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ON GENERALISED PRIME ESSENTIAL RINGS AND SPECIAL AND NONSPECIAL RADICALS

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For a supernilpotent radical α and a special class σ of rings we call a ring R (α, σ)-essential if R is α-semisimple and for each ideal P of R with $R/P \in \sigma$, $P \cap I \neq 0$ whenever I is a nonzero two-sided ideal of R. (α, σ)-essential rings form a generalisation of prime essential rings introduced by L. H. Rowen in his study of semiprime rings and their subdirect decompositions and they have been a subject of investigations of many prominent authors since. We show that many important results concerning prime essential rings are also valid for (α, σ)-essential rings and demonstrate how (α, σ)-essential rings can be used to determine whether a supernilpotent radical is special. We construct infinitely many supernilpotent nonspecial radicals whose semisimple class of prime rings is zero and show that such radicals form a sublattice of the lattice of all supernilpotent radicals. This generalises Yu.M. Ryabukhin's example.

1. INTRODUCTION

In this paper all rings are associative and all classes of rings are closed under isomorphisms and contain the one-element ring 0. The fundamental definitions and properties of radicals can be found in [?] and [?]. If μ is a hereditary class of rings, $\mathcal{U}(\mu)$ denotes the upper radical generated by μ, that is, the class of all rings which have no nonzero homomorphic images in μ. For any class μ of rings an ideal I of a ring R is called an μ-ideal if the factor ring R/I is in μ. For a radical ρ the class of all ρ-semisimple rings is denoted by $\mathcal{S}(\rho)$. π denotes the class of all prime rings and β the prime radical. The notation $I \triangleleft R$ means that I is a two-sided ideal of a ring R. For $I \triangleleft R \in \mathcal{S}(\beta)$, $\{r \in R : rI = 0\} = \{r \in R : Ir = 0\}$ is an ideal of R which we shall denote by I^*. An ideal I of a ring R is called essential in R if $I \cap J \neq 0$ for any nonzero two-sided ideal J of R. If $R \in \mathcal{S}(\beta)$ this is equivalent to $I^* = 0$. 0 is an inessential ideal. Hereditary and essentially closed class of prime rings (respectively semiprime rings) is called a special class (respectively weakly special class) and the upper radical generated by a special class (respectively weakly special class) is called a special radical (respecively supernilpotent radical). Unless otherwise stated, throughout this paper the letter α denotes a supernilpotent radical and σ denotes a special class of rings.

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A ring R is called (α, σ)-essential if R is α-semisimple and each σ-ideal P of R is essential in R. (α, σ)-essential rings form a generalisation of prime essential rings, that is, semiprime rings whose every π-ideal is essential. Prime essential rings were first introduced by Rowen [?] in his study of semiprime rings and their subdirect decompositions and they have been a subject of investigations of many authors (see [?, ?, ?, ?]) since.

In this paper we show that many important results concerning prime essential rings are also valid for (α, σ)-essential rings and demonstrate how (α, σ)-essential rings can be used to determine whether a supernilpotent radical is special. Using (α, σ)-essential rings, we generalise Ryabukhin's example of supernilpotent nonspecial radical by constructing infinitely many supernilpotent nonspecial radicals α such that $\mathcal{S}(\alpha) \cap \pi = \{0\}$ and show that such radicals form a sublattice of the lattice of all supernilpotent radicals.

2. Main Results

We start with a useful result which is a generalisation of [?, Proposition 1].

Proposition 1. *A ring R is (α, σ)-essential if and only if $R \in \mathcal{E}_{(\alpha,\sigma)}$, where $\mathcal{E}_{(\alpha,\sigma)} = \{R \in \mathcal{S}(\alpha) : 0 \neq I \lhd R \Rightarrow I \notin \sigma\}$.*

Proof: First assume that R is (α, σ)-essential and let $0 \neq I \lhd R$. Then $R \in \mathcal{S}(\alpha)$ and so $R \in \mathcal{S}(\beta)$ because $\beta \subseteq \alpha$ since α is supernilpotent. If $I \in \sigma$, then $R/I^* \in \sigma$ because σ is a special class. But then I^* is not essential in R because $I^* \cap I = 0$ which contradicts the assumption that R is (α, σ)-essential. Thus $I \notin \sigma$ which shows that $R \in \mathcal{E}_{(\alpha,\sigma)}$.

Conversely, assume that $R \in \mathcal{E}_{(\alpha,\sigma)}$ and suppose R contains an inessential σ-ideal I. Then $0 \neq I^* \simeq (I + I^*)/I \lhd R/I \in \sigma$ from which it follows that $I^* \in \sigma$ since σ is hereditary. But this contradicts our assumption that $R \in \mathcal{E}_{(\alpha,\sigma)}$. □

Our next result generalises most of [?, Theorem 1].

Theorem 2. *$\mathcal{E}_{(\alpha,\sigma)}$ is a weakly special class closed under extensions. Thus $\mathcal{U}(\mathcal{E}_{(\alpha,\sigma)})$ is a supernilpotent radical.*

Proof: Let $\mu = \mathcal{E}_{(\alpha,\sigma)}$. Since α is supernilpotent, $\mathcal{S}(\alpha) \subseteq \mathcal{S}(\beta)$ so $\mu \subseteq \mathcal{S}(\beta)$. To show hereditariness of μ, let $I \lhd R \in \mu$. Then $R \in \mathcal{S}(\alpha)$ and, since $\mathcal{S}(\alpha)$ is hereditary, it follows that $I \in \mathcal{S}(\alpha)$. Suppose $0 \neq J \lhd I$ and $J \in \sigma$. Consider the ideal J_R of R generated by J. As $\mathcal{S}(\alpha) \subseteq \mathcal{S}(\beta)$, it follows that $0 \neq J_R^3 \lhd J \in \sigma$ which implies that $J_R^3 \in \sigma$ because σ is hereditary. But this means that $R \notin \mu$, a contradiction. Thus $J \notin \sigma$ and so $I \in \mu$.

To show that μ is closed under essential extensions, let $I \in \mu$ be an essential ideal of a ring R. Then $I \in \mathcal{S}(\alpha)$ and it follows that $R \in \mathcal{S}(\alpha)$ because $\mathcal{S}(\alpha)$ is essentially closed. Now, let $0 \neq J \lhd R$ and suppose $J \in \sigma$. Then, since $I \cap J \lhd J$ and σ is hereditary, it follows that $I \cap J \in \sigma$. But, since I is an essential ideal of R, it follows that $I \cap J \neq 0$.

So, since $I \cap J \lhd I \in \mu$, $I \cap J \notin \sigma$, a contradiction. Thus $J \notin \sigma$ which implies that $R \in \mu$ and proves that μ is closed under essential extensions.

To show that μ is closed under extensions, let I and R/I be both in μ. Then I and R/I are both in $\mathcal{S}(\alpha)$ and, since $\mathcal{S}(\alpha)$ is closed under extensions, it follows that $R \in \mathcal{S}(\alpha)$. Let $0 \neq J \lhd R$. If $I \cap J = 0$, then $J \simeq (J+I)/I \lhd R/I$ and, since $R/I \in \mu$, it follows that $J \notin \sigma$. If $I \cap J \neq 0$, then $I \cap J \notin \sigma$ because $I \cap J \lhd I \in \mu$. Then $J \notin \sigma$ because $I \cap J \lhd J$ and σ is hereditary. Thus $R \in \mu$ which shows that μ is closed under extensions.

Since μ is a hereditary and essentially closed class of semiprime rings, it is weakly special and therefore $\mathcal{U}(\mu)$ is a supernilpotent radical. □

Corollary 3. *If σ contains a nonzero ring R such that $R/I \in \mathcal{S}(\beta)$ implies $R/I \in \sigma$ for every ideal $I \neq R$, then $\mathcal{U}(\mathcal{E}_{(\beta,\sigma)})$ is a supernilpotent radical that is not special.*

Proof: Since β is a supernilpotent radical, it follows from Theorem 2 that $\mathcal{U}(\mathcal{E}_{(\beta,\sigma)})$ is supernilpotent. Now, $\mathcal{U}(\mathcal{E}_{(\beta,\sigma)}) \neq \beta$ because any nonzero ring $R \in \sigma$ such that $R/I \in \mathcal{S}(\beta)$ implies $R/I \in \sigma$ for every ideal $I \neq R$ belongs to $\mathcal{U}(\mathcal{E}_{(\beta,\sigma)}) \cap \mathcal{S}(\beta)$. Moreover, the class $\mathcal{E}_{(\beta,\sigma)} \subseteq \mathcal{S}\big(\mathcal{U}(\mathcal{E}_{(\beta,\sigma)})\big)$ contains all prime essential rings since otherwise some prime essential ring S would contain a nonzero ideal $J \in \sigma \subsetneq \pi$ which is impossible. Since every radical $\rho \neq \beta$ whose semisimple class contains all prime essential rings is nonspecial by [**2**, Theorem 1], $\mathcal{U}(\mathcal{E}_{(\beta,\sigma)})$ is not a special radical. □

Corollary 4. ([**3**, Theorem 9(a)]) *For the class $\mathcal{E}$ of all prime essential rings the radical $\mathcal{U}(\mathcal{E})$ is supernilpotent but not special.*

Proof: Since $\mathcal{E} = \mathcal{E}_{(\beta,\pi)}$ and, as β is a supernilpotent radical and π is a special class, it follows from Theorem 2 that $\mathcal{U}(\mathcal{E})$ is supernilpotent. Taking $\alpha = \beta$, $\sigma = \pi$ and any nonzero simple prime ring for R in Corollary 3, we obtain the nonspeciality of $\mathcal{U}(\mathcal{E})$. □

Lemma 5. *If μ is a nonzero weakly special class of rings with $\mathcal{S}\big(\mathcal{U}(\mu)\big) \cap \pi = 0$, then $\mathcal{U}(\mu)$ is a supernilpotent nonspecial radical. However, the converse is not true.*

Proof: Since μ is a weakly special class, $\mathcal{U}(\mu)$ is supernilpotent. If $\mathcal{U}(\mu)$ were a special radical, then $\mathcal{U}(\mu) = \mathcal{U}\Big(\mathcal{S}\big(\mathcal{U}(\mu)\big) \cap \pi\Big) = \mathcal{U}\big(\{0\}\big)$ would be the class of all rings but this is impossible since $\{0\} \neq \mu \subseteq \mathcal{S}\big(\mathcal{U}(\mu)\big)$. Thus $\mathcal{U}(\mu)$ is not a special radical. □

Example 6. For a natural number $n > 1$, let $\mathcal{D}_n$ be the class of all rings which are subdirect sums of copies of the $n \times n$ matrix ring $M_n(\mathbf{Z}_2)$ over $\mathbf{Z}_2$ but which have no ideals isomorphic to $M_n(\mathbf{Z}_2)$ and let $\mathcal{F}$ be the class of all fields. Then $\rho = \mathcal{U}(\mathcal{D}_n \cup \mathcal{F})$ is a nonspecial radical by [**3**, Theorem 9(c)] but $\mathcal{S}(\rho) \cap \pi \neq \{0\}$ because any field F is in $\mathcal{F} \subseteq \mathcal{S}(\rho) \cap \pi$.

Let F be a finite field. For an integer $n \geqslant 1$ let $M_n(F)$ be the ring of all $n \times n$ matrices over F. It was shown in [**5**, Lemma 7] that $\mu = \big\{M_n(F)\big\}$ is a special class with

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$\mu = \mathcal{S}(\mathcal{U}(\mu)) \cap \pi$.

THEOREM 7. *Let σ be any class of simple rings with unity such that $\sigma = \mathcal{S}(\mathcal{U}(\sigma)) \cap \pi$. Then $\rho = \mathcal{U}(\mathcal{E}_{(\mathcal{U}(\sigma),\sigma)})$ is a supernilpotent nonspecial radical with $\mathcal{S}(\rho) \cap \pi = \{0\}$. Moreover, $\mathcal{S}(\mathcal{U}(\sigma)) \cap \mathcal{E} = \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$.*

PROOF: Since σ is a special class, it is weakly special. Hence $\alpha = \mathcal{U}(\sigma)$ is a supernilpotent radical and then Theorem 2 implies that so is ρ.

To show that $\mathcal{S}(\rho) \cap \pi = 0$, suppose $0 \neq R \in \mathcal{S}(\rho) \cap \pi$. Then $R \in \mathcal{S}(\rho)$ and, since the class $\mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$ is weakly special by Theorem 2, R is a subdirect sum of rings $R_i \in \mathcal{E}_{(\mathcal{U}(\sigma),\sigma} \subseteq \mathcal{S}(\mathcal{U}(\sigma))$. Then, as $\mathcal{S}(\mathcal{U}(\sigma))$ is subdirectly closed, it follows that $R \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \pi$. Thus $R \in \sigma$ since $\sigma = \mathcal{S}(\mathcal{U}(\sigma)) \cap \pi$. This implies that R is a simple ring with unity and $R \notin \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$. But R is also a subdirect sum of rings $R_i \in \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$. So there exist ideals I_i of R such that $\cap I_i = 0$ and $R/I_i \simeq R_i$ for all i. If all $I_i = R$, then $R = 0$, a contradiction. Thus some $I_i \neq R$ and then $I_i = 0$ since R is a simple ring. But this implies that $R \in \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$, a contradiction. Thus $\mathcal{S}(\rho) \cap \pi = 0$.

Now, since $\{0\} \neq \sigma = \mathcal{S}(\mathcal{U}(\sigma)) \cap \pi$, there exists a nonzero semiprime and $\mathcal{U}(\sigma)$-semisimple ring and hence prime essential $\mathcal{U}(\sigma)$-semisimple ring $S \neq 0$ by [**3**, Remark 4]. Then $S \in \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$ because otherwise S would contain a nonzero ideal $I \in \sigma$ and, since $\sigma \subseteq \pi$, this would show that S is not prime essential, a contradiction. This and Theorem 2 imply that $\mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$ is a nonzero weakly special class and, as $\mathcal{S}(\rho) \cap \pi = 0$, it follows from Lemma 5 that ρ is not a special radical.

To show that $\mathcal{S}(\mathcal{U}(\sigma)) \cap \mathcal{E} = \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$, first suppose $0 \neq R \in \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$ and $R \notin \mathcal{E}$. Then $R \in \mathcal{S}(\mathcal{U}(\sigma))$ and R contains a nonzero ideal $I \in \pi$. But, as $\mathcal{S}(\mathcal{U}(\sigma))$ is hereditary, this implies that $I \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \pi = \sigma$ which implies that $R \notin \mathcal{E}_{(\mathcal{U}(\sigma),\sigma)}$, a contradiction. Thus $\mathcal{E}_{(\mathcal{U}(\sigma),\sigma)} \subseteq \mathcal{S}(\mathcal{U}(\sigma)) \cap \mathcal{E}$. Now, if the inclusion were strict, then a ring $S \in \mathcal{S}(\mathcal{U}(\sigma)) \cap \mathcal{E}$ containing a nonzero ideal $J \in \sigma$ could be found and then, since $\sigma \subseteq \pi$, we would get $S \notin \mathcal{E}$, a contradiction. Thus $\mathcal{S}(\mathcal{U}(\sigma)) \cap \mathcal{E} = \mathcal{E}_{\mathcal{U}(\sigma),\sigma)}$. ∎

We shall now build infinitely many supernilpotent nonspecial radicals ρ with $\mathcal{S}(\rho) \cap \pi = \{0\}$.

COROLLARY 8. *Let $n \geqslant 1$ be an integer and let p be any prime number. Let $\sigma_p = \{M_n(\mathbb{Z}_p)\}$. Then $\rho_p = \mathcal{U}(\mathcal{E}_{(\mathcal{U}(\sigma_p),\sigma_p)})$ is a supernilpotent nonspecial radical with $\mathcal{S}(\rho_p) \cap \pi = \{0\}$.*

In particular, taking $n = 1$ and $p = 2$ in Corollary 8, we obtain

COROLLARY 9. ([8]) *If $\mathcal{D}$ is the class of all rings which are subdirect sums of copies of $\mathbf{Z}_2$ but which have no ideals isomorphic to $\mathbf{Z}_2$, then the radical $\rho = \mathcal{U}(\mathcal{D})$ is supernilpotent nonspecial with $\mathcal{S}(\rho) \cap \pi = \{0\}$.*

THEOREM 10. *The collection $\mathbf{N}$ of all supernilpotent radicals ρ with $\mathcal{S}(\rho) \cap \pi = \{0\}$ is a sublattice of the lattice $\mathbf{K}$ of all supernilpotent radicals.*

PROOF: Let $\rho, \gamma \in \mathbb{N}$. Since $\mathcal{S}(\rho \vee \gamma) = \mathcal{S}(\rho) \cap \mathcal{S}(\gamma)$, it follows that $\mathcal{S}(\rho \vee \gamma) \cap \pi = \{0\}$ which implies that $\rho \vee \gamma \in \mathbb{N}$.

To show that $\rho \wedge \gamma \in \mathbb{N}$, suppose $0 \neq R \in \mathcal{S}(\rho \wedge \gamma) \cap \pi$. If the ρ-radical $\rho(R)$ of R is zero, then, since $R \in \pi$, it follows that $R \in \mathcal{S}(\rho) \cap \pi$ which is impossible since $\mathcal{S}(\rho) \cap \pi = \{0\}$ because $\rho \in \mathbb{N}$. Thus $\rho(R) \neq 0$ and, similarly, $\gamma(R) \neq 0$. Now, since $\rho(R)$ and $\gamma(R)$ are nonzero ideals of R and R is a prime ring, it follows that $0 \neq \rho(R)\gamma(R) \lhd R$. But, since both ρ and γ are hereditary and $\rho(R)\gamma(R) \lhd \rho(R) \in \rho$ and $\rho(R)\gamma(R) \lhd \gamma(R) \in \gamma$, it then follows that $\rho(R)\gamma(R) \in \rho \wedge \gamma$ which contradicts the fact that $R \in \mathcal{S}(\rho \wedge \gamma)$. Thus $\mathcal{S}(\rho \wedge \gamma) \cap \pi = \{0\}$ which implies that $\rho \wedge \gamma \in \mathbb{N}$. ∎

REMARK 11. We do not know whether $\mathbb{N}$ is a complete sublattice of the lattice $\mathbb{K}$. To answer this question in the negative, it would suffice to show that $\wedge \rho_p \notin \mathbb{N}$, where ρ_p are described in Corollary 8.

In what follows for any ring R let $\Delta_\sigma = \cap\{I : I \text{ is a } \sigma\text{-ideal of } R \text{ with } I^* \neq 0\}$. If $I^* = 0$ for all σ-ideals I of R, we take $\Delta_\sigma = R$.

LEMMA 12. *Any ring $R \in \mathcal{S}(\alpha)$ is a subdirect sum of R/Δ_σ and R/Δ_σ^*. Moreover, $R/\Delta_\sigma^* \in \mathcal{E}_{(\alpha,\sigma)}$.*

PROOF: Since $\Delta_\sigma \cap \Delta_\sigma^* = 0$, it suffices to show that $R/\Delta_\sigma^* \in \mathcal{E}_{(\alpha,\sigma)}$.

Now, since $\Delta_\sigma \lhd R \in \mathcal{S}(\alpha)$ and $\mathcal{S}(\alpha)$ is hereditary, it follows that $\Delta_\sigma \in \mathcal{S}(\alpha)$. Now, if $\Delta_\sigma = 0$, then $R/\Delta_\sigma^* = 0 \in \mathcal{E}_{(\alpha,\sigma)}$ and we are done. So assume that $\Delta_\sigma \neq 0$. Then the factor ring R/Δ_σ^* is in $\mathcal{S}(\alpha)$ since it has an essential ideal $(\Delta_\sigma + \Delta_\sigma^*)/\Delta_\sigma^* \simeq \Delta_\sigma \in \mathcal{S}(\alpha)$ and $\mathcal{S}(\alpha)$ is essentially closed. Let $\overline{A} = A/\Delta_\sigma^*$ be a σ-ideal of $\overline{R} = R/\Delta_\sigma^*$ so that $R/A \in \sigma$. For any ideal $\overline{A\prime} = A\prime/\Delta_\sigma^*$ of $\overline{R}$ such that $\overline{A\prime}\,\overline{A} = 0$, we have $A\prime A \subseteq \Delta_\sigma^*$ so $\Delta_\sigma A\prime A \subseteq \Delta_\sigma \Delta_\sigma^* = 0$. It suffices to show that $\Delta_\sigma A\prime = 0$ since then $A\prime \subseteq \Delta_\sigma^*$ implying $\overline{A\prime} = \overline{0}$ in $\overline{R}$. If A is essential in R, then $\Delta_\sigma A\prime A = 0$ implies $\Delta_\sigma A\prime = 0$ and we are done. If A is inessential in R, then $\Delta_\sigma \subseteq A$, so $(\Delta_\sigma A\prime)^2 \subseteq \Delta_\sigma A\prime \Delta_\sigma \subseteq \Delta_\sigma A\prime A = 0$; hence $\Delta_\sigma A\prime = 0$ since $R \in \mathcal{S}(\alpha) \subseteq \mathcal{S}(\beta)$. Then $\overline{A}$ is essential in $\overline{R}$ which implies that $\overline{R} \in \mathcal{E}_{(\alpha,\sigma)}$. ∎

A supernilpotent radical ρ is special if every $R \in \mathcal{S}(\rho)$ is a subdirect sum of rings in $\mathcal{S}(\rho) \cap \pi$.

THEOREM 13. *A supernilpotent radical ρ that contains α is special if and only if every ring $A \in \mathcal{E}_{(\alpha,\sigma)} \cap \mathcal{S}(\rho)$ is a subdirect sum of rings in $\mathcal{S}(\rho) \cap \pi$.*

PROOF: If ρ is a special radical, then $\mathcal{S}(\rho) \cap \pi$ is a special class and $\rho = \mathcal{U}(\mathcal{S}(\rho) \cap \pi)$; hence the result follows.

Conversely, let $A \in \mathcal{S}(\rho)$. We need to show that A is a subdirect sum of rings in $\mathcal{S}(\rho) \cap \pi$. If $P^* = 0$ for all σ-ideals P of A, then Proposition 1 implies that $A \in \mathcal{E}_{(\alpha,\sigma)} \cap \mathcal{S}(\rho)$ and the result follows from the assumption. If $P^* \neq 0$, then $A/P \in \mathcal{S}(\rho)$ since it has an essential ideal $(P^* + P)/P \simeq P^* \in \mathcal{S}(\rho)$ and $\mathcal{S}(\rho)$ is essentially closed. But $A/P \in \sigma \subseteq \pi$

so $A/P \in \mathcal{S}(\rho) \cap \pi$ for every σ-ideal P of A with $P^* \neq 0$. Therefore the proof is completed if $\Delta_\sigma = 0$. So assume that $\Delta_\sigma \neq 0$. Then $A/\Delta_\sigma^* \in \mathcal{S}(\rho)$ because it contains an essential ideal $(\Delta_\sigma + \Delta_\sigma^*)/\Delta_\sigma^* \simeq \Delta_\sigma \in \mathcal{S}(\rho)$ and $\mathcal{S}(\rho)$ is hereditary and essentially closed. But, as $A \in \mathcal{S}(\rho) \subseteq \mathcal{S}(\alpha)$ Lemma 12 implies that A is a subdirect sum of A/Δ_σ and $A/\Delta_\sigma^* \in \mathcal{E}_{(\alpha,\sigma)}$. But then $A/\Delta_\sigma^* \in \mathcal{S}(\rho) \cap \mathcal{E}_{(\alpha,\sigma)}$ and it follows from the assumption that A/Δ_σ^* is a subdirect sum of rings in $\mathcal{S}(\rho) \cap \pi$, that is, there exist ideals J_i of A such that $J_i/\Delta_\sigma^* \lhd A/\Delta_\sigma^*$, $\cap J_i \subseteq \Delta_\sigma^*$ and $A/J_i \in \mathcal{S}(\rho) \cap \pi$. But A/Δ_σ is also a subdirect sum of rings from $\mathcal{S}(\rho) \cap \pi$ because $\cap\{P_j/\Delta_\sigma : P_j$ is a σ-ideal of A with $P_j^* \neq 0\} = 0$ in A/Δ_σ and, as it was shown above, $A/P_j \in \mathcal{S}(\rho) \cap \pi$ for every σ-ideal P_j of A with $P_j^* \neq 0$. Thus we get $(\cap J_i) \cap (\cap P_j) \subseteq \Delta_\sigma^* \cap \Delta_\sigma = 0$ which shows that A is a subdirect sum of rings in $\mathcal{S}(\rho) \cap \pi$. □

Taking $\alpha = \beta$ and $\sigma = \pi$ in Theorem 13, we obtain

COROLLARY 14. ([3, Theorem 6]) *A supernilpotent radical ρ is special if and only if every prime essential ρ-semisimple ring is a subdirect sum of prime ρ-semisimple rings.*

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