RUIN PROBABILITY DURING A FINITE TIME INTERVAL

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This paper was inspired by comments by H. L. Seal in a series of lectures given to the Actuaries Club in New York and by a paper of his recently published in the Swiss Actuarial Journal (Seal, 1972 [6]). In his lectures he showed that the probability $U(w, t)$ that a risk reserve at every epoch τ, where $0 < \tau \leqslant t$ will be non negative when the initial risk reserve is w is related to $F(w + \overline{1 + \eta} \cdot t, t)$, the probability that the aggregate claim outgo through epoch t does not exceed $w + \overline{1 + \eta} \cdot t$ by the relationship

$$U(w, t) = F(w + \overline{1 + \eta}\, t, t) - (1 + \eta) \int_0^t U(0, t - \tau) f(w + \overline{1 + \eta} \cdot \tau, \tau)\, d\tau \quad (1)$$

where η is the security loading and $f(x, t) = (\partial/\partial x)\, F(x, t)$.

It is assumed that the d.f. $F(x, t)$ is differentiable eith regard to x with a possible exception at the point $x = 0$.

Using an extension of the "ballot theorem" in Chapter III of Feller (1968 [4]) he showed that $U(0, t) = \frac{1}{(1 + \eta)t} \int_0^{(1+\eta)t} F(x, t) dx$ and observed that if numerical values of $F(x, t)$ were available values of $U(w, t)$ could be computed.

His paper in the Swiss Journal applied this technique to the Poisson/Exponential case and provided some numerical results obtained by quadrature. The formal simplicity of relation (1) suggested that it might be worth while investigating the problem in terms of the moments of the various functions with the object of finding approximations to $U(w, t)$ which could be useful in practice, particularly where the large scale computing facilities required for the quadrature were not available and under more general assumptions regarding the claim process. Furthermore, the numerical values provided by Seal formed a useful control at various stages. It may first be noted that the moments of the distribution of total claims about the mean in the Poisson/Exponential case are:

$$m_1 = t,\ \mu_2 = 2t,\ \mu_3 = 6t \text{ and } \mu_4 = 24t + 12t^2$$

https://doi.org/10.1017/S0515036100011193 Published online by Cambridge University Press

from which it follows that $\beta_1 = 4.5t^{-1}$ and $\beta_2 = 3 + 6t^{-1}$. These values of β_1 and β_2 are fairly close to the Pearson type III (Gamma function) values—(for a β_2 value of $3 + 6t^{-1}$ the β_1 value would be $4t^{-1}$) and a useful starting approximation is to assume that $F(x, t)$ can be approximated by a type III curve with the parameter $p = (4/\beta_1) - 1 = 8t - 1$. The approximation will clearly be worst for low values of t, apart from the error from ignoring the discontinuity at $t = 0$, but will improve as t increases. A few values for $t = 10$ give some idea of the closeness of the approximation:

x_0	$F(x_0, 10)$	$I(\sqrt{p+1} + x_0, p)$	$p = 7.\dot{8}$
−2	.00234	.00371	
−1	.15470	.15274	
0	.54489	.54461	
1	.84384	.84499	
2	.96236	.96248	
3	.99308	.99290	
4	.99897	.99888	
5	.99987	.99984	

These are, of course, values of the "non-ruin" probability and normally the values in the lower part of the table are those required in practical conditions.

A more useful model is however the Polya case in which the parameter of the Poisson distribution is assumed to be a random variable, distributed in gamma form. In this case the cumulants of the total claim distribution are:

$$\kappa_1 = t,\ \kappa_2 = tm_2 + \frac{t^2}{k},\ \kappa_3 = tm_3 + \frac{3t^2m_2}{k} + \frac{2t^3}{k^2},\ \kappa_4 = tm_4 + \\ + \frac{3t^2m_2^2}{k} + \frac{4t^2m_3}{k} + \frac{12t^3m_2}{k^2} + \frac{6t^4}{k^4}$$

where k is the dispersion coefficient of the claim frequency and the mean claim is the basic unit. If k is small, i.e. wide dispersion, and t not too small these cumulants are dominated by the last terms and we find $\beta_1 \sim (4/k)$, $\beta_2 \sim 3 + (6/k)$, again the values for a type III tribution and, incidentally approximating an exponential distribution when $k \sim 1$. Thus the type III could be expected to be a useful approximation in the Polya case.

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If, instead of a single negative exponential term for the claim distribution, we substitute the sum of a series of negative exponential terms, the cumulants become more complicated as they involve the convolution of the component terms but here again the type III would seem to be a useful approximation. This feature has, of course, been well known for some time—see e.g. Bohman and Esscher, 1964 [2].

Thus it would seem that in the generalised case where the claim frequency follows a Polya distribution and the claim distribution is a practical case defined by its moments, a reasonable assumption would be that $F(x, t)$ can be represented by an incomplete gamma function so that a reasonable approximation is available for the first term on the R.H.S. of relation (1).

The next step is to consider the calculation of $U(o, t) =$

$$= \frac{1}{(1+\eta)} \int_0^{(1+\eta)t} F(x, t)\, dx.$$

By noting the relation $\int_0^b I(u, p)\, du = b\, I(u, p) - \sqrt{p+1}$
$I\left(b\frac{p+1}{\sqrt{p+2}}, p+1\right)$ it will be found that $U(o, t)$ can be expressed in terms of 4 incomplete gamma functions. Two of these arise from the lower limit ($x = 0$) and can be ignored. A few representative values were calculated and found to be in close agreement with Seal's calculation. However, our ultimate object is the second term in (1) and the calculations of many values of $U(o, w)$ would be laborious. Accordingly, noting that this term is in the form of a convolution integral it was decided to try and find expressions for the moments of $U(o, t)$ and $f(w + \overline{1+\eta} \cdot t, t)$.

However, since $U(o, t)$ has a limit $\eta/(1+\eta)$ when $t \to \infty$, it is necessary to consider $U(o, t) - \eta/(1+\eta)$ and, making the lower limit zero, this can be shown to have the value

$$\frac{\sqrt{2}}{1+\eta}\sqrt{\dot{8}}\left\{\frac{1}{\Gamma(y)}\int_0^{(1+.75\eta)y} e^{-z}z^{y-1}dz - \frac{1}{\Gamma(y+1)}\int_0^{(1+.75\eta)y} e^{-z}z^{y}dz\right\} +$$

$$+ \frac{\eta}{1+\eta} \cdot \frac{1}{\Gamma(y)} \int_{(1+.75\eta)y}^{\infty} e^{-z}z^{y-1}dz \quad \text{where } y = .\dot{8}t \qquad (2)$$

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By expanding the integral in an Euler-Maclaurin series, noting that the terms involving the successive differential coefficients were relatively small and using the expansion (see, e.g. Bromwich 1947 [3], p. 160)

$$e^{ax} = 1 + ay + \frac{a(a+2b)}{2!} y^2 + \frac{a(a+3b)^2}{3!} y^2 + \ldots \text{where } y = xe^{-bx}$$

it was found that $\int_0^\infty \left(U(o, t) - \frac{\eta}{1+\eta} \right) dt$ was approximately

$$\frac{1.5}{(1+\eta)(1-x)} \tag{3}$$

where x is found from $x = e^{\beta(x-1)}$ and $\beta = 1 + .75\eta$

If $\int_0^\infty t^r \left\{ U(o, t) - \frac{\eta}{1+\eta} \right\} dt$ be denoted by S_r then it can be shown that $S_r = \frac{1.125x}{1-\beta x} \frac{d}{dx} S_{r-1}$ and the moments of $\left(U(o, t) - \frac{\eta}{1+\eta} \right)$ thus determined.

The expressions rapidly become complicated but the first few are as follows:

$$S_0 = \frac{1.5}{(1+\eta)(1-x)}$$

$$S_1 = \frac{1.125 . 1.5x}{(1+\eta)(1-x)^2(1-\beta x)}$$

$$S_2 = \frac{(1.125)^2 \, 1.5x(1 + x - 2\beta x^2)}{(1+\eta)(1-x)^3(1-\beta x)^3}$$

$$S_3 = \frac{(1.125)^3 \, 1.5x(1 + 4x + 2\beta x + x^2 - 10\beta x^2 - 4\beta x^3 + 6\beta^2 x^4)}{(1+\eta)(1-x)^4(1-\beta x)^5}$$

For $\eta = .1$, $x = .8638$ and the moment functions derived from the foregoing are mean $= 99.908$, $\sigma = 178.94$, $\beta_1 = 15.409$, $\beta_2 = 27.251$.

The β_1, β_2 values are appropriate for a Pearson Type VI, although they are close to the Type III values. However, both of these curves

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start at a small positive value of t so that representation for small t would be poor. The type III curve from these moments is .01713 $\left(1+\frac{x}{91.171}\right)^{-.74041} e^{-.0028473x}$, x measured from 99.908, which leads to the following values for $U(o, t)$

t	Type III	True
20	.1682	.1976
50	.1284	.1264
100	.1100	.1080
1000	.0911	.0910

The Type III starts with an infinite ordinate at about $t = 9$ whereas the $U(0, t)$ curve starts with a zero ordinate at $t = 0$. For the higher values of t the representation is fairly close but other methods of estimating the curve would be needed for low values. These values are, of course, not needed by the present approach.

The next stage is to find the moments (with respect to t) of $f(w + \overline{1+\eta}t, t)$ where

$$f = \frac{e^{-z} z^{.8t-1}}{\Gamma(.8t)} \text{ and } z = .8t + \frac{\sqrt{.8t}}{\sqrt{2t}}(w + \eta\, t).$$

Using the same method as for $\left(U(0, t) - \frac{\eta}{1+\eta}\right)$ it can be shown that approximately

$$\eta \int_0^\infty f(w + \overline{1+\eta}t, t)\, dt = I_0 = .75\eta \frac{e^{a(x-1)}}{1 - \beta x} .x \text{ where } a = 6w \quad (4)$$

and

$$\eta \int_0^\infty t^r f(.)\, dt = I_r = \frac{1.125x}{1-\beta x} \frac{d}{dx} I_{r-1}$$

Thus

$$I_1 = \left(\frac{1}{(1-\beta x)^2} + \frac{a}{\beta}\frac{1}{(1-\beta x)} - \frac{a}{\beta}\right) 1.125\, I_0 \text{ etc.}$$

Now the function whose value we are seeking is the second term on the R.H.S. of relation (1) which may be written as

$$(1+\eta)\left\{\int_0^t f(\cdot)\left(U_{(0,\, t-\tau)} - \frac{\eta}{1+\eta}\right) d\tau + \frac{\eta}{1+\eta}\int_0^t f(\cdot)\, d\tau\right\} \quad (5)$$

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We know the moments of the second term and we can find the appropriate values of the moments of the first term by convolution, i.e. from the product of the two moment generating functions, which are both known. These can be added to obtain the moments of the whole expression (5).

Calculations were made using $w = 10$ which led to the following result:

$$(1 + \eta) \left\{ \int_0^\infty f(\cdot) \left(U_{(0,\, t-\tau)} - \frac{\eta}{1 + \eta} \right) d\tau + \frac{\eta}{1 + \eta} \int_0^\infty f(\cdot)\, d\tau \right\} =$$

$$= \cdot 75 \eta \frac{e^{a(x-1)} x}{1 - \beta x} = .36589$$

mean $= 2.0116 . 10^2$
$\sigma = 2.761 . 10^2$
$\beta_1 = 9.323$
$\beta_2 = 16.287$

These are the moments of a Pearson Type I curve, but close to a type III. The final stage is to find values of the distribution function for values of t, having given these moments. The values of β_1 and β_2 are within the range of the tables of percentage points calculated by Amos (1971) but the results will again be poor for low values of t because the type I curve starts with an infinite ordinate at a small positive value of t. A Type III curve (with a negative value of p) could be used, but interesting values are crowded together at a very inconvenient part of the tabulated values in Pearson's tables of the Incomplete Gamma function.

For $\sqrt{\beta_1} = 3.0$ and $\beta_2 = 16.4$ Amos gives the following values:

p	.0000	.0010	.0025	.0050	.0100	.0250	0.500	.1000	.2500	.500
	—.6617	—6617	—6617	—6617	—6617	—6615	—6606	—6559	—6130	—397
	291.6617	7.1421	5.9025	4.9746	4.0585	2.8737	2.0071	1.1802	.1924	

x being measured in units of standard deviation from the mean.

This curve starts at $t = 18$ approximately. By interpolation for selected values of t the following values are found for $U(w, t)$ where $F(\cdot)$ is calculated from the type III approximation and the true values of U are interpolated from Seal's values of $U(10, t)$:

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t	$F(.)$	adj	$U_{(10,-)}$	True Value
20	.956	.037	.919	.918
50	.921	.135	.786	.816
100	.911	.194	.717	.738
200	.921	.753	.678	.681
1000	.982	.357	.625	.634

The accuracy is sufficient for many practical purposes so that the primary object of this paper is achieved.

I am conscious that paper is very untidy and that there are a number of directions in which improvement is possible or further research is indicated. Probably the most untidy aspect is the inadequacy of the Pearson system to cope with the distribution for low values of t. There is some indication that a functional form $t^{\alpha}(t + a)^{(\alpha+r)} e^{-\beta t}$, i.e. a confluent hypergeometric function, would be suitable but the simplicity of the numerical inversion using Amos' table would be lost. The evaluation of the integrals is also incomplete, the form of the answers suggesting that there is an approach via. the calculus of residues. Finally, the moments of the "adjustment" terms derived numerically from the moment generating functions may possibly be obtained in a more direct fashion.

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