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# Asymptotics for symmetrized positive moments of odd ranks

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*Abstract.* In 2007, Andrews introduced Durfee symbols and $k$-marked Durfee symbols so as to give a combinatorial interpretation for the symmetrized moment function $\eta_{2k}(n)$ of ranks of partitions. He also considered the relations between odd Durfee symbols and the mock theta function $\omega(q)$, and proved that the $2k$th moment function $\eta^0_{2k}(n)$ of odd ranks of odd Durfee symbols counts $(k+1)$-marked odd Durfee symbols of $n$. In this paper, we first introduce the definition of symmetrized positive odd rank moments $\eta^{0+}_k(n)$ and prove that for all $1 \le i \le k+1$, $\eta^{0+}_{2k-1}(n)$ is equal to the number of $(k+1)$-marked odd Durfee symbols of $n$ with the $i$th odd rank equal to zero and $\eta^{0+}_{2k}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$th odd rank being positive. Then we calculate the generating functions of $\eta^{0+}_k(n)$ and study its asymptotic behavior. Finally, we use Wright's variant of the Hardy–Ramanujan circle method to obtain an asymptotic formula for $\eta^{0+}_k(n)$.

## 1 Introduction and statement of results

This paper is concerned with combinatorial interpretations and asymptotic formulas for the symmetrized positive moments of odd ranks of odd Durfee symbols and $k$-marked Durfee symbols. Odd Durfee symbols and $k$-marked Durfee symbols were first introduced by Andrews [2] in 2007, and were used to give a natural combinatorial explanation to an identity associated with Watson's third-order mock theta function $\omega(q)$ [11], which is defined as

$$\omega(q) := \sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(q;q^2)^2_{n+1}},$$

where

$$(a;q)_n := \prod_{m=0}^{n-1} (1 - aq^m).$$

***Definition 1.1*** (Odd Durfee symbols) An odd Durfee symbol of $n$ is a two-rowed array with a subscript

$$(\alpha,\beta)_d := \begin{pmatrix} \alpha_1\ \alpha_2\ \dots\ \alpha_s \\ \beta_1\ \beta_2\ \dots\ \beta_t \end{pmatrix}_d,$$

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where $\alpha_i$ and $\beta_i$ are all positive odd numbers, $2d+1 \geq \alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_s > 0$, $2d+1 \geq \beta_1 \geq \beta_2 \geq \cdots \geq \beta_t > 0$ and $n = \sum_{i=1}^{s} \alpha_i + \sum_{i=1}^{t} \beta_i + 2d^2 + 2d + 1$.

Here, we use the rephrased version of the definition of odd Durfee symbols due to the work of Ji [9].

Recall that $q\omega(q)$ is also the generating function for partitions in which at least all but one instance of the largest part is one of a pair of consecutive nonnegative integers. It is trivial to see that the summation of two consecutive nonnegative integers is odd, and conversely that every odd number is a sum of two consecutive integers. We also observe that any odd positive integer can been decomposed into some 2's and one 1. Then there is a simple bijection between the set of partitions of $n$ in which at least all but one instance of the largest part is one of a pair of consecutive nonnegative integers and the set of odd Durfee symbols of $n$. For example, the partition $8 + (4+3) + (3+2) + (3+2) + (1+0) + (1+0)$ can been seen as "odd Ferrers diagram"

$$\begin{array}{ccc:ccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 & & & & \\
1 & 2 & 2 & & & & & \\
\hdashline
1 & 2 & 2 & & & & & \\
1 & & & & & & & \\
1 & & & & & & &
\end{array}$$

of shape $(8, 4, 3, 3, 1, 1)$. The top row of 1's represents 8; the second row is $(4+3)$; the third row is $(3+2)$ as is the fourth row; the last two rows represent $(1+0)$. As it shown, this odd Ferrers diagram has a Durfee square of size 3 indicated by the dotted lines. Then we see that it is mapped to the odd Durfee symbol:

$$\begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 5 & 1 & 1 & & \end{pmatrix}_2.$$

Analogues to ordinary rank of partitions, Andrews [2] defined *odd rank* of an odd Durfee symbol to be the number of entries in the top row minus the number of entries in the bottom row. Let $N^0(m,n)$ denote the number of odd Durfee symbols of $n$ with odd rank $m$. According to this definition, it is easy to see that $N^0(m,n) = N^0(-m,n)$. Wang [10] showed that $N^0(m,n) = 0$ for $n \equiv m \pmod 2$. Denote the number of odd Durfee symbols of $n$ with odd rank congruent to $a$ modulo $b$ by $N^0(a,b;n)$. Wang [10] also obtained some generating functions for $N^0(a,b;cn+d)$, where $c \in \{2,4,8\}$, and thus derived many congruences for odd rank modulo powers of 2.

In order to study the relations between cranks and ranks of partitions, Atkin and Garvan [3] considered the $k$th moment of the rank which is defined by

$$N_k(n) := \sum_{m=-\infty}^{\infty} m^k N(m,n).$$

Here, $N(m,n)$ denotes the number of partitions of $n$ with ordinary rank $m$. Andrews [2] found that there is a rich combinatorial and enumerative structure related to the

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moments of ranks. He defined the $k$th symmetrized moment of rank as

$$\eta_k(n) := \sum_{m=-\infty}^{\infty} \binom{m+\lfloor \frac{k}{2} \rfloor}{k} N(m,n),$$

and discovered that $\eta_{2k}(n)$ counts $(k+1)$-marked Durfee symbols of $n$. For our purpose, we do not give the definitions of Durfee symbols and $k$-marked Durfee symbols here. Bringmann et al. [5] introduced two-parameter generalizations of $k$-marked Durfee symbols and the $k$th symmetrized rank moment and studied automorphic properties of their generating functions. In 2014, Chen, Ji, and Shen [7] defined the $k$th symmetrized positive moment $\overline{\eta}_k(n)$ of ranks of partitions of $n$ by

$$\overline{\eta}_k(n) := \sum_{m=1}^{\infty} \binom{m+\lfloor \frac{k-1}{2} \rfloor}{k} N(m,n).$$

They also gave a combinatorial interpretation of $\overline{\eta}_k(n)$. More precisely, for $1 \le i \le k+1$, $\overline{\eta}_{2k-1}(n)$ counts those $(k+1)$-marked Durfee symbols of $n$ with the $i$th rank equal to 0; $\overline{\eta}_{2k}(n)$ counts those $(k+1)$-marked Durfee symbols of $n$ with the $i$th rank being positive.

Furthermore, Andrews [2] considered the symmetrized $k$th moment function of odd ranks of odd Durfee symbols

$$\eta_k^0(n) := \sum_{m=-\infty}^{\infty} \binom{m+\lfloor \frac{k}{2} \rfloor}{k} N^0(m,n),$$

and showed that $\eta_{2k}^0(n)$ is equal to the number of $(k+1)$-marked odd Durfee symbols of $n$. It is the right time to give the definition of $k$-marked odd Durfee symbols.

***Definition 1.2*** ($k$-marked odd Durfee symbols) A $k$-marked odd Durfee symbol of $n$ is composed of $k$-pairs of partitions into odd parts with the subscript, which is defined as

$$\eta^0 := \begin{pmatrix} \alpha^k, \alpha^{k-1}, \dots, \alpha^1 \\ \beta^k, \beta^{k-1}, \dots, \beta^1 \end{pmatrix}_d,$$

where $\alpha^i$ (resp. $\beta^i$) are all partitions with odd parts and $\sum_{i=1}^k (|\alpha^i| + |\beta^i|) + 2d^2 + 2d + 1$=n. Furthermore, the partitions $\alpha^i$ and $\beta^i$ must satisfy the following three conditions:

(i) For $1 \le i < k$, $\alpha^i$ must be non-empty partition, while $\alpha^k$ and $\beta^i$ could be empty;
(ii) $\beta_1^{i-1} \le \alpha_1^{i-1} \le \beta_{\ell(\beta^i)}^i$ for $2 \le i \le k$;
(iii) $\beta_1^k, \alpha_1^k \le 2d+1$,

where $\alpha_1^i$ (resp. $\beta_1^i$) is the largest part of the partition $\alpha^i$ (resp. $\beta^i$) and $\alpha_{l(\alpha^i)}^i$ (resp. $\beta_{l(\beta^i)}^i$) is the smallest part of the partition $\alpha^i$ (resp. $\beta^i$), and $\ell(\beta)$ denotes the number of parts of the partition $\beta$. Moreover, the $i$th odd rank of $\eta^0$ is defined as

$$r_i(\eta^0) := \begin{cases} \ell(\alpha^i) - \ell(\beta^i) - 1, & \text{for } 1 \le i < k, \\ \ell(\alpha^k) - \ell(\beta^k), & \text{for } i = k. \end{cases}$$

Some open problems and conjectures proposed by Andrews in [2] have since driven the study of this combinatorial structure. For example, Bringmann [4] gave effective asymptotic formulas for the symmetrized second moment functions $\eta_2(n)$ and $\eta_2^0(n)$. Namely, we have

$$\eta_2(n) = \sum_{k=1}^{\lfloor\sqrt{n}\rfloor} A_k(n)\left[-\frac{3}{2(24n-1)^{\frac{1}{4}}} I_{1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + \frac{\pi(24n-1)^{\frac{1}{4}}}{12k} \times I_{-1/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right) + \frac{\pi}{12k(24n-1)^{\frac{3}{4}}} I_{3/2}\left(\frac{\pi}{6k}\sqrt{24n-1}\right)\right] + O(n^{1+\varepsilon}),$$

and

$$\eta_2^0(n) = -\frac{i}{\sqrt{2}} \sum_{\substack{k=1\\ k\,odd}}^{\lfloor\sqrt{n}\rfloor} A_k^0(n)\left[-\frac{3\pi^2(3n-1)^{\frac{1}{4}}}{4} I_{-1/2}\left(\frac{\pi}{3k}\sqrt{3n-1}\right) + \frac{\pi(3n-1)^{\frac{3}{4}}}{16k} I_{-3/2}\left(\frac{\pi}{3k}\sqrt{3n-1}\right)\right] + O(n^{2+\varepsilon}),$$

where $A_k^0(n)$ is a certain Kloosterman type summation and the function $I_k(x)$ is the usual $I$-Bessel function of order $k$. Alfes et al. [1] introduced two-parameter generalizations of $(k+1)$-marked odd Durfee symbols and the $2k$th symmetrized odd rank moment and studied automorphic properties of their generating functions. With the help of Mittag-Leffler decomposition and Wright's variant of the Hardy–Ramanujan circle method, Bringmann and Mahlburg [6] found an asymptotic formula of the symmetrized positive moments $\eta_k^+(n)$ of ranks of partitions, where

$$\eta_k^+(n) := \sum_{m=1}^{\infty} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N(m,n).$$

If the $k$th symmetrized positive moment $\eta_k^{0+}(n)$ of odd ranks of odd Durfee symbols of $n$ is given by

$$\eta_k^{0+}(n) := \sum_{m=1}^{\infty} \binom{m+\lfloor\frac{k-1}{2}\rfloor}{k} N^0(m,n),$$

it is natural to ask which combinatorial structure counted by $\eta_k^{0+}(n)$ and study the asymptotic behavior of $\eta_k^{0+}(n)$.

One of main objectives of this paper is to give an combinatorial interpretation of $\eta_k^{0+}(n)$ in terms of $k$-marked Durfee symbols. Denote the number of $k$-marked odd Durfee symbols of $n$ with the $i$th odd rank equal to $m_i$ by $D_k^0(m_1, m_2, \ldots, m_k; n)$. Andrews [2] showed that $D_k^0(m_1, m_2, \ldots, m_k; n)$ is symmetric with respect to $m_1, m_2, \ldots, m_k$ for $k \geq 2$. Moreover, Ji [9] found the following relation:

$$D_k^0(m_1, m_2, \ldots, m_k; n) = \sum_{j=0}^{\infty} \binom{j+k-2}{k-2} N^0\left(\sum_{i=1}^{k} |m_i| + 2j + k - 1, n\right).$$

With the aid of symmetry of $D_k^0(m_1, m_2, \ldots, m_k; n)$ and the relation above, we find the following combinatorial interpretations of $\eta_k^{0+}(n)$.

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**Theorem 1.3** *For any $k \geq 1$ and fixed $i \geq 1$, we have:*

*(1) $\eta_{2k-1}^{0+}(n)$ is equal to the number of $(k+1)$-marked odd Durfee symbols of $n$ with the ith odd rank equal to zero;*

*(2) $\eta_{2k}^{0+}(n)$ is equal to the number of $(k+1)$-marked odd Durfee symbols of $n$ with the ith odd rank being positive.*

Notice that the generating function of the odd rank function $N^0(m,n)$ of odd Durfee symbols is

$$\begin{aligned} R^0(w;q) &:= \sum_{n=1}^{\infty} \sum_{m=-\infty}^{\infty} N^0(m,n) w^m q^n \\ &= \sum_{n=0}^{\infty} \frac{q^{2n(n+1)+1}}{(wq;q^2)_{n+1}(w^{-1}q;q^2)_{n+1}} \\ &= \frac{1}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n+1}}{1-wq^{2n+1}}. \end{aligned} \tag{1.1}$$

The last identity is equivalent to Watson's first identity on page 66 of [11]. By expanding the summand of (1.1), we find that for any $m$,

$$\sum_{n=0}^{\infty} N^0(m,n) q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} (-1)^n q^{3n^2+3n+1+|m|(2n+1)}. \tag{1.2}$$

Using this generating function (1.2) of $N^0(m,n)$ and the combinatorial interpretation of $\eta_r^{0+}(n)$ in Theorem 1.3, we obtain the following generating functions of $\eta_r^{0+}(n)$.

**Theorem 1.4** *For any $r \geq 1$, we have*

$$\sum_{n=0}^{\infty} \eta_r^{0+}(n) q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+3n+1+(2n+1)\lfloor \frac{r+1}{2} \rfloor}}{(1-q^{2n+1})^{r+1}}. \tag{1.3}$$

Although Wright's approach is lesser-known, and gives much weaker results than Hardy and Ramanujan in the study of the coefficients of modular forms, it is powerful enough to provide an asymptotic expansion for the coefficients and flexible enough that it applies to nonmodular generating functions such as the symmetrized positive odd rank moments. Let $I_\ell$ denote the usual $I$-Bessel function of order $\ell$. The Dirichlet beta function $\beta(s)$ (also known as the Catalan beta function) is defined as

$$\beta(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^s},$$

where we assume $\mathrm{Re}(s) > 0$. Based on the generating function of $\eta_r^{0+}(n)$ in Theorem 1.4, we apply Wright's variant of the Hardy–Ramanujan circle method to study the asymptotic behavior of the symmetrized positive moments.

**Theorem 1.5** *Suppose that $r \geq 2$. As $N \to \infty$,*

$$\eta_r^{0+}(N) = \lambda_r N^{\frac{2r-1}{4}} I_{r-\frac{1}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) + \mu_r N^{\frac{2r-3}{4}} I_{r-\frac{3}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) + O\left(N^{\frac{2r-5}{4}} e^{\frac{\pi\sqrt{N}}{\sqrt{3}}}\right),$$

*where the constants $\lambda_r$ and $\mu_r$ are given by*

$$\lambda_r := 2^{r-\frac{1}{2}} 3^{\frac{2r-1}{4}} \pi^{-r} \beta(r+1),$$

$$\mu_r := -\frac{\pi}{24\sqrt{3}}\lambda_r - 3^{\frac{2r-3}{4}} 2^{r-\frac{7}{2}} \pi^{-r+1} \left[3\beta(r-1) - 2\left(1+r-2\left\lfloor \frac{r+1}{2} \right\rfloor\right)\beta(r) + \beta(r+1)\right].$$

Utilizing the well-known asymptotic formula for the modified Bessel functions:

$$I_s(x) = \frac{e^x}{\sqrt{2\pi x}} + O\left(\frac{e^x}{x^{\frac{3}{2}}}\right), \quad \text{as} \quad x \to \infty,$$

we obtain the following asymptotic formula of symmetrized positive moments of odd ranks.

***Corollary 1.6*** *Suppose that $r \geq 2$. As $N \to \infty$,*

$$\eta_r^{0+}(N) \sim 2^{r-1} 3^{\frac{r}{2}} \pi^{-r-1} \beta(r+1) N^{\frac{r-1}{2}} e^{\pi\sqrt{\frac{N}{3}}}.$$

***Remark 1.7*** It should be noted that the asymptotic formula for $\eta_r^{0+}(N)$ in Corollary 1.6 also holds for $r = 1$, although the proof is simpler, but slightly different from those for $r \geq 2$ in the proof of Theorem 1.5.

This paper is organized as follows: In Section 2, we first define the $k$th symmetrized positive moment $\eta_k^{0+}(n)$ of odd ranks by

$$\eta_k^{0+}(n) := \sum_{m=1}^{\infty} \binom{m + \left\lfloor \frac{k-1}{2} \right\rfloor}{k} N^0(m,n).$$

Then we prove that for all $1 \leq i \leq k+1$, $\eta_{2k-1}^{0+}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$th odd rank equal to zero and $\eta_{2k}^{0+}(n)$ is equal to the number of $(k+1)$-marked Durfee symbols of $n$ with the $i$th odd rank being positive, which are stated in Theorem 1.3. In Section 3, we show that the generating function, given in Theorem 1.4, of $\eta_k^{0+}(n)(k \geq 1)$ is

$$\sum_{n=0}^{\infty} \eta_k^{0+}(n) q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+3n+1+(2n+1)\left\lfloor \frac{k+1}{2} \right\rfloor}}{(1-q^{2n+1})^{k+1}}.$$

In Section 4, we first study the asymptotic behavior of the generating function of $\eta_k^{0+}(n)$, and then use Wright's variant of the Hardy–Ramanujan circle method to obtain an asymptotic formula for $\eta_k^{0+}(n)(k \geq 2)$ in Corollary 1.6:

$$\eta_k^{0+}(n) \sim 2^{k-1} 3^{\frac{k}{2}} \pi^{-k-1} \beta(k+1) n^{\frac{k-1}{2}} e^{\pi\sqrt{\frac{n}{3}}},$$

as $n$ tends to infinity in which $\beta(k)$ is the Dirichlet beta function.

## 2 Combinatorial interpretations for $\eta_k^{0+}(n)$

Motivated the work of Chen, Ji, and Shen [7], we consider the $k$th symmetrized positive moment $\eta_k^{0+}(n)$ of odd ranks of odd Durfee symbols of $n$. Before giving combinatorial interpretations of $\eta_k^{0+}(n)$, we need some definitions and related results. Denote the number of $k$-marked odd Durfee symbols of $n$ with the $i$th odd rank equal to $m_i$ by $D_k^0(m_1, m_2, \ldots, m_k; n)$. Andrews [2] showed that $D_k^0(m_1, m_2, \ldots, m_k; n)$ is symmetric in $m_1, m_2, \ldots, m_k$ for $k \geq 2$. Ji [9] found the following relation between $D_k^0(m_1, m_2, \ldots, m_k; n)$ and $N^0(m, n)$.

**Theorem 2.1** *[9, Theorem 4.9] For $k \geq 2$, we have*

$$D_k^0(m_1, m_2, \ldots, m_k; n) = \sum_{j=0}^{\infty} \binom{j+k-2}{k-2} N^0\left(\sum_{i=1}^{k} |m_i| + 2j + k - 1, n\right). \tag{2.1}$$

Note that $\binom{j+k-2}{k-2}$ is the number of nonnegative integer solutions to $t_1 + t_2 + \cdots + t_{k-1} = j$. Then we find that the identity (2.1) is equivalent to

$$D_k^0(m_1, m_2, \ldots, m_k; n) = \sum_{t_1, \ldots, t_{k-1}=0}^{\infty} N^0\left(\sum_{i=1}^{k} |m_i| + 2\sum_{i=1}^{k-1} t_i + k - 1, n\right). \tag{2.2}$$

Based on this relation above, we give the combinatorial interpretations of $\eta_k^{0+}(n)$ in Theorem 1.3.

**Proof of Theorem 1.3** (1) Since $D_k^0(m_1, m_2, \ldots, m_k; n)$ is symmetric, it suffices to show that

$$\sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} D_{k+1}^0(0, m_2, m_3, \ldots, m_{k+1}; n) = \eta_{2k-1}^{0+}(n).$$

From the relation (2.2), we derive that

$$\begin{aligned} &\sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} D_{k+1}^0(0, m_2, m_3, \ldots, m_{k+1}; n) \\ &= \sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} \sum_{t_1, \ldots, t_{k-1}=0}^{\infty} N^0\left(\sum_{i=2}^{k+1} |m_i| + 2\sum_{i=1}^{k} t_i + k, n\right). \end{aligned} \tag{2.3}$$

For fixed $k$ and $n$, let $c_k(n)$ denote the number of integer solutions to the equation

$$|m_2| + |m_3| + \cdots + |m_{k+1}| + 2t_1 + \cdots + 2t_k = n,$$

where $t_i \geq 0$ for $1 \leq i \leq k$. Then the generating function of $c_k(n)$ is

$$\begin{aligned} \sum_{n=0}^{\infty} c_k(n) q^n &= (1 + 2q + 2q^2 + 2q^3 + \ldots)^k (1 + q^2 + q^4 + q^6 + \ldots)^k \\ &= \left(\frac{1+q}{1-q}\right)^k \left(\frac{1}{1-q^2}\right)^k = \frac{1}{(1-q)^{2k}} \\ &= \sum_{n=0}^{\infty} \binom{n+2k-1}{2k-1} q^n. \end{aligned}$$

Therefore, the identity (2.3) is equivalent to

$$\sum_{m_2, m_3, \ldots, m_{k+1}=-\infty}^{\infty} D_{k+1}^0(0, m_2, m_3, \ldots, m_{k+1}; n) = \sum_{m=0}^{\infty} \binom{m+k-1}{2k-1} N^0(m, n), \tag{2.4}$$

which is equal to $\overline{\eta}_{2k-1}(n)$.

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(2) It suffices to show that

$$\sum_{\substack{m_1=1\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{+\infty} D^0_{k+1}(m_1,m_2,\ldots,m_{k+1};n)$$

$$=\sum_{\substack{m_1=1\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{+\infty}\ \sum_{t_1,\ldots,t_k=0}^{\infty} N^0\left(m_1+\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k,n\right).$$

For fixed $k$ and $n$, let $c_k(n)$ denote the number of integer solutions to the equation

$$m_1+|m_2|+|m_3|+\cdots+|m_{k+1}|+2t_1+\cdots+2t_k=n,$$

where $m_1$ is a positive integer and $t_i\ge 0$ for $0\le i\le k$. Then the generating function of $c_k(n)$ is

$$\sum_{n=0}^{\infty} c_k(n)q^n=\frac{q}{(1-q)^{2k+1}}=\sum_{n=0}^{\infty}\binom{n+2k-1}{2k}q^n.$$

It follows that

$$\sum_{\substack{m_1=1\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty} D^0_{k+1}(m_1,m_2,\ldots,m_{k+1};n)=\sum_{m=0}^{\infty}\binom{m+k-1}{2k}N^0(m,n),$$

which is equal to $\eta^{0+}_{2k}(n)$. ∎

## 3 Generating functions of $\eta^{0+}_k(n)$

In this section, we use the generating function of $N^0(m,n)$ to calculate the generating functions of $D_{k+1}(0,m_2,\ldots,m_{k+1};n)$ and $D_{k+1}(m_1,m_2,\ldots,m_{k+1};n)(m_1>0)$. Then we obtain the generating functions of $\eta^{0+}_{2k-1}(n)$ and $\eta^{0+}_{2k}(n)$.

***Theorem 3.1*** *For $k\ge 1$, we have*

$$\sum_{m_2,m_3,\ldots,m_{k+1}=-\infty}^{\infty}\ \sum_{n=0}^{\infty} D^0_{k+1}(0,m_2,\ldots,m_{k+1};n)x_1^{m_2}\ldots x_k^{m_{k+1}}q^n$$

$$=\frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n\frac{q^{3n^2+3n+1+k(2n+1)}}{\prod_{j=1}^{k}(1-x_jq^{2n+1})(1-x_j^{-1}q^{2n+1})}.$$

**Proof** Define

$$G_k(x_1,\ldots,x_k;q):=\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\ \sum_{n=0}^{\infty} D^0_{k+1}(0,m_2,\ldots,m_{k+1};n)x_1^{m_2}\ldots x_k^{m_{k+1}}q^n.$$

With the help of the generating function (1.2) of $N^0(m,n)$ and the relation (2.2), we find that

$$G_k(x_1,\ldots,x_k;q)$$

$$=\sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}\ \sum_{t_1,\ldots,t_k=0}^{\infty} x_1^{m_2}x_2^{m_3}\ldots x_k^{m_{k+1}}\sum_{n=0}^{\infty}N^0\left(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k}t_i+k,n\right)q^n$$

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$$= \frac{1}{(q^2;q^2)_\infty} \sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty} \sum_{t_1,\ldots,t_k=0}^{\infty} x_1^{m_2}\ldots x_k^{m_{k+1}}$$

$$\times \sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n+1+\left(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k} t_i+k\right)(2n+1)}$$

$$= \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty}(-1)^n q^{3n^2+3n+1+k(2n+1)} \sum_{m_2,\ldots,m_{k+1}=-\infty}^{\infty}$$

$$\times \sum_{t_1,\ldots,t_k=0}^{\infty} x_1^{m_2}\ldots x_k^{m_{k+1}} q^{(2n+1)\left(\sum_{i=2}^{k+1}|m_i|+2\sum_{i=1}^{k} t_i\right)}.$$

Applying the formula

$$\sum_{a=-\infty}^{\infty}\sum_{b=0}^{\infty} x^a q^{(2n+1)(|a|+2b)} = \frac{1}{(1-xq^{2n+1})(1-x^{-1}q^{2n+1})}$$

repeatedly, we find that

$$G_k(x_1,\ldots,x_k;q) = \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n \frac{q^{3n^2+3n+1+k(2n+1)}}{\prod_{j=1}^{k}(1-x_jq^{2n+1})(1-x_j^{-1}q^{2n+1})}.$$

■

Setting $x_i = 1 (1 \le i \le k)$ in Theorem 3.1 and using Theorem 1.3, we derive the following generating function of $\eta_{2k-1}^{0+}(n)$.

**Theorem 3.2** *For $k \ge 1$, we have*

$$\sum_{n=0}^{\infty} \eta_{2k-1}^{0+}(n)q^n = \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n^2+(2k+3)n+k+1}}{(1-q^{2n+1})^{2k}}.$$

Similar to the proof of Theorem 3.1, one may obtain the generating function of $D_{k+1}^{0}(m_1,m_2,\ldots,m_{k+1};n)$.

**Theorem 3.3** *For $k \ge 1$, we have*

$$\sum_{\substack{m_1>0\\ m_2,m_3,\ldots,m_{k+1}=-\infty}}^{\infty}\sum_{n=0}^{\infty} D_{k+1}^{0}(m_1,m_2,\ldots,m_{k+1};n)x_1^{m_1}\ldots x_{k+1}^{m_{k+1}}q^n$$

$$= \frac{1}{(q^2;q^2)_\infty}\sum_{n=0}^{\infty}(-1)^n \frac{x_1 q^{3n^2+3n+1+k(2n+1)}}{(1-x_1q^{2n+1})\prod_{j=2}^{k+1}(1-x_jq^{2n+1})(1-x_j^{-1}q^{2n+1})}.$$

Setting $x_i = 1 (1 \le i \le k+1)$ in Theorem 3.3 and using Theorem 1.3, we obtain the following generating function of $\eta_{2k}^{0+}(n)$.

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*Theorem 3.4* *For $k \geq 1$, we have*

$$\sum_{n=0}^{\infty} \eta_{2k}^{0+}(n)q^n = \frac{1}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+(2k+3)n+k+1}}{(1-q^{2n+1})^{2k+1}}.$$

**Proof of Theorem 1.4** Combining Theorem 3.3 with Theorem 3.4 yields Theorem 1.4. ∎

# 4 Asymptotic behavior for $\eta_k^{0+}(n)$

In this section, we apply Wright's variant of the Hardy–Ramanujan circle method to study the asymptotic behavior of the symmetrized positive moments. Throughout this section, we set $q = e^{2\pi i\tau}$ with $\tau = x + iy \in \mathcal{H}$, where $\mathcal{H}$ denotes the upper half complex plane.

## 4.1 Asymptotic behavior of generating functions

Now, we use these generating functions of symmetrized positive moments of odd ranks to study the asymptotic behavior analytically. For convenience, we define a "false" Appell-type sum by

$$S_r(q) := \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+(2\lfloor \frac{r}{2} \rfloor+3)n+\lfloor \frac{r}{2} \rfloor+1}}{(1-q^{2n+1})^r}. \tag{4.1}$$

For $r \geq 1$, we set

$$F_r(q) := \sum_{n=0}^{\infty} \eta_r^{0+}(n)q^n = \frac{1}{(q^2;q^2)_\infty} S_{r+1}(q).$$

From the factor $\frac{1}{(q^2;q^2)_\infty}$, we see that the dominant poles are at $q = \pm 1$. Since the asymptotic behavior of $\frac{1}{(q^2;q^2)_\infty}$ can be easily understood through modular transformations, the asymptotic behavior of $F_r(q)$ is reduced to study that of $S_{r+1}(q)$ near the dominant poles $q = \pm 1$.

## 4.2 Bounds near the dominate poles

To investigate the asymptotic behavior of $S_r(q)$ at the poles $q = 1$ and $q = -1$, we need the Mittag-Leffler partial fraction decomposition (for example, see [6, Equation (3.1)]) as follows:

$$\left(\frac{e^{\pi i w}}{1-e^{2\pi i w}}\right)^r = \frac{1}{(-2\pi i w)^r} + \sum_{\substack{0<j<r \\ j\equiv r \pmod 2}} \frac{\alpha_j}{(-2\pi i w)^j}$$

$$+ \sum_{\substack{0<j\leq r \\ j\equiv r \pmod 2}} \frac{\alpha_j}{(-2\pi i)^j} \sum_{m\geq 1} (-1)^{mr} \left[\frac{1}{(w-m)^j} + \frac{1}{(w+m)^j}\right], \tag{4.2}$$

where $w \in \mathbb{C}$ and $\alpha_j$ are certain constants. In particular, $\alpha_r = 1$.

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We start with investigating $S_r(q)$ near $q = 1$. Setting $w = (2n+1)\tau$, we see that

$$S_r(q) = \sum_{n=0}^{\infty} (-1)^n q^{3n^2 + \left(2\lfloor \frac{r}{2} \rfloor + 3 - r\right) n + \lfloor \frac{r}{2} \rfloor - \frac{r}{2} + 1} \Bigg[ \frac{1}{(-2\pi i(2n+1)\tau)^r}$$

$$+ \sum_{\substack{0<j<r \\ j \equiv r \pmod 2}} \frac{\alpha_j}{(-2\pi i(2n+1)\tau)^j} \tag{4.3}$$

$$+ \sum_{\substack{0<j\leq r \\ j \equiv r \pmod 2}} \frac{\alpha_j}{(-2\pi i)^j} \sum_{m\geq 1} (-1)^{rm} \left( \frac{1}{((2n+1)\tau - m)^j} + \frac{1}{((2n+1)\tau + m)^j} \right) \Bigg].$$

Now, we first consider the contribution from the first bracketed term in (4.3) for $r \geq 3$ and $|x| \leq y = \frac{1}{4\sqrt{3N}}$. To this end, we define

$$g_j(\tau) := \sum_{n=0}^{\infty} (-1)^n (2n+1)^{-j} q^{3n^2 + (2\lfloor \frac{r}{2} \rfloor + 3 - r)n + \lfloor \frac{r}{2} \rfloor - \frac{r}{2} + 1}.$$

Then the first term of $S_r(q)$ in (4.3) can be expressed as $(-2\pi i\tau)^{-r} g_r(\tau)$. Thus, it is sufficient to study the functions $g_j(\tau)$. For convenience, we set $a := 2\lfloor \frac{r}{2} \rfloor + 3 - r$ and $b := \lfloor \frac{r}{2} \rfloor - \frac{r}{2} + 1$. If $j \geq 1$, then $g_j(\tau)$ is convergent at $\tau = 0$, and

$$g_j(0) = \lim_{\tau \to 0} g_j(\tau) = \beta(j),$$

where $\beta(j)$ is the Dirichlet beta function. Notice that if $j \geq 2$, then $g_j$ is absolutely (and uniformly) convergent for all $|q| \leq 1$, since

$$|g_j(\tau)| \leq \sum_{n=0}^{\infty} \frac{1}{(2n+1)^j} = \frac{2^j - 1}{2^j} \zeta(j).$$

Here, $\zeta(j)$ denotes the Riemann $\zeta$-function. We will apply Taylor's theorem to obtain lower-order asymptotic terms in $g_j$, and make use of the fact that the resulting derivatives can be expressed recursively using

$$\frac{1}{2\pi i} \frac{\partial}{\partial \tau} g_j(\tau) = \frac{3}{4} g_{j-2}(\tau) + \frac{a-3}{2} g_{j-1}(\tau) + \frac{1}{4} g_j(\tau),$$

$$\begin{aligned} \left(\frac{1}{2\pi i}\right)^2 \frac{\partial^2}{\partial \tau^2} g_j(\tau) =& \frac{9}{16} g_{j-4}(\tau) + \frac{3(a-3)}{4} g_{j-3}(\tau) \\ &+ \frac{2a^2 - 18a + 12b + 27}{8} g_{j-2}(\tau) \\ &+ \frac{a-3}{4} g_{j-1}(\tau) + \frac{1}{16} g_j(\tau). \end{aligned}$$

We use Taylor's theorem for $j \geq 6$ and get the following truncated expansion:

$$g_j(\tau) - g_j(0) - g_j'(0)\tau \ll |\tau|^2 \sup_{w \in \mathbb{H}} |g_j''(w)| \ll |\tau|^2 \ll N^{-1}. \tag{4.4}$$

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A simple calculation shows that for $j \geq 6$,

$$g_j(0) = \beta(j), \tag{4.5}$$

and

$$\frac{1}{2\pi i} g_j'(0) = \frac{3}{4}\beta(j-2) + \frac{a-3}{2}\beta(j-1) + \frac{1}{4}\beta(j). \tag{4.6}$$

Additionally, we should individually consider the small value of $j$, that is, $-1 \leq j \leq 1$. Here, we make use of Zagier's technical tool [13, Proposition 3] to study the asymptotic expansions of series such as $g_j$. Bringmann and Mahlburg [6] gave a generalization of Zagier's result.

***Lemma 4.1*** [6, Proposition A.1] *Suppose that $f(t)$ is a smooth function on the positive reals and has the asymptotic expansion*

$$f(t) = \sum_{n=0}^{s} b_n t^n + O(t^{S+1}),$$

*for any $S \geq 0$, as $t \to 0^+$. Assume also that $f(t)$ and all of its derivatives are of rapid decay at infinity and that $I_f := \int_0^\infty f(u)du$ converges. Then we have the following asymptotic expansion:*

$$\sum_{m=0}^{\infty} f((m+a)t) = \frac{I_f}{t} - \sum_{n=0}^{s} b_n \frac{B_{n+1}(a)}{n+1} t^n,$$

*where $B_n(x)$ is the nth Bernoulli polynomial, defined by $\frac{te^{xt}}{e^t-1} = \sum_{n\geq 0} B_n(x)\frac{t^n}{n!}$.*

Now, we aim to prove that $g_j(\tau)(-1 \leq j \leq 1)$ can be uniformly bounded in a neighborhood of $\tau = 0$ with $|x| \leq y$. With the aid of Lemma 4.1, we first deal with the case of $j = 0$.

***Proposition 4.2*** *Suppose that $y = \frac{1}{4\sqrt{3N}}$ and $|x| \leq y$. Then*

$$g_0(\tau) \ll 1 \text{ and } g_{-1}(\tau) \ll 1.$$

**Proof** Observe that

$$g_0(\tau) = \sum_{n=0}^{\infty} (-1)^n q^{3n^2+an+b} = q^{b-\frac{a^2}{12}} \sum_{n=0}^{\infty} (-1)^n q^{3(n+\frac{a}{6})^2}.$$

We separate the odd terms and even terms, and consider only the real part, as the imaginary part can be treated in the same way. We have

$$\begin{aligned}&\mathrm{Re}(g_0(\tau))\\ &= \cos\left(2\pi\left(b-\frac{a^2}{12}\right)x\right)e^{-2\pi y(b-a^2/12)} \sum_{n=0}^{\infty}\left[e^{-24\pi y(n+a/12)^2}\cos\left(24\pi x(n+a/12)^2\right)\right.\end{aligned}$$

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$$- e^{-24\pi y(n+\frac{a+6}{12})^2} \cos\left(24\pi x\left(n+\frac{a+6}{12}\right)^2\right)\Bigg]$$

$$= \cos(2\pi(b-a^2/12)x)e^{-2\pi y(b-a^2/12)} \sum_{n\geq 0}\left(f_{\frac{x}{y}}((n+a/12)\sqrt{y}) - f_{\frac{x}{y}}\left(\left(n+\frac{a+6}{12}\right)\sqrt{y}\right)\right),$$

where

$$f_v(t) := e^{-24\pi t^2}\cos(24\pi v t^2).$$

Note that $f_v(t)$ is an even function, and thus its Taylor series at $t = 0$ only has even powers of $t$ as follows:

$$f_v(t) = 1 - 24\pi t^2 + O_v\left(t^4\right).$$

It is easy to see that

$$I_f := \int_0^\infty f_v(t)dt = \int_0^\infty e^{-24\pi t^2}\cos(24\pi v t^2)dt$$

converges and has a uniform bound, since

$$|I_f| \leq \int_0^\infty e^{-24\pi t^2} dt < \infty.$$

Therefore, we apply Zagier's lemma to $\mathrm{Re}(g_0(\tau))$ and obtain the asymptotic expansion

$$\mathrm{Re}(g_0(\tau)) = \left(y^{-\frac{1}{2}}I_f - B_1\left(\frac{a}{12}\right) + O_{\frac{x}{y}}(y) - \left(y^{-\frac{1}{2}}I_f - B_1\left(\frac{a+6}{12}\right) + O_{\frac{x}{y}}(y)\right)\right)$$
$$\cdot \cos\left(2\pi(b-a^2/12)x\right) e^{-2\pi y(b-a^2/12)}.$$

Since $|\frac{x}{y}| \leq 1$, we see that

$$\mathrm{Re}(g_0(\tau)) \ll 1.$$

The proof of the case $j = -1$ is similar as that of $j = 0$. ■

We turn to deal with the case for $j = 1$.

***Proposition 4.3*** *Suppose that $y = \frac{1}{4\sqrt{3N}}$ and $|x| \leq y$. Then*

$$g_1(\tau) \ll \sqrt{y}.$$

**Proof** We first define

$$\widetilde{g_1}(\tau) = \sum_{n\geq 0}(-1)^n(2n+1)^{-1}q^{3(n+\frac{a}{6})^2}.$$

Then it follows that $g_1(\tau) = q^{b-\frac{a^2}{12}} \cdot \widetilde{g_1}(\tau)$. As above, we split $\widetilde{g_1}(\tau)$ into odd terms and even terms, and consider only the real part, as the imaginary part is treated identically. So we have

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$$\operatorname{Re}(\widetilde{g}_1(\tau)) = \sum_{n\geq 0}\Bigg(\frac{e^{-24\pi y(n+a/12)^2}\cos(24\pi x(n+a/12)^2)}{4n+1} - \frac{e^{-24\pi y\left(n+(a+6)/12\right)^2}\cos(24\pi x(n+(a+6)/12)^2)}{4n+3}\Bigg). \tag{4.7}$$

In order to combine these two terms, we write

$$\frac{1}{4n+1} = \frac{1}{4n+3} + O\left(\frac{1}{n^2}\right).$$

If the big-O term is inserted into (4.7), the sum is absolutely and uniformly convergent, so we may discard this error term without affecting the overall convergence.

Here, we present two trivial but useful estimates:

$$|\cos(x+a) - \cos(x)| \leq \min\{|a|, 2\}, \text{ for } \quad x, a \in \mathbb{R},$$

$$|1 - e^{-x}| \leq \min\{x, 1\}, \quad \text{for} \quad x \geq 0.$$

To apply these bounds, we rewrite the second term of (4.7) in the following way:

$$\begin{aligned}
&\frac{e^{-24\pi y\left(n+\frac{a+6}{12}\right)^2}\cos\left(24\pi x\left(n+\frac{a+6}{12}\right)^2\right)}{4n+1}\\
&= \frac{e^{-24\pi y\left(n+\frac{a}{12}\right)^2}}{4n+1}\left(e^{-24\pi y\left(n+\frac{a+3}{12}\right)}-1\right)\cos\left(24\pi x\left(n+\frac{a+6}{12}\right)^2\right)\\
&\quad + \frac{e^{-24\pi y\left(n+\frac{a}{12}\right)^2}}{4n+1}\left(\cos\left(24\pi x\left(n+\frac{a+6}{12}\right)^2\right) - \cos\left(24\pi x\left(n+\frac{a}{12}\right)^2\right)\right)\\
&\quad + \frac{e^{-24\pi y\left(n+\frac{a}{12}\right)^2}}{4n+1}\cos\left(24\pi x\left(n+\frac{a}{12}\right)^2\right).
\end{aligned} \tag{4.8}$$

The contribution of the first pair of terms in (4.8) to the sum in (4.7) is asymptotically bounded by

$$\begin{aligned}
&\sum_{n\geq 0}\frac{e^{-24\pi y\left(n+\frac{a}{12}\right)^2}}{4n+1}\min\left\{1, 24\pi y\left(n+\frac{a+3}{12}\right)\right\}\\
&\ll y\sum_{0\leq n\leq \frac{1}{24\pi y}-\frac{a+3}{12}} e^{-n^2 y} + \sum_{n\geq \frac{1}{24\pi y}-\frac{a+3}{12}} n^{-1}e^{-n^2 y}\\
&\ll \sqrt{y} + y\sum_{n\geq \frac{1}{24\pi y}-\frac{a+3}{12}} e^{-n^2 y} \ll \sqrt{y}.
\end{aligned}$$

For the second pair of terms in (4.8), we note that $|x| \leq y$ and obtain that

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$$\sum_{n\geq 0}\frac{e^{-24\pi y\left(n+\frac{a}{12}\right)^2}}{4n+1}\min\left\{2,24\pi|x|\left(n+\frac{a+3}{12}\right)\right\}$$
$$\leq \sum_{n\geq 0}\frac{e^{-24\pi y\left(n+\frac{a}{12}\right)^2}}{4n+1}\min\left\{2,24\pi y\left(n+\frac{a+3}{12}\right)\right\}\ll\sqrt{y}.$$

Since the third term in (4.8) cancels the first term in (4.7), and the real part is bounded overall, we complete the proof. ∎

The second bracketed term in (4.3) is a finite summation on $j$. Employing the following simple uniform bound:

$$g_j(\tau)\ll\sum_{n=0}^{\infty}\frac{e^{-Cn^2y}}{(2n+1)^j}\ll\begin{cases}1, & \text{if } j\geq 2,\\ 1+\frac{1}{\sqrt{y}}\ll N^{\frac14}, & \text{if } j=1,\end{cases}\tag{4.9}$$

for some constants $C$, we conclude that the contribution of the second bracketed term is absorbed into the error term.

The final bracketed term in (4.3) is also a finite summation on $j$. Noticing the constraint $|x|\leq y$, we easily deduce that for $m\in\mathbb{Z}\backslash\{0\}$,

$$\frac{1}{(2n+1)\tau+m}\ll\frac{1}{m}.$$

So when $j>1$, we see that

$$\sum_{n=0}^{\infty}(-1)^n q^{3n^2+an+b}\sum_{m\geq 1}(-1)^{mr}\left(\frac{1}{((2n+1)\tau-m)^j}+\frac{1}{((2n+1)\tau+m)^j}\right)$$
$$\ll\sum_{m\geq 1}m^{-j}\sum_{n\geq 0}e^{-6\pi n^2y}\ll y^{-1/2}\ll N^{\frac14}.\tag{4.10}$$

Moreover, for $j=1$, we have

$$\frac{1}{(2n+1)\tau-m}+\frac{1}{(2n+1)\tau+m}\ll\frac{(2n+1)\tau}{m^2}.$$

Thus, the corresponding contribution can be bounded by

$$|\tau|\sum_{n\geq 0}(2n+1)e^{-6\pi n^2y}\ll N^{-\frac12}y^{-1}\ll 1.\tag{4.11}$$

As a consequence, combining equations (4.4)–(4.6) and (4.9)–(4.11) with Propositions 4.2 and 4.3, we could determine the first terms in the asymptotic expansion of $S_r(q)$ near $q=1$.

***Proposition 4.4*** *Assume $r\geq 3$, $|x|\leq y=\frac{1}{4\sqrt{3N}}$. As $N\to\infty$, we have*

$$S_r(q)-c_r(-2\pi i\tau)^{-r}+d_r(-2\pi i\tau)^{-r+1}\ll N^{\frac{r}{2}-1}$$

*with*

$$c_r=\beta(r)\quad \textit{for } r\geq 3,$$
$$d_r=\frac34\beta(r-2)+\frac{a-3}{2}\beta(r-1)+\frac14\beta(r)\quad \textit{for } r\geq 3.$$

***Corollary 4.5*** *Assume that $r \geq 2$, $y = \frac{1}{4\sqrt{3N}}$, and $|x| \leq y$. As $N \to \infty$, we have*

$$F_r(q) - c^*_{r+1}(-2\pi i\tau)^{-\frac{1}{2}-r} e^{\frac{\pi i}{24\tau}} - d^*_{r+1}(-2\pi i\tau)^{\frac{1}{2}-r} e^{\frac{\pi i}{24\tau}} \ll N^{\frac{2r-3}{4}} e^{\pi\sqrt{\frac{N}{12}}},$$

*where*

$$c^*_r := \frac{c_r}{\sqrt{\pi}} \quad \text{and} \quad d^*_r := \frac{1}{\sqrt{\pi}}\left(-\frac{c_r}{12} - d_r\right).$$

**Proof** Recall the modular inversion formula for Dedekind's eta-function, which states that

$$\eta(-\frac{1}{\tau}) = \sqrt{-i\tau}\eta(\tau).$$

Then we have

$$\frac{1}{(q^2;q^2)_\infty} = \sqrt{-2i\tau} e^{\frac{2\pi i}{24}(2\tau + \frac{1}{2\tau})}\left(1 + O(e^{-\pi\sqrt{12N}})\right) \tag{4.12}$$

$$= \sqrt{-2i\tau} e^{\frac{\pi i}{24\tau}}\left(1 + \frac{\pi i\tau}{6} + O(N^{-1})\right), \quad \text{for} \quad |x| \leq y. \tag{4.13}$$

Combining this with Proposition 4.4, we find that

$$\begin{aligned}\frac{1}{(q^2;q^2)_\infty} S_{r+1}(q) &= \sqrt{-2i\tau} e^{\frac{\pi i}{24\tau}}\left(1 - \frac{-2\pi i\tau}{12} + O(N^{-1})\right) \\ &\quad \cdot \left(c_{r+1}(-2\pi i\tau)^{-r-1} - d_{r+1}(-2\pi i\tau)^{-r} + O(N^{\frac{r+1}{2}-1})\right) \\ &= c^*_{r+1}(-2\pi i\tau)^{-\frac{1}{2}-r} e^{\frac{\pi i}{24\tau}} + d^*_{r+1}(-2\pi i\tau)^{\frac{1}{2}-r} e^{\frac{\pi i}{24\tau}} + O\left(N^{\frac{2r-3}{4}} e^{\pi\sqrt{\frac{N}{12}}}\right).\end{aligned}$$

∎

Now, we turn to investigate the asymptotic behavior of $F_r(q)$ near $q = -1$.

***Proposition 4.6*** *Assume that $r \geq 1$, $y = \frac{1}{4\sqrt{3N}}$, and $|x - \frac{1}{2}| \leq y$. As $N \to \infty$, we have*

$$S_r(q) \ll 1.$$

**Proof** By setting $z := \tau - \frac{1}{2} = x - \frac{1}{2} + iy$ and $Q := e^{2\pi iz} = -q$, we derive that

$$\begin{aligned}S_r(q) = S_r(-Q) &= \sum_{n\geq 0}(-1)^n \frac{(-Q)^{3n^2+(2\lfloor\frac{r}{2}\rfloor+3)n+\lfloor\frac{r}{2}\rfloor+1}}{(1-(-Q)^{2n+1})^r} \\ &= \sum_{n\geq 0}(-1)^{n+\lfloor\frac{r}{2}\rfloor+1} \frac{Q^{3n^2+(2\lfloor\frac{r}{2}\rfloor+3)n+\lfloor\frac{r}{2}\rfloor+1}}{(1+Q^{2n+1})^r}.\end{aligned}$$

We recall that $a = 2\lfloor\frac{r}{2}\rfloor + 3 - r$ and $b = \lfloor\frac{r}{2}\rfloor - \frac{r}{2} + 1$, and then obtain that

$$S_r(q) = \sum_{n\geq 0}(-1)^{n+\lfloor\frac{r}{2}\rfloor+1} Q^{3n^2+an+b} \frac{Q^{(n+\frac{1}{2})r}}{(1+Q^{2n+1})^r}.$$

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By setting $w := (2n+1)z + \frac{1}{2}$ in (4.2), we find that

$$\begin{aligned}&\frac{i^r e^{\pi i(2n+1)rz}}{\left(1+e^{2\pi i(2n+1)z}\right)^r}\\ &\quad= \sum_{\substack{0<j\le r\\ j\equiv r \pmod 2}} \frac{\alpha_j}{(-\pi i)^j\left((4n+2)z+1\right)^j} + \sum_{\substack{0<j\le r\\ j\equiv r \pmod 2}} \frac{\alpha_j}{(-\pi i)^j}\sum_{m\ge 1}(-1)^{rm}\\ &\qquad\times\left(\frac{1}{((4n+2)z+1-2m)^j}+\frac{1}{((4n+2)z+1+2m)^j}\right).\end{aligned}$$

We first consider the contribution of the final term. It is clear that for $m \in \mathbb{Z}$ (note that $z = (x-\frac{1}{2}) + iy$ and $|x-\frac{1}{2}| \le y$),

$$\frac{1}{(4n+2)z+2m+1} \ll \frac{1}{2m+1}.$$

Notice that for $j > 1$,

$$\sum_{m\ge 1}(-1)^m\left(\frac{1}{(1-2m)^j}+\frac{1}{(1+2m)^j}\right) \ll \sum_{m\ge 1} m^{-j} \ll 1.$$

If $r$ is odd, then, for the case $j = 1$, we have

$$\begin{aligned}&\frac{1}{(4n+2)z+1} + \sum_{m\ge 1}(-1)^m\left(\frac{1}{(4n+2)z+1-2m}+\frac{1}{(4n+2)z+1+2m}\right)\\ &= \sum_{m\ge 1}(-1)^m\left(\frac{1}{(4n+2)z+1-2m}-\frac{1}{(4n+2)z-1+2m}\right)\\ &= \sum_{m\ge 1}(-1)^m\frac{4m-2}{(4n+2)^2z^2-(2m-1)^2}.\end{aligned}$$

By splitting this fraction into two terms:

$$\frac{4m-2}{(4n+2)^2z^2-(2m-1)^2} = (4m-2)\left(\frac{1}{(4n+2)^2z^2-(2m-1)^2}+\frac{1}{(2m-1)^2}\right) - \frac{2}{2m-1},$$

and noting that

$$\begin{aligned}&\left|\frac{1}{(4n+2)^2z^2-(2m-1)^2}+\frac{1}{(2m-1)^2}\right|\\ &\quad= \left|\frac{(4n+2)^2z^2}{((4n+2)^2z^2-(2m-1)^2)(2m-1)^2}\right| \le \frac{2(4n+2)^2y^2}{(2m-1)^4}.\end{aligned}$$

Therefore, we derive that

$$\begin{aligned}&\sum_{m\ge 1}(-1)^m\frac{4m-2}{(4n+2)^2z^2-(2m-1)^2}\\ &\quad\le 4(4n+2)^2y^2\left|\sum_{m\ge 1}\frac{1}{(2m-1)^3}\right| + \left|\sum_{m\ge 1}\frac{(-1)^k}{2k-1}\right|\\ &\quad\le \frac{\pi^3(2n+1)^2y^2}{2}+\frac{\pi}{4}.\end{aligned}$$

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Moreover, we observe that

$$\sum_{\substack{1<j\leq r\\ j\equiv r(\mathrm{mod}\ 2)}} \frac{\alpha_j}{(-\pi i)^j\left((4n+2)z+1\right)^j} \ll 1.$$

Therefore, we have

$$\frac{i^r e^{\pi i(2n+1)rz}}{\left(1+e^{2\pi i(2n+1)z}\right)^r} \ll 1+\frac{\pi^3(2n+1)^2y^2}{2}+\frac{\pi}{4}+1 \ll \frac{\pi^3(2n+1)^2y^2}{2}.$$

Thus, by using Lemma 4.1, we find that the contribution of the final term to $S_r(q)$ can be bound by

$$S_r(q)=\sum_{n\geq 0}(-1)^{n+\lfloor\frac{r}{2}\rfloor+1}Q^{3n^2+an+b}\frac{Q^{(n+\frac{1}{2})r}}{(1+Q^{2n+1})^r} \ll \sum_{n\geq 1}(2n+1)^2y^2e^{-6\pi y n^2} \ll 1,$$

which completes the proof. ∎

***Corollary 4.7*** *Assume that $r\geq 0$, $y=\frac{1}{4\sqrt{3N}}$, and $|x-1/2|\leq y$. As $N\to\infty$, we have*

$$F_r(e^{2\pi i\tau}) \ll N^{-\frac{1}{4}}e^{\pi\sqrt{\frac{N}{12}}}.$$

**Proof** We note that

$$\frac{1}{(q^2;q^2)_\infty}=\frac{1}{(Q^2;Q^2)_\infty}=\sqrt{-2iz}e^{\frac{2\pi i}{24}(2z+\frac{1}{2z})}\left(1+O(e^{-\pi\sqrt{12N}})\right), \quad \text{for} \quad |x-\frac{1}{2}|\leq y,$$

and the proof follows. ∎

## 4.3 Bounds away from the dominate poles

Finally, there is a position to estimate the asymptotic behavior of $F_r(q)$ away from the dominate poles $q=\pm 1$.

***Proposition 4.8*** *For $r\geq 0$ and $y>0$ with $y\leq |x|\leq \frac{1}{2}-y$,*

$$S_r(q) \ll y^{-r-1/2}.$$

**Proof** From the Taylor expansion, we see that

$$\frac{1}{1-|q|}=\frac{1}{1-e^{-2\pi y}}=O(y^{-1}).$$

So we have

$$|S_r(q)|\leq \frac{|q|^{\lfloor\frac{r}{2}\rfloor+1}}{(1-|q|)^r}\sum_{n\geq 0}|q|^{3n^2+(2\lfloor\frac{r}{2}\rfloor+3)n} \ll \frac{1}{y^r}\cdot y^{-1/2}=y^{-r-1/2},$$

where the final summation follows through a comparison with a Gaussian integral. ∎

***Corollary 4.9*** *Assume that $r\geq 0$, $y=\frac{1}{4\sqrt{3N}}$, and $y\leq |x|\leq\frac{1}{2}-y$. As $N\to\infty$, we have*

$$F_r(q) \ll N^{\frac{r+1}{2}}e^{\sqrt{3N}\left(\frac{\pi}{6}-\frac{1}{\pi}\left(1-\frac{1}{\sqrt{2}}\right)\right)}.$$

**Proof** From [8, Equation (4.4)], we know that

$$\left|\frac{1}{(q^2;q^2)_\infty}\right| \ll \sqrt{2y}\exp\left[\frac{1}{y}\left(\frac{\pi}{24}-\frac{1}{4\pi}\left(1-\frac{1}{\sqrt{2}}\right)\right)\right].$$

Combined with Proposition 4.8, this gives the claimed expression. ∎

## 4.4 The circle method

Now, we apply Wright's variant of the Hardy–Ramanujan circle method. Cauchy's theorem gives an integral representation for the coefficients of $F_r(q)$, namely,

$$\eta_r^{0+}(N) = \frac{1}{2\pi i}\int_{\mathcal{C}} \frac{F_r(q)}{q^{N+1}}dq = \int_{-\frac{1}{2}}^{\frac{1}{2}} F_r(e^{2\pi i x-\frac{\pi}{2\sqrt{3N}}})e^{-2\pi i N x+\frac{\pi\sqrt{N}}{2\sqrt{3}}}dx, \tag{4.14}$$

where the contour is the counterclockwise traversal of the circle $\mathcal{C} := \{|q| = e^{-\frac{\pi}{2\sqrt{3N}}}\}$. We sperate (4.14) into three ranges

$$\begin{aligned}\eta_r^{0+}(N) =& \int_{|x|\le y} F_r(e^{2\pi i x-\frac{\pi}{2\sqrt{3N}}})e^{-2\pi i N x+\frac{\pi\sqrt{N}}{2\sqrt{3}}}dx\\ &+\int_{y\le|x|\le\frac{1}{2}-y} F_r(e^{2\pi i x-\frac{\pi}{2\sqrt{3N}}})e^{-2\pi i N x+\frac{\pi\sqrt{N}}{2\sqrt{3}}}dx\\ &+\int_{|x-\frac{1}{2}|\le y} F_r(e^{2\pi i x-\frac{\pi}{2\sqrt{3N}}})e^{-2\pi i N x+\frac{\pi\sqrt{N}}{2\sqrt{3}}}dx\\ =&:\mathcal{J}_1+\mathcal{J}_2+\mathcal{J}_3,\end{aligned}$$

where $y = \frac{1}{4\sqrt{3N}}$.

Next, we will show that the integral $\mathcal{J}_1$ contributes the main term and the integrals $\mathcal{J}_2$ and $\mathcal{J}_3$ are absorbed in the error term.

We first evaluate the integral $\mathcal{J}_1$ and aim to rewrite the integral $\mathcal{J}_1$ in terms of Bessel functions up to an error term. To this end, we need to introduce an auxiliary function, which is defined by Wright [12],

$$P_s(u) := \frac{1}{2\pi i}\int_{1-Mi}^{1+Mi} v^s e^{u(v+\frac{1}{v})}dv,$$

where $M > 0$ is fixed and $u \in \mathbb{R}^+$. Adopting the similar approach of [6, Lemma 4.2], one may find that the auxiliary function $P_s(u)$ can be rewritten in terms of the $I$-Bessel function up to an error term.

***Lemma 4.10*** *Assume that $u = \pi\sqrt{\frac{N}{6}}$. As $N \to \infty$, we have*

$$P_s(u) - I_{-s-1}(2u) \ll e^{\frac{3}{2}u}.$$

With the help of this lemma, we can evaluate the integral $\mathcal{J}_1$.

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***Proposition 4.11*** *Assume that $r \geq 2$. As $N \to \infty$,*

$$\mathcal{J}_1 - c^*_{r+1}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{1}{2}-r} I_{r-\frac{1}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) + d^*_{r+1}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{3}{2}-r} I_{r-\frac{3}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) \ll N^{\frac{2r-5}{4}} e^{\pi\sqrt{\frac{N}{3}}}.$$

**Proof** By making the change of variables $v = 1 - i4\sqrt{3N}x$, we arrive at

$$\begin{aligned}
\mathcal{J}_1 = \int_{1-i}^{1+i} \frac{1}{i4\sqrt{3N}}\Bigg(c^*_{r+1}\left(\frac{\pi v}{2\sqrt{3N}}\right)^{-\frac{1}{2}-r} e^{\frac{\pi\sqrt{N}}{2\sqrt{3}v}} \\
+ d^*_{r+1}\left(\frac{\pi v}{2\sqrt{3N}}\right)^{\frac{1}{2}-r} e^{\frac{\pi\sqrt{N}}{2\sqrt{3}v}} + O\left(N^{\frac{2r-3}{4}} e^{\pi\sqrt{\frac{N}{12}}}\right)\Bigg) e^{\frac{\pi\sqrt{N}v}{2\sqrt{3}}}\,\mathrm{d}v \\
= \frac{1}{i4\sqrt{3N}}\int_{1-i}^{1+i}\left[c^*_{r+1}\left(\frac{\pi v}{2\sqrt{3N}}\right)^{-\frac{1}{2}-r} e^{\frac{\pi\sqrt{N}}{2\sqrt{3}}(v+v^{-1})} + d^*_{r+1}\left(\frac{\pi v}{2\sqrt{3N}}\right)^{\frac{1}{2}-r} e^{\frac{\pi\sqrt{N}}{2\sqrt{3}}(v+v^{-1})}\right]\mathrm{d}v \\
+ O\left(N^{\frac{2r-5}{4}} e^{\pi\sqrt{\frac{N}{3}}}\right) \\
= c^*_{r+1}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{1}{2}-r} P_{-\frac{1}{2}-r}\left(\pi\sqrt{\frac{N}{12}}\right) + d^*_{r+1}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{3}{2}-r} P_{\frac{1}{2}-r}\left(\pi\sqrt{\frac{N}{12}}\right) \\
+ O\left(N^{\frac{2r-5}{4}} e^{\pi\sqrt{\frac{N}{3}}}\right).
\end{aligned}$$

By applying Lemma 4.10, we conclude that

$$\begin{aligned}
\mathcal{J}_1 = c^*_{r+1}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{1}{2}-r} I_{r-\frac{1}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) + d^*_{r+1}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{3}{2}-r} I_{r-\frac{3}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) \\
+ O\left(N^{\frac{2r-5}{4}} e^{\pi\sqrt{\frac{N}{3}}}\right).
\end{aligned}$$

■

We now turn to the integrals $\mathcal{J}_2$ and $\mathcal{J}_3$ and find that they are exponentially smaller than the main asymptotic terms.

***Proposition 4.12*** *As $N \to \infty$, we have*

$$\mathcal{J}_2 \ll N^{\frac{r+1}{2}} e^{\pi\sqrt{\frac{N}{3}} - \frac{\sqrt{3N}}{\pi}\left(1-\frac{\sqrt{2}}{2}\right)} \quad \textit{and} \quad \mathcal{J}_3 \ll N^{-\frac{3}{4}} e^{\pi\sqrt{\frac{N}{3}}}.$$

**Proof**

$$\begin{aligned}
|\mathcal{J}_2| \leq \int_{\frac{1}{4\sqrt{3N}} \leq |x| \leq \frac{1}{2} - \frac{1}{4\sqrt{3N}}} \left| F_r\left(e^{2\pi i x - \frac{\pi}{2\sqrt{3N}}}\right) e^{-2\pi i N x + \frac{\pi\sqrt{N}}{2\sqrt{3}}}\right| \mathrm{d}x \\
\ll N^{\frac{r+1}{2}} e^{\sqrt{3N}\left(\frac{\pi}{6} - \frac{1}{\pi}\left(1-\frac{1}{\sqrt{2}}\right)\right)} e^{\frac{\pi\sqrt{N}}{2\sqrt{3}}} = N^{\frac{r+1}{2}} e^{\pi\sqrt{\frac{N}{3}} - \frac{\sqrt{3N}}{\pi}\left(1-\frac{\sqrt{2}}{2}\right)}.
\end{aligned}$$

For the integral $\mathcal{J}_3$, we have

$$
\begin{aligned}
|\mathcal{J}_3| &= \int_{|x-\frac{1}{2}|\leq\frac{1}{4\sqrt{3N}}}\left|F_r\left(e^{2\pi ix-\frac{\pi}{2\sqrt{3N}}}\right)e^{-2\pi iNx+\frac{\pi\sqrt{N}}{2\sqrt{3}}}\right|dx\\
&\ll N^{-\frac{1}{4}}e^{\pi\sqrt{\frac{N}{12}}}\int_{|x-\frac{1}{2}|\leq\frac{1}{4\sqrt{3N}}}e^{\frac{\pi\sqrt{N}}{2\sqrt{3}}}dx\\
&\ll N^{-\frac{3}{4}}e^{\pi\sqrt{\frac{N}{3}}}.
\end{aligned}
$$

■

Combining with Propositions 4.11 and 4.12, we obtain the following asymptotic formula.

***Theorem 4.13*** *Assume that $r \geq 2$. As $N \to \infty$,*

$$
\eta_r^{0+}(N) - c_{r+1}^{*}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{1}{2}-r} I_{r-\frac{1}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) - d_{r+1}^{*}\left(\frac{\pi}{2\sqrt{3N}}\right)^{\frac{3}{2}-r} I_{r-\frac{3}{2}}\left(\pi\sqrt{\frac{N}{3}}\right) \ll N^{\frac{2r-5}{4}}e^{\frac{\pi\sqrt{N}}{\sqrt{3}}}.
$$

This completes the proof of Theorem 1.5.

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