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A FLAT NONMETRIZABLE CONNEXION

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A linear connexion ∇ on a smooth manifold M with vanishing torsion and curvature is called flat. It is well known that such a connexion need not be the Levi-Civita connexion of a Riemannian (nor pseudo-Riemannian) structure (cf. [1]).

The following example with M homeomorphic to a cylinder is particularly simple. Using polar coordinates the punctured plane $M=R^2\setminus\{0\}$ is covered by a smooth atlas with two charts:

$$\begin{aligned} x^1 &= r, &\quad 0 < x^2 &= \theta < 2\pi, \\ y^1 &= r, &\quad -\pi < y^2 &= \theta < \pi. \end{aligned}$$

The following system of connexion coefficients is compatible with the coordinate transformation and hence defines a linear connexion ∇ on M.

$$\begin{aligned} \Gamma^1_{11} &= 2, &\quad \Gamma^1_{12} = \Gamma^1_{21} &= 1, &\quad \Gamma^1_{22} &= -2, \\ \Gamma^2_{11} &= -2, &\quad \Gamma^2_{12} = \Gamma^2_{21} &= 4, &\quad \Gamma^2_{22} &= -3. \end{aligned}$$

It is a trivial, though lengthy, computation to verify that torsion and curvature of ∇ vanish.

To see that ∇ is not induced by a pseudo-Riemannian structure of any arbitrary signature we observe that the equations for parallel displacement can be explicitly integrated: If the vector $\lambda(\partial/\partial r)+\mu(\partial/\partial\theta)$ is transported parallel to itself along a circle, $r=\text{const}$, we obtain

$$\begin{aligned} \lambda(\theta) &= e^{\theta}[\lambda(0)\cos 2\theta+(\mu(0)-\lambda(0))\sin 2\theta], \\ \mu(\theta) &= e^{\theta}[\mu(0)\cos 2\theta+(\mu(0)-2\lambda(0))\sin 2\theta]. \end{aligned}$$

Thus passing from $\theta=0$ to $\theta=2\pi$ every vector is multiplied by $e^{2\pi}$, which is impossible if parallel transport were to preserve a bilinear nondegenerate form on the tangent spaces.

This also shows that the holonomy group H_p of ∇ at $p\in M$ contains an infinite cyclic group. Since the fundamental group $\pi_1(M)$ is Z_∞ and the holonomy homomorphism $h:\pi_1(M)\to H_p$ is surjective for a connected flat manifold we conclude that H_p is infinite cyclic.

Note also that the geodesics of ∇ can be determined by quadratures.

Topologically our example cannot be simplified, according to the following

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THEOREM. *Let M be a connected manifold with a finite fundamental group and a flat linear connexion ∇. Then ∇ is the Levi-Civita connexion of a Riemannian structure.*

Proof. For any point $p \in M$ the holonomy group H_p must be finite because of subjectivity of h, say

$$H_p = \{h_1, h_2, \ldots, h_k\},$$

with each h_j a linear isomorphism of the tangent space M_p. Choose an arbitrary Euclidean scalar product $\langle\ ,\ \rangle$ on M_p and define a bilinear form on M_p by

$$\beta(X_p, Y_p) = \sum_{j=1}^{k} \langle h_j(X_p), h_j(Y_p)\rangle, \quad X_p, Y_p \in M_p. \tag{1}$$

For any $x \in M$ we choose a smooth path σ from p to x and denote parallel translation along σ by τ_σ. Letting

$$g_x(X, Y) = \beta(\tau_\sigma^{-1}X, \tau_\sigma^{-1}Y), \quad X, Y \in M_x \tag{2}$$

we obtain a Riemannian metric tensor g on M: Since β is invariant under H_p, g_x is independent of σ, symmetric, positive definite, and depends differentiably on x.

If α is an arbitrary smooth path on M and $X_{\alpha(t)}, Y_{\alpha(t)} \in M_{\alpha(t)}$ denote two parallel vector fields along α, we choose as σ in (2) the composition of a path ρ from p to $\alpha(0)$ and α. Then

$$\begin{aligned} g_{\alpha(t)}[X_{\alpha(t)}, Y_{\alpha(t)}] &= \beta[\tau_\rho^{-1}\tau_\alpha^{-1}X_{\alpha(t)}, \tau_\rho^{-1}\tau_\alpha^{-1}Y_{\alpha(t)}] \\ &= \beta[\tau_\rho^{-1}X_{\alpha(0)}, \tau_\rho^{-1}Y_{\alpha(0)}] \end{aligned}$$

is independent of t. Hence scalar products are preserved under parallel transport. Since ∇ is torsion free it coincides with the Levi-Civita connexion determined by g. Q.E.D.

(Hence there exist exactly as many Riemannian structures on M compatible with ∇ as there are Euclidean metrics on M_p invariant under H_p.)

In particular a simply connected M cannot carry a flat nonmetric connexion.

Replacing the sum in (1) by HAAR integration we can prove similarly:

Let M be a connected manifold with a torsion free connexion ∇ (not necessarily flat) whose holonomy group H_p is compact with respect to the compact-open topology. Then ∇ can be metrized by a Riemannian structure.

REFERENCE

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