid geological materials can be described by the same model. A description of this model is presented in this section. To make the presentation of the model easier, we develop it in two parts and then combine the parts. Thus, in the following section a theory of surface remodeling is presented and, in the section following, a theory of internal remodeling is presented.

The theories of surface and internal remodeling use a simple two constituent model. The solid matrix is modeled as a porous anisotropic linear elastic solid. The basic model is then a porous, anisotropic linear elastic solid perfused with a fluid. In the model of the stress adaptation process chemical reactions convert the fluid into the porous solid matrix and *vice versa*. As a result of the chemical reactions mass, momentum, energy and entropy are transferred to or from the porous solid matrix. The rates of these chemical reactions depend on matrix strain and are very slow compared to the characteristic time of inertia effects. Thus, inertia effects are neglected and the stress in the matrix considered here is the

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actual stress averaged over a time period greater than any inertia effects.

At this point the discussion naturally bifurcates into the consideration of the two parts which constitute the complete model, namely the surface remodeling theory and the internal remodeling theory. The distinction between the two theories is made upon the locations at which the chemical reactions occur and the way in which mass is added or removed from a material body. In the theory of surface remodeling the chemical reactions occur only on the external surfaces of the body and mass is added or removed from the body by changing the external shape of the body. During surface remodeling the interior of the body remains at constant bulk density. In the theory of internal remodeling the chemical reactions occur everywhere within the porous solid matrix of the body and mass is added by changing the bulk density of the matrix and without changing the exterior dimensions of the body. In both cases the rate and direction of the chemical reaction at a point are determined by the strain at the point. It is important to note that these two theories are neither contradictory nor incompatible, but combine easily for a single body in which there are both overall shape changes and density changes.

The theory of surface remodeling acknowledges the observed fact that external changes in body shape are induced by changes in the loading environment of the body. This theory postulates a causal relationship between the rate of surface deposition or resorption and the strain in the surface of the body. The body is considered to be an open system with regard to mass transport and the mass of the body will vary as the external shape of the body varies. This theory is described by Cowin and Van Buskirk [8] and Cowin and Firoozbakhsh [4]. The theory of internal remodeling postulates a causal relationship between the rate of deposition or resorption of the solid matrix at any point and the strain at that point in the solid matrix. A schematic diagram of this model is shown in Figure 1 (p. 66). The fact that living bone and living wood tissue are encased in a living organism and that the geological materials are saturated, is reflected in the model by setting the elastic porous solid in a bath of the perfusant. The mechanical load is applied directly to the porous structure across the walls of the perfusant bath as illustrated. The system consisting of the elastic porous solid and its perfusant bath is considered to be closed with respect to mass, heat energy, and entropy transfer, but

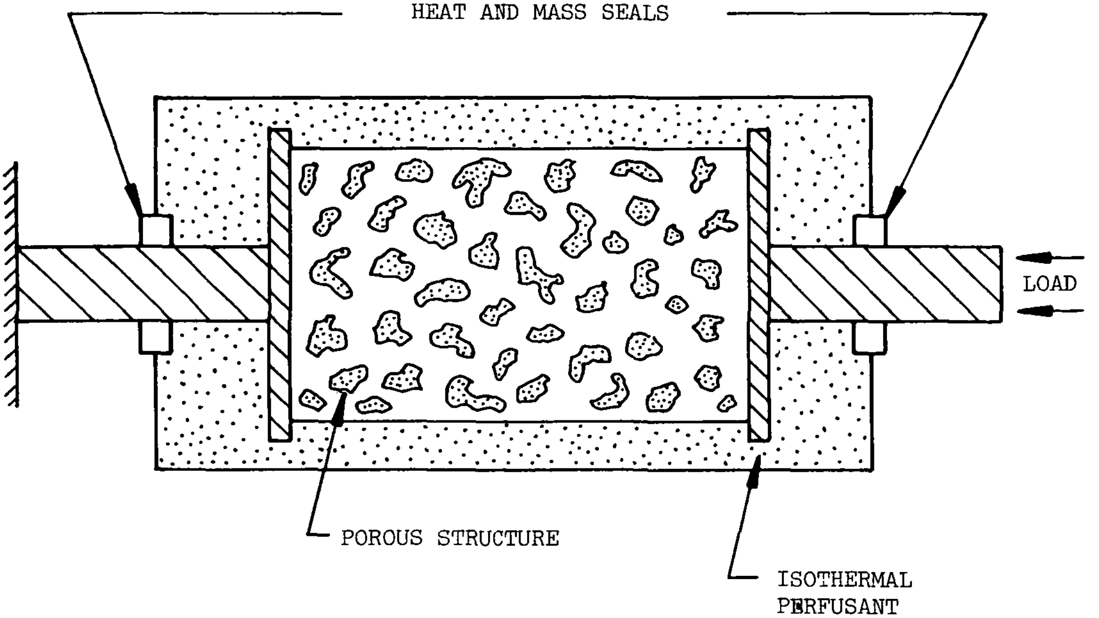

FIGURE 1. A schematic model of the remodeling mechanism postulated.

open with respect to momentum transfer from loading. The system consisting of only the elastic porous solid without its entrained perfusant is open with respect to momentum transfer as well as mass, energy, and entropy transfer. The solid matrix is taken as the control system since the changes in the mechanical properties of the solid matrix alone determine the changes in the mechanical properties of the whole body.

The theory of internal remodeling is developed in a series of papers: Cowin and Hegedus [5], Hegedus and Cowin [15], Cowin and Nachlinger [6], and Cowin and Van Buskirk [7].

6. The theory of surface remodeling

The model for surface remodeling employed assumes that solid matrix can be modeled as a linear elastic body whose free surfaces move according to an additional specific constitutive relation. The additional constitutive relation for the movement of the free surface is the result of a postulate that the rate of surface deposition or resorption is proportional to the change in the strain in the surface from a reference

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value of strain. At the reference value of strain there is no movement of the surface. In order to express the constitutive equation for the surface movement in equation form some notation is introduced. Let Q denote a surface point on the body and let $\mathbf{n}$ denote an outward unit normal vector of the tangent plane to the surface of the body at Q. Let U denote the speed of the remodeling surface normal to the surface, that is to say U is the velocity of the surface in the $\mathbf{n}$ direction. The velocity of the surface in any direction in the tangent plane is zero because the surface is not moving tangentially with respect to the body. Let $E_{ij}(Q)$ denote the cartesian components of the strain tensor at Q. Small strains are assumed. The hypothesis for surface remodeling is that the speed of the remodeling surface is linearly proportional to the strain tensor,

$$U = C_{ij}(\mathbf{n}, Q)\left[E_{ij}(Q)-E^{0}_{ij}(Q)\right] , \tag{1}$$

where $E^{0}_{ij}(Q)$ is a reference value of strain where no remodeling occurs and $C_{ij}(\mathbf{n}, Q)$ are surface remodeling rate coefficients which are, in general, dependent upon the point Q and the normal $\mathbf{n}$ to the surface at Q. The surface remodeling rate coefficients and the reference values of strain are phenomenological coefficients of the body surface and must be determined by experiment. It is assumed here that the surface remodeling rate coefficients C_{ij} are not site specific, that is to say they are independent of the position of the surface point Q. It is also postulated above that surface remodeling rate coefficients are independent of strain. Equation (1) gives the normal velocity of the surface at the point Q as a function of the existing strain state at Q. If the strain state at Q, $E_{ij}(Q)$, is equal to the reference strain state $E^{0}_{ij}(Q)$, then the velocity of the surface is zero and no remodeling occurs. If the right hand side of (1) is positive, the surface will be growing by deposition of material. If, on the other hand, the right hand side of (1) is negative, the surface will be resorbing. Equation (1) by itself does not constitute the complete theory. The theory is completed by assuming that the body is composed of a linearly elastic material. Thus, the complete theory is a modification of linear elasticity in which the external surfaces of the body move according to the rule prescribed by

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equation (1). A boundary value problem will be formulated in the same manner as a boundary value problem in linear elastostatics, but it will be necessary to specify the boundary conditions for a specific time period. As the body evolves to a new shape, the stress and strain states will be varying quasi-statically. At any instant the body will behave exactly as an elastic body, but moving boundaries will cause local stress and strain to redistribute themselves slowly with time.

This theory has been applied to the problem of a hollow circular cylinder subjected to an axial load by Cowin and Firoozbakhsh [4]. The results suggest that the stable response of the cylinder to an increased compressive axial load is to increase its cross sectional area by movement of the external surface of the cylinder outward and the internal surface inward. On the other hand, the stable response of the cylinder to a decreased compressive axial load is to decrease its cross sectional area by movement of the external surface of the cylinder inward and the internal surface outward. When the theory is applied to the bending of a rod, it predicts deposition of material on the concave side and resorption on the convex side.

7. The theory of internal remodeling

The rationale underlying the theory of internal remodeling was outlined in Section 5. The adapting body is modeled as a chemically reacting elastic porous medium in which the rate of reaction is strain controlled (Cowin and Hegedus [5]). The porous medium has two components: a porous elastic solid representing the matrix structure and a perfusant. Mass is transferred from the porous elastic solid to the fluid perfusant and *vice versa* by the chemical reaction whose rate is strain controlled. The mass of the porous elastic solid is changed by increasing or decreasing its porosity, but not by changing the overall dimensions of the body.

The small strain theory of internal bone remodeling is an adaptation of the theory of equilibrium of elastic bodies. The theory models the bone matrix as a chemically reacting porous elastic solid. The bulk density ρ of the porous solid is written as the product of γ and ν,

(2) $$\rho = \gamma\nu$$

where γ is the density of the material that composes the matrix structure

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and ν is the volume fraction of that material present. Both γ and ν are considered to be field variables. We let ξ denote the value of the volume fraction ν of the matrix material in an unstrained reference state. The density γ of the material composing the matrix is assumed to be constant, hence the conservation of mass will give the equation governing ξ. It is also assumed that there exists a unique zero-strain reference state for all values of ξ. Thus ξ may change without changing the reference state for strain. One might imagine a block of porous elastic material with the four points, the vertices of a tetrahedron, marked on the block for the purpose of measuring the strain. When the porosity changes, material is added or taken away from the pores, but if the material is unstrained it remains so and the distances between the four vertices marked on the block do not change. Thus ξ can change while the zero-strain reference state remains the same. The change in volume fraction from a reference volume fraction ξ_0 is denoted by e,

(3) $$e = \xi - \xi_0 .$$

The basic kinematic variables, and also the independent variables, for the theory of internal bone remodeling are the six components of the strain matrix E_{ij} and the change in volume fraction e of the matrix material from a reference value ξ_0.

The governing system of equations for this theory are (Hegedus and Cowin [15])

(4) $$2E_{ij} = (u_{i,j}+u_{j,i}) ,$$

(5) $$T_{ij,j} + \gamma(\xi_0+e)b_i = 0 ,$$

(6) $$T_{ij} = (\xi_0+e)C_{ijkm}(e)E_{km} ,$$

(7) $$\dot{e} = a(e) + A_{ij}(e)E_{km} ,$$

where $a(e)$, $A_{ij}(e)$ and $C_{ijkm}(e)$ are material coefficients dependent upon the change in volume fraction e of the adaptive elastic material from the reference volume fraction ξ_0, and where the superimposed dot indicates the material time derivative. Equation (4) represents the strain-displacement relations for small strain, valid in the present theory as well as in the theory of elasticity. Equation (5) represents the condition of equilibrium in terms of stress. Equation (6) is a generalization of Hooke's law, $T_{ij} = C_{ijkm}E_{km}$, in which the elastic coefficients C_{ijkm} now have a dependence upon the change in the reference volume fraction e and are denoted by $(\xi_0+e)C_{ijkm}(e)$. In the case when the change in volume fraction e vanishes and the reference volume fraction ξ_0 is one, (6) coincides with the generalized Hooke's law. Equation (7) is the remodeling rate equation and it specifies the rate of change of the volume fraction as a function of the volume fraction and strain. A positive value of $\dot{e}$ means the volume fraction of elastic material is increasing while a negative value means the volume fraction is decreasing. Equation (7) is obtained from the conservation of mass and the constitutive assumption that the rate of mass deposition or absorption is dependent upon only the volume fraction e and the strain state E_{ij}. The linear dependence upon strain shown in (7) occurs as a result of the small strain assumption. A uniqueness theorem for this theory was given by Cowin and Nachlinger [6].

The system of equations (6) and (7) is an elementary mathematical model of the volumetric stress adaptation process. Equation (6) is a statement that the moduli occurring in Hooke's law actually depend upon the volume fraction of solid matrix material present. Equation (7) is an evolutionary law for the volume fraction of matrix material. We will now describe how the model works in terms of equations (6) and (7) using, to fix ideas, a hollow circular cylinder subjected to an axial compressive load. Suppose for $t < 0$ the hollow circular cylinder has been under a constant stress for a long time. The body is then in remodeling equilibrium and the volume fraction field e is steady $(\dot{e} = 0)$. From equation (7) we can see that this means there is a particular steady strain associated with the steady stress and from equation (6) we can see that the elastic coefficients C_{ijkm} are steady. Now, at $t = 0$ the axial applied compressive stress is increased to a new value and held at the new value for all $t > 0$. This stress history is illustrated in Figure 2. From equation (6) it follows that the strain will jump to a new value at $t = 0$

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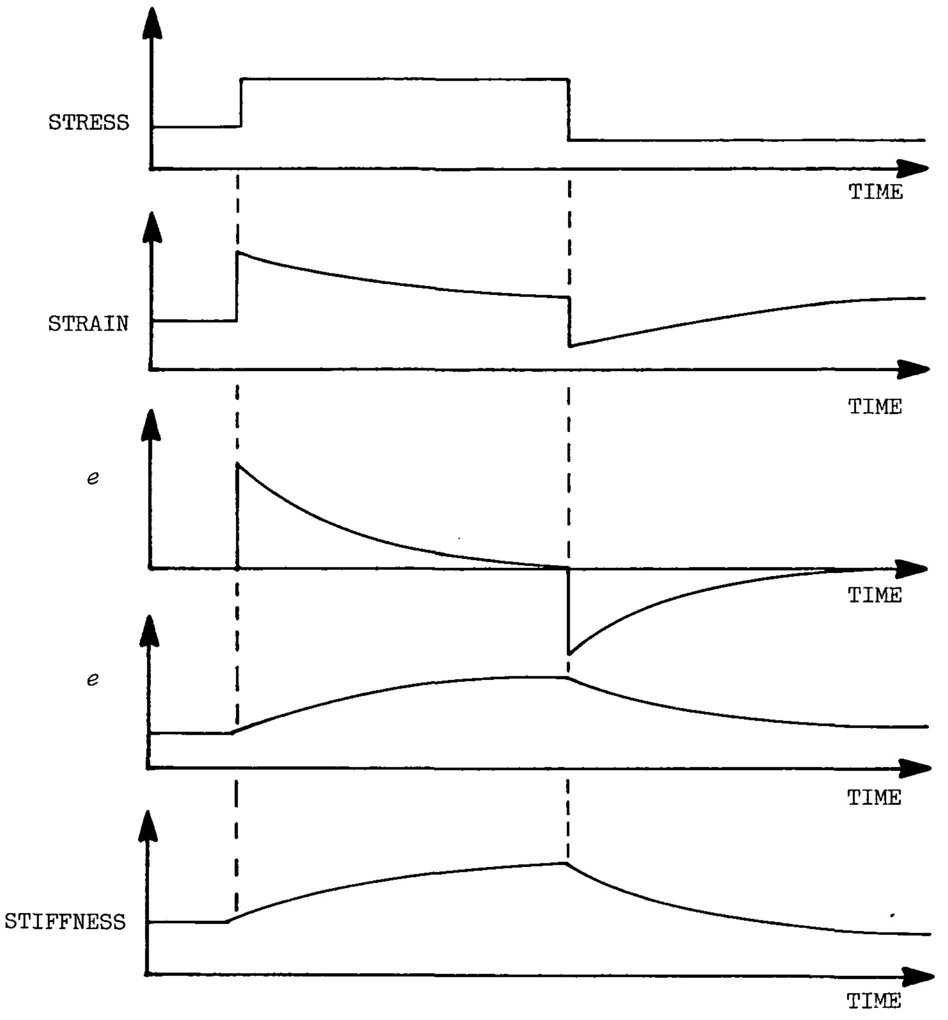

FIGURE 2. An illustration of the internal remodeling response to jumps in the constant applied stress. The response of the strain, $\dot{e}$, e and bone stiffness to the illustrated stress history is shown.

and from equation (7) it follows that $\dot{e}$ will jump to a non zero value. These jumps are also illustrated in Figure 2. Since $\dot{e}$ is non zero for $t > 0$, equation (7) shows that e will change with time and the normal physics of remodeling suggests that e will increase as a result of the increased strain. Since e is increasing for $t > 0$, the elastic

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coefficients $C_{ijkm}(e)$ will increase (see Figure 2). It follows from (6) that if the stress is constant and the elastic coefficients are increasing, then the strain must decrease in time. Since the strain is decreasing in time, it follows from (7) that one remodeling rate e will decrease and that e will continue to change, but more and more slowly. Thus for very large times $\dot{e}$ will tend to zero and the body will evolve to a new value of e and a new value of strain that is compatible with the increased stress state applied at $t = 0$. Remodeling of the cylinder is then complete. The process just described as well as the process associated with a jump reduction in stress are illustrated in Figure 2.

The theory described above involves the functions $a(e)$, $A_{ij}(e)$ and $C_{ijkm}(e)$ characterizing the material properties. There is no data in the literature on the values of the functions $a(e)$ and $A_{ij}(e)$ and the data on $C_{ijkm}(e)$ suggests that it can be approximated as a linear function of e. Hegedus and Cowin [15] introduced an approximation scheme that gave $C_{ijkm}(e)$ as a linear function of e. This scheme involved a series expansion in which terms of the order e^3, $|E|e^2$, and $|E|^2e$ were neglected and terms of the order e, $|E|$, $|E|e$ and e^2 retained. The scheme showed that $A_{ij}(e)$ was also linear in e while $a(e)$ was quadratic in e, thus the constitutive relations (6) and (7) were approximated by

$$T_{ij} = \left(\xi_0 C^0_{ijkm} + eC^1_{ijkm}\right)E_{km} , \tag{8}$$

and

$$\dot{e} = c_0 + c_1 e + c_2 e^2 + A^0_{ij}E_{ij} + eA^1_{ij}E_{ij} , \tag{9}$$

where C^0_{ijkm}, C^1_{ijkm}, c_0, c_1, c_2, A^0_{ij} and A^1_{ij} are constants.

8. Devolution of inhomogeneities under steady homogeneous stress

To study the devolution of an inhomogeneous body into a homogeneous one we subject an inhomogeneous body which is in mechanical and remodeling equilibrium for $t < 0$ to a homogeneous steady stress state for $t > 0$.

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We consider an inhomogeneous, adaptive elastic material body which is in mechanical and remodeling equilibrium for time $t < 0$. By mechanical and remodeling equilibrium it is meant that there exists a steady stress $T^0_{ij}(x)$, a steady strain $E^0_{ij}(x)$ and a steady volume fraction field $e_0(x)$ which satisfy equations (4), (5), (8) and (9) identically. At $t = 0$ the stress field $T^0_{ij}(x)$ is changed to the steady homogeneous field T^*_{ij} and the body force, if it was not zero for $t < 0$, is set to zero for $t > 0$. Since the body force is zero and the stress is homogeneous, the equation of equilibrium (5) is satisfied identically. The strain tensor for $t > 0$ can be determined by inversion of (8), thus

$$(10) \qquad E_{ij}(x,\ t) = \left(K^0_{ijkm} - eK_{ijrs}C^1_{rspq}K^0_{pqkm}\right)T^*_{km}$$

where K^0_{ijkm} is defined by

$$(11) \qquad \xi_0 K^0_{ijkm}C^0_{kmpq} = \delta_{ip}\delta_{jq}\ .$$

The representation (10) for the strain tensor is then substituted into the remodeling rate equation (9) to obtain the differential equation governing the evolution of $e(x,\ t)$, thus

$$(12) \qquad \dot{e}(x,\ t) = A\{e^2(x,\ t) - 2Be(x,\ t) + C\}\ , \quad e(x,\ 0) = e_0(x)\ ,$$

where A, B and C are constants, $A = c_2$,

$$(13) \qquad B = -\frac{1}{2c_2}\left(c_1 + A^1_{ij}K^0_{ijmk}T^*_{mk} - A^0_{ij}K^0_{ijrs}C^1_{rspq}K^0_{pqmk}T^*_{mk}\right)$$

and

$$(14) \qquad C = \frac{1}{c_2}\left(c_0 + A^0_{ij}K^0_{ijmk}T^*_{mk}\right)\ .$$

The initial condition indicated as the second part of (12) requires that the inhomogeneity at time $t = 0$ be that associated with the mechanical and remodeling equilibrium state that existed for $t < 0$.

The solution to the differential equation (12) is presented by Firoozbakhsh and Cowin [11]. In order to discuss this solution we

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introduce the notation e_1, e_2 for the solutions to the quadratic equation

$$e^2 - 2Be + C = 0 ,$$

(15) $$e_1, e_2 = B \pm (B^2 - C)^{\frac{1}{2}} ,$$

and we employ the convention $e_1 \geq e_2$ when e_1 and e_2 are real. Insight into the behavior predicted by the differential equation (12) at a fixed value of X can be obtained by considering its representation in the phase plane. In Figure 3 the solution to the remodeling rate equation (12) is plotted in the case when e_1 and e_2 are real and distinct, for both $A > 0$ and $A < 0$. In this case the remodeling rate equation is a parabola in $\dot{e}$ and e which crosses the e axis at two points and opens up or down depending on the sign of A . These parabolas are sketched in Figure 3. The arrowheads on the parabolas indicate the direction a solution will evolve in a positive time for a given value of e . Thus, for example, the fact that $e \to e_2$ in infinite time for e_1 and e_2 real, $A > 0$ and $e_1 \geq e_0$ is indicated by the arrowheads on the parabola to the left and the right of e_2 being directed towards e_2 . The arrowheads to the right of e_1 are oppositely directed for $A > 0$ indicating that $e \to \infty$ in finite time for e_1 and e_2 real and distinct. A completely analogous description holds for the case $A < 0$ shown in Figure 3. Hegedus and Cowin [15] have shown that the solution to (12) is stable only if e_1 and e_2 are real and distinct and, under those conditions on e_1 and e_2 , $e(X, t)$ is given by

(16) $$e(X, t) = \frac{e_1[e_0(X)-e_2]+e_2[e_1-e_0(X)]\exp[(\mathrm{sgn}A)(t/\tau)]}{[e_0(X)-e_2]+[e_1-e_0(X)]\exp[(\mathrm{sgn}A)(t/\tau)]}$$

where sgn A means the sign of A and where

(17) $$\tau = \frac{1}{|A|(e_1-e_2)}$$

is called the remodeling time constant. The precise conditions under which the differential equation (12) yields stable solutions which tend to finite, physiologically possible values for $e(X, t)$ are discussed by

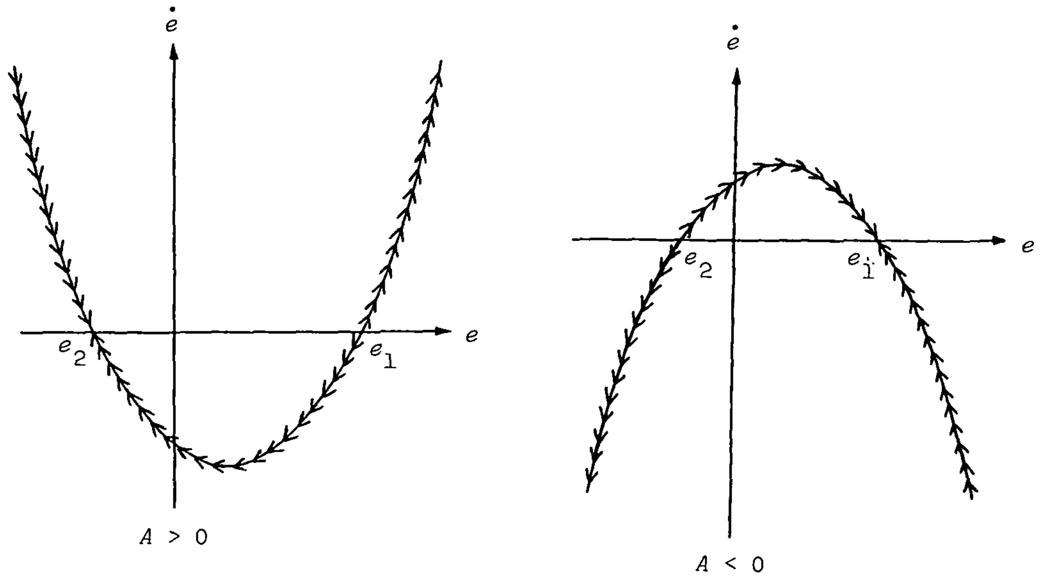

FIGURE 3. Phase plane representations of the remodeling rate equation. The arrowheads indicate the evolution of the solution in positive time.

Firoozbakhsh and Cowin [11]. The result (16) shows that an inhomogeneous adaptive elastic body will become homogeneous under the action of a steady homogeneous stress field.

In order to illustrate this result we consider a particular case. Specifically we consider a cylindrical body of length $2l$ which is initially inhomogeneous along the axis of the cylinder, but which is homogeneous in each transverse plane of the cylinder. For the purposes of this illustration the initial inhomogeneity is assumed to be sinusoidal,

$$e(x, 0) = 0.1 \sin \frac{\pi x_3}{l} , \tag{18}$$

where we have taken the axis of the cylinder to be in the x_3 direction. The steady homogeneous stress this body is subjected to for all $t > 0$ is a constant compressive stress of magnitude P along its axis, thus

$$T^*_{33} = -P \text{ , all other } T^*_{ij} = 0 \text{ .} \tag{19}$$

The orthotropic elastic constants for human cortical bone are taken from

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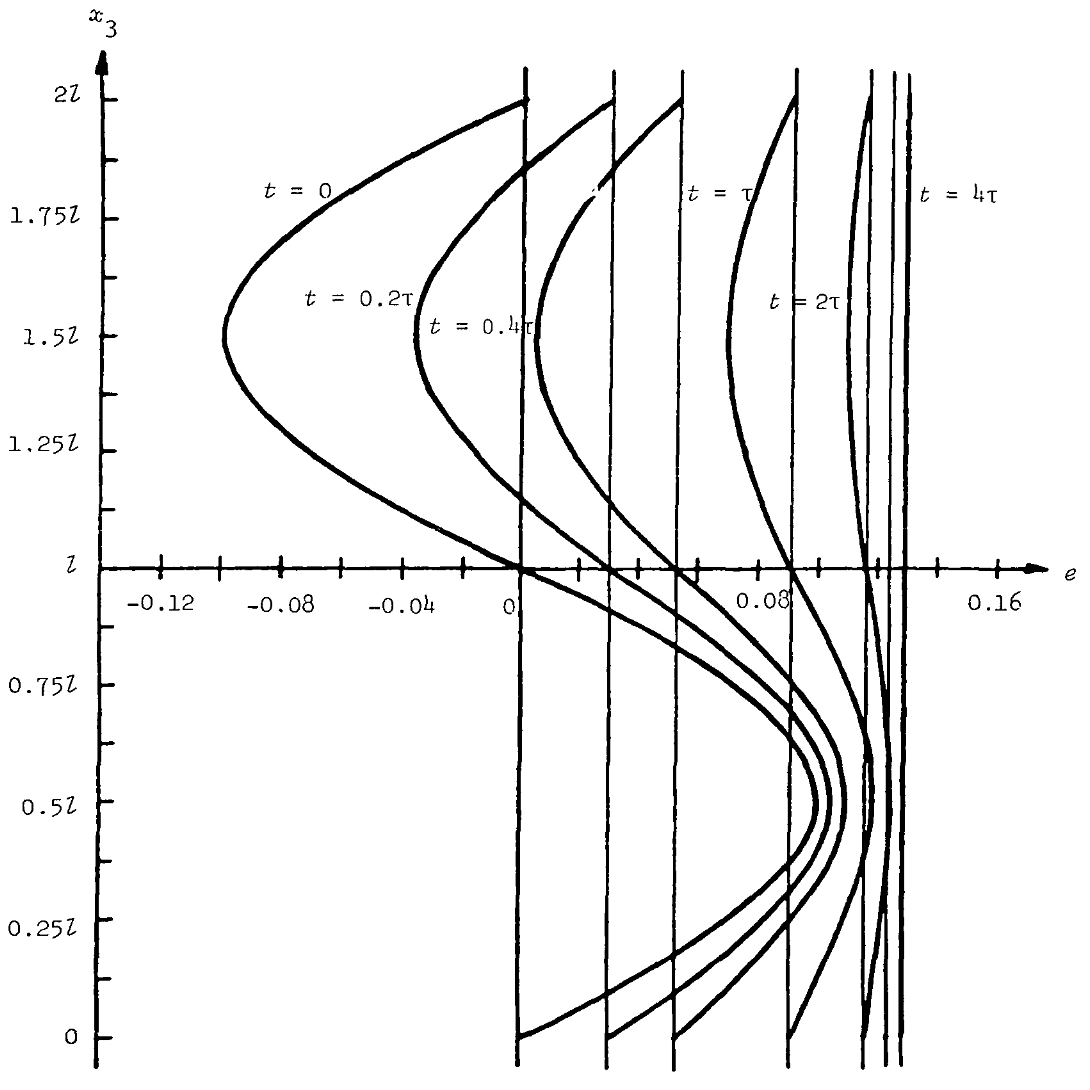

FIGURE 4. Devolution of an initial inhomogeneity. A graphical representation of the solution $e(x_3, t)$ given by (22).

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Knets and Malmeister [19] and the dependence of these constants upon the change in volume fraction e is estimated from the microstructure dependence data given by Wright and Hayes [34], thus

$$(20)\quad \frac{1}{E_3} = 0.055(1-11e)\ ,\quad \frac{\nu_{31}}{E_3} = 0.0165(1-11e)\ ,\quad \frac{\nu_{32}}{E_3}\ 0.033(1-11e)\ ,$$

where the units are $(GPa)^{-1}$. The reference volume fraction ξ_0 is 0.892 for (20). We assume that $A > 0$ and that the rate coefficients have the following values:

$$(21)\quad \begin{aligned} &c_0 = 1.5 \times 10^{-8}\ \text{sec}^{-1}\ ,\quad c_1 = -15 \times 10^{-8}\ \text{sec}^{-1}\ ,\quad c_2 = 2.5 \times 10^{-7}\ \text{sec}^{-1}\ ,\\ &A^0_{11} = A^0_{22} = A^0_{33} = A^1_{11} = A^1_{22} = A^1_{33} = 10^{-4}\ \text{sec}^{-1}\ . \end{aligned}$$

Since $A > 0$ the solution for $e(x_3,\ t)$ is

$$(22)\quad e(x_3,\ t) = \frac{0.129\big(4.4-\sin(\pi x_3/l)\big)+0.44\big(\sin(\pi x_3/l)-1.29\big)e^{-t/\tau}}{\big(4.4-\sin(\pi x_3/l)\big)+\big(\sin(\pi x_3/l)-1.29\big)e^{-t/\tau}}\ ,$$

where τ is 12.73×10^6 sec or about 147 days. This result is plotted in Figure 4 for various values of time. From this illustration of the temporal evolution of the sine wave inhomogeneity one can see that, as time progresses, the amplitude of the sine wave decreases, rapidly at first and then more slowly. At large times the sine wave becomes a straight line signifying that the cylinder has become homogeneous.

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