Bull. Aust. Math. Soc. **95** (2017), 209–213
doi:10.1017/S0004972716000630

FIXED ELEMENTS OF NONINJECTIVE ENDOMORPHISMS OF POLYNOMIAL ALGEBRAS IN TWO VARIABLES

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(Received 8 June 2016; accepted 25 June 2016; first published online 20 October 2016)

Abstract

Let A_2 be a free associative algebra or polynomial algebra of rank two over a field of characteristic zero. The main results of this paper are the classification of noninjective endomorphisms of A_2 and an algorithm to determine whether a given noninjective endomorphism of A_2 has a nontrivial fixed element for a polynomial algebra. The algorithm for a free associative algebra of rank two is valid whenever an element is given and the subalgebra generated by this element contains the image of the given noninjective endomorphism.

2010 *Mathematics subject classification*: primary 13B10; secondary 16W20.

Keywords and phrases: fixed element, free associative algebra, polynomial algebra, retract.

1. Introduction

In this paper, K always denotes a field of characteristic zero. Let A_n be a free associative algebra or polynomial algebra of rank n over K. Automorphisms (endomorphisms) always mean K-automorphisms (K-endomorphisms). As in the case of other free objects, an element $p \in A_n$ is called a test element if every endomorphism of A_n fixing p is an automorphism. A subalgebra R of A_n is called a retract if there is an idempotent endomorphism π ($\pi^2 = \pi$) of A_n (called a retraction or a projection) such that $\pi(A_n) = R$. A proper retract means a retract that is different from K and A_n. Test elements and retracts of groups and other algebras are defined in a similar way. For more details, we refer the reader to [6].

A test element is not contained in any proper retract for any algebra or group since the corresponding noninjective idempotent endomorphism is not an automorphism. The converse, that an element which is not contained in any proper retract is a test element, is proved by Turner [14] for free groups, by Mikhalev and Zolotykh [10] and Mikhalev and Yu [8, 9] for free Lie algebras and free Lie superalgebras, by Mikhalev *et al.* [7] for free nonassociative algebras and by Gong and Yu [5] for free associative algebras and polynomial algebras of rank two. This means that the elements of these free algebras or free groups are exactly divided into two parts: the set of test elements and the set of elements in proper retracts.

The research of Yueyue Li was supported by NSF of China (Grant No. 11371165).

A proper retract of A_2 has been characterised in [13]. If R is a proper retract of A_2, then $R = K[r]$ for some $r \in A_2$ and there exists an automorphism α of A_2 such that $\alpha(r) = x + \omega(x, y)$, where $\omega(x, y)$ is contained in the ideal of A_2 generated by y. The following result was proved by Gong and Yu.

Theorem 1.1 [5, Theorem 1.4]. *An element $p(x, y) \in A_2 - K$ is contained in a proper retract of A_2 if $p(x, y)$ is fixed by a noninjective endomorphism ϕ of A_2. Moreover, in this case, there exists a positive integer m such that ϕ^m is a retraction of A_2.*

This theorem was also proved for $A_2 = \mathbb{C}[x, y]$ in Drensky and Yu [2]. The following question arises naturally.

Question 1.2. Let ϕ be a noninjective endomorphism of A_2. Determine whether ϕ has a fixed element $p \in A_2 - K$. If it does, find the minimal positive integer m such that ϕ^m is a retraction and find a nontrivial fixed element of ϕ.

In this paper, we present a classification of noninjective endomorphisms of A_2 and an algorithm to determine whether a given noninjective endomorphism of A_2 has nontrivial fixed elements in the polynomial algebra. The algorithm is valid for free associative algebras whenever $u \in A_2$ is given with $\phi(A_2) \subseteq K[u]$.

2. Main results

As a consequence of a result of Bergman [1] for free associative algebras and a result of Shestakov and Umirbaev [12] for polynomial algebras, the following lemma follows from [3].

Lemma 2.1. *If f and g are two algebraically dependent elements of A_n over K, then f and g are in $K[u]$ for some $u \in A_n$.*

The following lemma is clearly motivated by the proof of [5, Theorem 1.4].

Lemma 2.2. *Suppose that ϕ is a noninjective endomorphism of A_2. If $\phi^2(A_2) \neq K$, then $\phi|_{\phi(A_2)}$ is an injective endomorphism of $\phi(A_2)$. Moreover, $\phi|_{\phi(A_2)} : \phi(A_2) \to \phi^2(A_2)$ is an isomorphism.*

Proof. Since $\phi = (f, g)$ is a noninjective endomorphism of A_2, f and g are algebraically dependent. By Lemma 2.1 and since $\phi(A_2) \neq K$, there exists $u' \in A_2 - K$ such that $f, g \in K[u']$. Therefore the transcendence degree of the quotient field of $\phi(A_2)$ over K is one. If $\phi^2(A_2) \neq K$, there exists $r \in \phi^2(A_2) - K$. So there exists $t \in \phi(A_2) - K$ such that $\phi(t) = r$. If there exists $0 \neq q \in \ker \phi \cap \phi(A_2)$, then q and t are algebraically dependent. Hence there exists an irreducible polynomial $h(u, v) \in K[u, v]$ such that $h(t, q) = 0$. Suppose that

$$h(u, v) = a_0(v)u^n + a_1(v)u^{n-1} + \cdots + a_{n-1}(v)u + a_n(v).$$

Then $\phi(h(t, q)) = h(\phi(t), \phi(q)) = h(r, 0) = 0$: that is,

$$a_0(0)r^n + a_1(0)r^{n-1} + \cdots + a_{n-1}(0)r + a_n(0) = 0.$$

Therefore $a_0(0) = a_1(0) = \cdots = a_n(0) = 0$. It follows that the polynomials $a_i(v)$ have no constant terms and so $h(u, v)$ is divisible by v, which contradicts the irreducibility of $h(u, v)$. Thus $\ker\phi \cap \phi(A_2) = 0$, which implies that $\phi|_{\phi(A_2)} : \phi(A_2) \to \phi^2(A_2)$ is an isomorphism. □

THEOREM 2.3. *Suppose that $\phi = (f, g)$ is a noninjective endomorphism of A_2 and $\phi(A_2) \neq K$. Then there exists $\omega \in A_2 - K$ such that $f, g \in K[\omega]$ and $K(f, g) = K(\omega)$.*

- (i) *$\phi^2(A_2) = K$ if and only if $\phi(\omega) \in K$.*
- (ii) *$K \subsetneq \phi^2(A_2) \subsetneq \phi(A_2)$ if and only if $\phi(\omega)$ is a polynomial in ω with degree larger than one. In this case, $\bigcap_{s=1}^{\infty} \phi^s(A_2) = K$.*
- (iii) *$K \subsetneq \phi^2(A_2) = \phi(A_2)$ if and only if $\phi(\omega)$ is a polynomial in ω with degree one. In this case, $\phi(A_2) = k[\omega]$.*

PROOF. Since $\phi = (f, g)$ is a noninjective endomorphism of A_2, f and g are algebraically dependent. Since $\phi(A_2) \neq K$, by Lemma 2.1, there exists $u \in A_2 - K$ such that $f, g \in K[u]$. Then $K \subsetneq K(f, g) \subseteq K(u)$. By Lüroth's Theorem and Theorem 4 in [11, page 16], there exists $\omega \in K[u] \subseteq A_2$ such that $K(f, g) = K(\omega)$. Obviously, $K(f, g)$ is the quotient field of $\phi(A_2)$ and $f, g \in K[\omega]$.

Suppose that $\phi(\omega) \notin K$. Let $f = f_1(\omega)$ and $g = g_1(\omega)$. Then $\phi(f) = f_1(\phi(\omega))$ and $\phi(g) = g_1(\phi(\omega))$. Since $\phi(A_2) \neq K$, either $\deg f_1 \geq 1$ or $\deg g_1 \geq 1$. Thus $\phi^2(A_2) \supsetneq K$. Hence if $\phi^2(A_2) = K$, then $\phi(\omega) \in K$. The converse is obvious. This proves (i).

Now suppose that $K \subsetneq \phi^2(A_2)$. By Lemma 2.2, $\phi|_{\phi(A_2)}$ can be extended to its quotient field $K(\omega)$. We denote by $\bar{\phi}$ this endomorphism of $K(\omega)$. Obviously, $\bar{\phi}(K(\omega))$ is a subfield of $K(\omega)$ and it is also the quotient field of $\phi^2(A_2)$. For $\omega \in A_2$, $\phi(\omega) = \bar{\phi}(\omega)$. To see this, consider $\omega = l(f, g)/h(f, g)$ with $l(x, y), h(x, y) \in K[x, y]$ and $h(f, g) \neq 0$. Then $l(f, g) = \omega h(f, g)$. Thus

$$\phi(\omega)h(\phi(f), \phi(g)) = l(\phi(f), \phi(g)) = l(\bar{\phi}(f), \bar{\phi}(g)) = \bar{\phi}(\omega)h(\bar{\phi}(f), \bar{\phi}(g)).$$

By Lemma 2.2, $h(\phi(f), \phi(g)) = h(\bar{\phi}(f), \bar{\phi}(g)) \neq 0$. Hence $\phi(\omega) = \bar{\phi}(\omega)$. So $\bar{\phi}(\omega) \in \phi(A_2) \subseteq K[\omega]$. Let r be the degree of $\phi(\omega)$ in ω. Then $r \geq 1$ by assumption.

If $r = 1$, then $\bar{\phi}(K(\omega)) = K(\omega)$. Since $\omega \in \phi(A_2)$, $\phi(A_2) = K[\omega]$ and $\phi^2(A_2) = K[\phi(\omega)] = K[\omega]$. Hence $\phi^2(A_2) = \phi(A_2)$. Conversely, if $\phi^2(A_2) = \phi(A_2)$, then $\bar{\phi}(K(\omega)) = K(\omega)$. So $r = 1$. This proves (iii).

If $r > 1$, then $\bar{\phi}(K(\omega)) \subsetneq K(\omega)$ and $\phi^2(A_2) \subsetneq \phi(A_2)$. Hence $K \subsetneq \phi^2(A_2) \subsetneq \phi(A_2)$ if and only if $r > 1$. In this case, $[K(\omega) : \bar{\phi}(K(\omega))] = r$ and $[K(\omega) : \bar{\phi}^s(K(\omega))] = r^s$. By Lüroth's Theorem, $\bigcap_{s=0}^{\infty} \bar{\phi}^s(K(\omega)) = K$. So $\bigcap_{s=1}^{\infty} \phi^s(A_2) \subseteq \bigcap_{s=0}^{\infty} \bar{\phi}^s(K(\omega)) = K$. This proves (ii). □

REMARK 2.4. In the first two cases of Theorem 2.3, ω is not necessarily contained in $\phi(A_2)$. For example, for the first case, $\phi = ((x^2 - y^3)^3, (x^2 - y^3)^2)$ is an endomorphism with $\omega \notin \phi(A_2)$ and, for the second case, $\phi = (x^2, x^3)$ is an endomorphism with $\omega \notin \phi(A_2)$.

COROLLARY 2.5. *In Theorem 2.3, ω can be replaced by u with $f, g \in K[u]$.*

https://doi.org/10.1017/S0004972716000630 Published online by Cambridge University Press

Proof. Obviously, this is correct for (i). For (iii), if $\phi(u)$ is a polynomial in u with degree one, then $u \in K[\omega]$. Since $K(\omega) = K(f, g) \subseteq K(u)$, $K[u] = K[\omega] = \phi(A_2)$ and $K \subsetneq \phi^2(A_2) = \phi(A_2)$. Conversely, if $K \subsetneq \phi^2(A_2) = \phi(A_2)$, then $\omega \in \phi(A_2) \subseteq K[u]$. Suppose that ω can be expressed as a polynomial in u with degree larger than one. Then, by $\phi(u) \in K[\omega] - K$, we see that $\phi(\omega)$ is a polynomial in ω with degree larger than one, which is a contradiction. Hence ω can be expressed as a polynomial in u with degree one. That is, $\phi(u)$ is a polynomial in u with degree one. This proves (iii). Since $\phi(u) \in \phi(A_2) \subseteq K[u]$, (ii) can be deduced from (i) and (iii). □

Theorem 2.6. *Suppose that $\phi = (f, g)$ is a noninjective endomorphism of A_2 and that $f, g \in K[\omega]$ for some $\omega \in A_2 - K$. Then ϕ has a nontrivial fixed element if and only if $\phi(\omega) = a\omega + b$, where a is a primitive kth root of unity in K for some positive integer k and b is in K such that $a = 1$ and $b = 0$ whenever $k = 1$. In this case, $K[q]$ is the set of fixed elements of ϕ, where $q = \omega\phi(\omega)\cdots\phi^{k-1}(\omega)$. Furthermore, $\pi := \phi^k$ is a retraction of A_2 and $\pi(A_2) = \phi(A_2) = K[\omega]$.*

Proof. If ϕ has a nontrivial fixed element, then, by Corollary 2.5, $\phi(\omega) = a\omega + b$ with $0 \neq a \in K$, $b \in K$ and $\phi(A_2) = K[\omega]$. By Theorem 1.1, there exists a minimal positive integer k such that $\pi := \phi^k$ is a retraction of A_2. So $\pi(A_2) = \phi(A_2) = K[\omega]$. Then, by $\phi^k(\omega) = \omega$, $a^k = 1$ and $(1 + a + \cdots + a^{k-1})b = 0$. It follows that $a = 1$ and $b = 0$, or $a \neq 1$ is a kth root of unity. By the minimality of k, a is a primitive kth root of unity.

If $a = 1$ and $b = 0$, then ϕ is a retraction and the corresponding retract is $k[\omega]$. Now suppose that a is a primitive kth root of unity for some $k > 1$. As in the proof of Theorem 2.3, $\bar{\phi}$ is an automorphism of $K(\omega)$. Let $G = \{\bar{\phi}^i \mid i = 0, 1, \ldots, k-1\}$, which is an automorphism group of $K(\omega)$. Obviously, $q = \omega\phi(\omega)\cdots\phi^{k-1}(\omega)$ is fixed by ϕ. So it is fixed by the automorphism group G. Thus $G \subseteq \mathrm{Gal}(K(\omega)/K(q))$. Since $|\mathrm{Gal}(K(\omega)/K(q))| \leq [K(\omega) : K(q)] = k = |G|$, $G = \mathrm{Gal}(K(\omega)/K(q))$. Thus $K(q)$ is the fixed field of G and also that of $\bar{\phi}$. Since $K(q) \cap \phi(A_2) = K[q]$, it follows that $K[q]$ is the set of fixed elements of ϕ. □

For a polynomial algebra, ω can be obtained by means of functional decomposition of polynomials [4]. To verify that ϕ has nontrivial fixed elements, it is sufficient to check that $\phi(\omega)$ has the form in Theorem 2.6. For a free associative algebra, given $u \in A_2$ such that $f, g \in K[u]$, we can use a similar method to determine whether ϕ has nontrivial fixed elements.

We conclude with some simple examples in $\mathbb{C}[x, y]$ to illustrate that the three cases of Theorem 2.3 may occur and few noninjective endomorphisms of A_2 have nontrivial fixed elements.

For $\phi = (y, 0)$, take $\omega = y$. Then $\phi(\omega) = 0$. So $\phi^2(A_2) = \mathbb{C}$ and ϕ does not have nontrivial fixed elements.

For $\phi = (x^2, 0)$, take $\omega = x^2$. Then $\phi(\omega) = x^4 = \omega^2$. So $\mathbb{C} \subsetneq \phi^2(A_2) \subsetneq \phi(A_2)$ and ϕ does not have nontrivial fixed elements.

For $\phi = (2x + b, 0)$ with $b \in \mathbb{C}$, take $\omega = x$. Then $\phi(\omega) = 2\omega + b$. So $\mathbb{C} \subsetneq \phi^2(A_2) = \phi(A_2)$ and ϕ does not have nontrivial fixed elements.

For $\phi = (-x + b, 0)$ with $b \in \mathbb{C}$, take $\omega = x$. Then $\phi(\omega) = -\omega + b$. So $\mathbb{C} \subsetneq \phi^2(A_2) = \phi(A_2)$, $k = 2$, $q = bx - x^2$, and $\mathbb{C}[q]$ is the set of fixed elements of ϕ.

Finally, for $\phi = (1, \frac{1}{2}(1 + \sqrt{3}i)y + x^2 - xy)$, take $\omega = \frac{1}{2}(1 + \sqrt{3}i)y + x^2 - xy$. Then $\phi(\omega) = \frac{1}{2}(\sqrt{3}i - 1)\omega + 1$. So $\mathbb{C} \subsetneq \phi^2(A_2) = \phi(A_2)$, $k = 3$, and $\mathbb{C}[q]$, generated by $q = \omega^3 - \frac{1}{2}(3 + \sqrt{3}i)\omega^2 + \frac{1}{2}(1 + \sqrt{3}i)\omega$, is the set of fixed elements of ϕ.

Acknowledgements

The first author is grateful to Professor Xiankun Du for helpful discussions. The authors would like to thank the referee for his careful reading and constructive comments.

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https://doi.org/10.1017/S0004972716000630 Published online by Cambridge University Press