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THE SUBEXPONENTIAL PRODUCT CONVOLUTION OF TWO WEIBULL-TYPE DISTRIBUTIONS

YAN LIU and QIHE TANG ✉

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Abstract

Let X_1 and X_2 be two independent and nonnegative random variables with distributions F_1 and F_2, respectively. This paper proves that if both F_1 and F_2 are of Weibull type and fulfill certain easily verifiable conditions, then the distribution of the product X_1X_2, called the product convolution of F_1 and F_2, belongs to the class $\mathcal{S}^*$ and, hence, is subexponential.

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1. Introduction

Throughout this paper, all limit relations are for $x \to \infty$ unless otherwise stated and the symbol $\sim$ means that the quotient of both sides tends to 1. Let X_1 and X_2 be two independent and nonnegative random variables with distributions F_1 and F_2, respectively. The distribution of the sum $X_1 + X_2$, written as $F_1 * F_2$, is called the sum convolution of F_1 and F_2; that is,

$$F_1 * F_2(x) = \int_{0-}^{x} F_1(x-y)\, dF_2(y) \quad \forall x \geq 0. \tag{1.1}$$

The distribution of the product X_1X_2, written as $F_1 \otimes F_2$, is called the product convolution of F_1 and F_2; that is,

$$F_1 \otimes F_2(x) = \int_{0}^{\infty} F_1(x/y)\, dF_2(y) \quad \forall x \geq 0. \tag{1.2}$$

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Comparing (1.1) and (1.2), our experience tells us that the product convolution is usually much more intractable than the sum convolution.

A distribution F on $[0, \infty)$ is said to be subexponential, written as $F \in \mathcal{S}$, if $\overline{F}(x) = 1 - F(x) > 0$ for all $x \geq 0$ and

$$\overline{F * F}(x) \sim 2\overline{F}(x). \tag{1.3}$$

A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{S}^*$ if $\overline{F}(x) > 0$ for all $x \geq 0$, $\mu = \int_0^\infty \overline{F}(y)\, dy < \infty$, and

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\, dy \sim 2\mu\overline{F}(x).$$

The class $\mathcal{S}^*$ forms an important subclass of the subexponential class $\mathcal{S}$.

Klüppelberg [16] first introduced the class $\mathcal{S}^*$ and pointed out that the class $\mathcal{S}^*$ contains almost all cited subexponential distributions with finite means. In recent studies in applied probability, researchers have discovered that the class $\mathcal{S}^*$ enjoys a lot of nicer properties than the class $\mathcal{S}$. In studies concerning asymptotic tail probabilities in many fields such as queueing theory and risk theory, it is often a standard assumption that underlying distributions belong to the class $\mathcal{S}^*$. Recent in-depth studies revealing new properties and proposing important applications of the class $\mathcal{S}^*$ can be found in [3, 4, 8, 14, 15], among others. All these works look at the asymptotic tail probabilities of sums and maxima of sums of random variables.

The study of subexponentiality of product convolutions was initiated by [7], reactivated by [11, 13], and further extended by [12, 20–22]. This study is important because, like sums, products of random variables are a basic element of modeling in applied fields and because the study of the tail behavior of certain stochastic quantities of complicated structure can usually be reduced to the study of the tail behavior of sums and products. The reader is referred to [11] for further discussion. However, the study of subexponentiality of products is often much more difficult than the study of subexponentiality of sums. This is not surprising since, as shown in (1.3), subexponentiality is defined in terms of sums and not products.

In this paper we prove that the product convolution of two Weibull-type distributions fulfilling certain mild conditions belongs to the class $\mathcal{S}^*$. Our work is motivated by an interesting observation of [22] that the product convolution of two exponential distributions is subexponential.

The rest of this paper consists of three sections. After showing a main result and two related consequences in Section 2, we prepare two propositions in Section 3 and prove the main result in Section 4.

2. Main result

A distribution F on $[0, \infty)$ is said to be of Weibull type if

$$\overline{F}(x) = c(x)\exp\{-b(x)x^p\} \quad \forall x \geq 0, \tag{2.1}$$

where $p > 0$ is a constant, called the shape parameter, and $c(\cdot) : [0, \infty) \mapsto (0, \infty)$ and $b(\cdot) : [0, \infty) \mapsto (-\infty, \infty)$ are two measurable functions such that the limits $c(x) \to c$ and $b(x) \to b$ exist and are positive. If only for the purpose of definition, the function $c(\cdot)$ in (2.1) can be eliminated by replacing $b(x)$ by $\widetilde{b}(x) = b(x) - x^{-p} \log c(x)$. Nevertheless, we still separate the two functions $c(\cdot)$ and $b(\cdot)$ as in (2.1) because in (2.5) we make an assumption on the differentiability of the function $b(\cdot)$.

Let X_1 and X_2 be two independent and nonnegative random variables with distributions F_1 and F_2, and let G be the distribution of their product

$$Y = X_1 X_2. \tag{2.2}$$

Thus, $G = F_1 \otimes F_2$. Assume that both F_1 and F_2 are of Weibull type with tails

$$\overline{F_i}(x) = c_i(x) \exp\{-b_i(x) x^{p_i}\} \quad \forall x \geq 0,\ i = 1, 2, \tag{2.3}$$

for some constants $p_i > 0$ and some measurable functions $c_i(\cdot) : [0, \infty) \mapsto (0, \infty)$ and $b_i(\cdot) : [0, \infty) \mapsto (-\infty, \infty)$ satisfying $c_i(x) \to c_i > 0$ and $b_i(x) \to b_i > 0$. Further assume that

$$p_1^{-1} + p_2^{-1} > 1 \tag{2.4}$$

and that, for $i \in \{1, 2\}$ determined by $p_i = \max\{p_1, p_2\}$, the function $b_i(\cdot)$ is eventually continuously differentiable with derivative $b_i'(\cdot)$ satisfying

$$-b_i p_i < \liminf_{x \to \infty} b_i'(x) x \leq \limsup_{x \to \infty} b_i'(x) x < b_i p_i (p_1^{-1} + p_2^{-1} - 1). \tag{2.5}$$

THEOREM 2.1. *Consider the product in (2.2). Under conditions (2.3)–(2.5) we have* $G = F_1 \otimes F_2 \in \mathcal{S}^*$.

Note that condition (2.4) cannot be removed from Theorem 2.1. A simple counterexample is that $F_1(x) = F_2(x) = \Phi(x \vee 0)$ for $x \geq 0$, where $\Phi(\cdot)$ is the standard normal distribution. In this case, $p_1 = p_2 = 2$ and it is easy to verify that $F_1 \otimes F_2$ has certain finite exponential moments and, hence, it is even not heavy-tailed.

Professor Enkelejd Hashorva has kindly brought to our attention a closely related but different result obtained by [2]. According to this reference, a distribution F on $[0, \infty)$ is said to have a Weibullian tail if

$$\overline{F}(x) \sim C x^{\gamma} \exp\{-\beta x^{\alpha}\} \quad \text{where } \alpha, \beta, C > 0,\ \gamma \in (-\infty, \infty).$$

Lemma 2.1 of [2] shows that if both F_1 and F_2 have Weibullian tails then the product convolution $F_1 \otimes F_2$ also has a Weibullian tail with explicitly given parameters.

A distribution F on $[0, \infty)$ is said to belong to the class $\mathcal{L}(\gamma)$ for $\gamma \geq 0$ if the relation

$$\overline{F}(x - y) \sim e^{\gamma y} \overline{F}(x)$$

holds for all y. When $\gamma > 0$ the distribution $F \in \mathcal{L}(\gamma)$ is usually said to have an exponential tail, while when $\gamma = 0$ the class $\mathcal{L}(\gamma)$ reduces to the well-known class of

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long-tailed distributions. Studies on this and related distribution classes can be found in [1, 9, 10, 18, 19, 24], among others. Applying Karamata's representation theorem for regularly varying functions (see [5, 17]), we know that $F \in \mathcal{L}(\gamma)$ if and only if $\overline{F}(\cdot)$ can be expressed as

$$\overline{F}(x) = c(x) \exp\left\{-\int_0^x \gamma(y)\, dy\right\} \quad \forall x \geq 0, \tag{2.6}$$

where $c(\cdot) : [0, \infty) \mapsto (0, \infty)$ and $\gamma(\cdot) : [0, \infty) \mapsto (-\infty, \infty)$ are measurable functions such that the limits $c(x) \to c > 0$ and $\gamma(x) \to \gamma$ exist. As can be seen from the proof of Corollary 2.1 below, the class $\mathcal{L}(\gamma)$ for $\gamma > 0$ forms a subclass of Weibull-type distributions with shape parameter 1.

COROLLARY 2.1. *If $F_1 \in \mathcal{L}(\gamma_1)$ for some $\gamma_1 > 0$ and F_2 is of Weibull type with shape parameter $0 < p_2 \leq 1$, then $G = F_1 \otimes F_2 \in \mathcal{S}^*$. In particular, if $F_i \in \mathcal{L}(\gamma_i)$ for some $\gamma_i > 0$ for $i = 1, 2$, then $G = F_1 \otimes F_2 \in \mathcal{S}^*$.*

PROOF. According to (2.6), $\overline{F_1}(\cdot)$ can be expressed as

$$\overline{F_1}(x) = c_1(x) \exp\left\{-\int_0^x \gamma_1(y)\, dy\right\} \quad \forall x \geq 0,$$

for measurable functions $c_1(\cdot) : [0, \infty) \mapsto (0, \infty)$ and $\gamma_1(\cdot) : [0, \infty) \mapsto (-\infty, \infty)$ satisfying $c_1(x) \to c_1 > 0$ and $\gamma_1(x) \to \gamma_1$. We can always construct a distribution F_0 with

$$\overline{F_0}(x) = \exp\left\{-\int_0^x \gamma_0(y)\, dy\right\} \quad \forall x \geq 0, \tag{2.7}$$

for some continuous function $\gamma_0(\cdot) : [0, \infty) \mapsto (0, \infty)$ such that

$$\int_0^\infty |\gamma_1(y) - \gamma_0(y)|\, dy < \infty.$$

Clearly, there exists some positive constant c_* such that $\overline{F_1}(x) \sim c_* \overline{F_0}(x)$. By [23, Lemma A.5], we obtain

$$\overline{G}(x) \sim c_* \overline{F_0 \otimes F_2}(x).$$

Rewrite (2.7) in the style of (2.3) so that

$$\overline{F_0}(x) = \exp\{-b_0(x)x\} \quad \forall x \geq 0,$$

with $b_0(x) = x^{-1} \int_0^x \gamma_0(y)\, dy$. Note that $(b_0(x))' x \to 0$. Hence, by Theorem 2.1, $F_0 \otimes F_2 \in \mathcal{S}^*$. By [22, Corollary 1.1(C1)], $G \in \mathcal{L}(0)$. Therefore, by the closure of $\mathcal{S}^*$ under tail equivalence as shown in [16, Theorem 2.1(b)], we know that $G \in \mathcal{S}^*$. $\square$

Let $\{B(t), t \geq 0\}$ be a standard Brownian motion and let τ be a nonnegative random variable independent of $\{B(t), t \geq 0\}$. Denote by

$$B^*(\tau) = \sup_{0 \leq t \leq \tau} B(t)$$

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the maximum of the Brownian motion over a random time interval $[0, \tau]$. It is well known that if τ is exponentially distributed, so is $B^*(\tau)$; see, for example, [6, (1.1.2)]. The following example catches a subexponential tail of $B^*(\tau)$ when τ follows a heavy-tailed Weibull-type distribution.

EXAMPLE 2.1. If τ follows a Weibull-type distribution with shape parameter p, such that $0 < p < 1$, then the distribution of $B^*(\tau)$ belongs to the class $\mathcal{S}^*$.

PROOF. By conditioning on τ,

$$\Pr(B^*(\tau) > x) = 2\Pr(B(\tau) > x) \quad \forall x \geq 0.$$

Thus, we only need to prove that the distribution of $B^+(\tau) = B(\tau) \vee 0$ belongs to the class $\mathcal{S}^*$. Clearly,

$$B^+(\tau) \stackrel{\mathrm{D}}{=} X\sqrt{\tau}, \tag{2.8}$$

where X and τ are independent, X follows the distribution $\Phi(x \vee 0)$ for $x \geq 0$ where $\Phi(\cdot)$ denotes the standard normal distribution, and $\stackrel{\mathrm{D}}{=}$ means equality in distribution. Elementary calculation gives

$$\Pr(X > x) \sim \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}\left(1 + \frac{2\log x}{x^2}\right)x^2\right\},$$

which shows that the distribution of X is of Weibull type with shape parameter 2. By the assumption on τ, it is easy to see that $\sqrt{\tau}$ follows a Weibull-type distribution too with shape parameter $2p$. Therefore by Theorem 2.1, the distribution of $X\sqrt{\tau}$ in (2.8) belongs to the class $\mathcal{S}^*$. $\square$

3. Preliminaries

In the rest of this paper, for two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) \asymp b(x)$ if $0 < \liminf a(x)/b(x) \leq \limsup a(x)/b(x) < \infty$ and write $a(x) \lesssim b(x)$ or $b(x) \gtrsim a(x)$ if $\limsup a(x)/b(x) \leq 1$. Let C denote an absolute positive constant whose value may vary from line to line.

PROPOSITION 3.1. *Let F_1 and F_2 be as given in (2.3). Assume that $b_1 = b_2 = 1$ and $0 < p_2 \leq p_1 < \infty$, and write $a = (p_1/p_2)^{1/(p_1+p_2)}$. If $b_1(\cdot)$ is differentiable and its derivative satisfies*

$$-p_1 < \liminf_{x\to\infty} b_1'(x)x \leq \limsup_{x\to\infty} b_1'(x)x < \infty, \tag{3.1}$$

then, for all $\varepsilon > 0$,

$$\overline{G}(x) \asymp \int_{(1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1} - b_2(y)y^{p_2}\}y^{p_2-1}\,dy. \tag{3.2}$$

PROOF. For any fixed $\varepsilon > 0$, we choose $\delta > 0$ such that $(2+\varepsilon)(1-\delta) > 2(1+\delta)$. Since

$$\begin{aligned}\overline{G}(x) &\geq \Pr(X_1 > x^{p_2/(p_1+p_2)})\Pr(X_2 > x^{p_1/(p_1+p_2)}) \\ &\asymp \exp\{-b_1(x^{p_2/(p_1+p_2)})x^{p_1p_2/(p_1+p_2)} - b_2(x^{p_1/(p_1+p_2)})x^{p_1p_2/(p_1+p_2)}\} \\ &\gtrsim \exp\{-2(1+\delta)x^{p_1p_2/(p_1+p_2)}\},\end{aligned}$$

then

$$\begin{aligned}&\overline{F_1}((2+\varepsilon)^{1/p_2}ax^{p_2/(p_1+p_2)}) \\ &\qquad \asymp \exp\{-b_1((2+\varepsilon)^{1/p_2}ax^{p_2/(p_1+p_2)})(2+\varepsilon)^{p_1/p_2}a^{p_1}x^{p_1p_2/(p_1+p_2)}\} \\ &\qquad \lesssim \exp\{-(2+\varepsilon)(1-\delta)x^{p_1p_2/(p_1+p_2)}\} \\ &\qquad = o(\overline{G}(x)).\end{aligned} \tag{3.3}$$

Likewise,

$$\overline{F_2}((2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}) = o(\overline{G}(x)). \tag{3.4}$$

Therefore,

$$\overline{G}(x) = \int_0^\infty \overline{F_1}(x/y)\,dF_2(y) \sim \int_{(2+\varepsilon)^{-1/p_2}a^{-1}x^{p_1/(p_1+p_2)}}^{(2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \overline{F_1}(x/y)\,dF_2(y). \tag{3.5}$$

Using integration by parts and substituting (3.3) and (3.4), we obtain that

$$\begin{aligned}\overline{G}(x) &\asymp \int_{(2+\varepsilon)^{-1/p_2}a^{-1}x^{p_1/(p_1+p_2)}}^{(2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1}\}\,dF_2(y) \\ &= o(\overline{G}(x)) + \int_{(2+\varepsilon)^{-1/p_2}a^{-1}x^{p_1/(p_1+p_2)}}^{(2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \overline{F_2}(y)\,d\exp\{-b_1(x/y)(x/y)^{p_1}\}.\end{aligned}$$

By (3.1), for all large x, the exponential function behind the differential operator is strictly increasing in y in the indicated interval. It follows from (2.3) and (3.1) that

$$\overline{G}(x) \asymp \int_{(2+\varepsilon)^{-1/p_2}a^{-1}x^{p_1/(p_1+p_2)}}^{(2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1} - b_2(y)y^{p_2}\}y^{p_2-1}\,dy.$$

Split the integral on the right-hand side above into three parts as

$$\begin{aligned}I_1(x) + I_2(x) + I_3(x) &= \int_{(2+\varepsilon)^{-1/p_2}a^{-1}x^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)}} + \int_{(1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \\ &\quad + \int_{(1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}}^{(2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}}.\end{aligned}$$

To obtain (3.2), it suffices to prove that

$$I_1(x) + I_3(x) = o(I_2(x)). \tag{3.6}$$

Note that $(x/y)^{p_1} + y^{p_2}$, as a function of y, decreases when $0 < y \le ax^{p_1/(p_1+p_2)}$ and increases when $y \ge ax^{p_1/(p_1+p_2)}$. On the one hand,

$$\begin{aligned}
I_1(x) &\lesssim \int_{(2+\varepsilon)^{-1/p_2}a^{-1}x^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-(1-\varepsilon^3)((x/y)^{p_1} + y^{p_2})\}y^{p_2-1}\,dy \\
&\le Cx^{p_1p_2/(p_1+p_2)} \\
&\qquad \times \exp\{-(1-\varepsilon^3)((1+\varepsilon)^{p_1/p_2}a^{-p_1} + (1+\varepsilon)^{-1}a^{p_2})x^{p_1p_2/(p_1+p_2)}\}
\end{aligned}$$

and

$$\begin{aligned}
I_3(x) &\lesssim \int_{(1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}}^{(2+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-(1-\varepsilon^3)((x/y)^{p_1} + y^{p_2})\}y^{p_2-1}\,dy \\
&\le Cx^{p_1p_2/(p_1+p_2)} \\
&\qquad \times \exp\{-(1-\varepsilon^3)((1+\varepsilon)^{-p_1/p_2}a^{-p_1} + (1+\varepsilon)a^{p_2})x^{p_1p_2/(p_1+p_2)}\}.
\end{aligned}$$

On the other hand, it is easy to see that

$$\begin{aligned}
I_2(x) &\gtrsim \int_{(1+\varepsilon/2)^{-1/p_2}ax^{p_1/(p_1+p_2)}}^{ax^{p_1/(p_1+p_2)}} \exp\{-(1+\varepsilon^3)((x/y)^{p_1} + y^{p_2})\}y^{p_2-1}\,dy \\
&\qquad + \int_{ax^{p_1/(p_1+p_2)}}^{(1+\varepsilon/2)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-(1+\varepsilon^3)((x/y)^{p_1} + y^{p_2})\}y^{p_2-1}\,dy \\
&\gtrsim Cx^{p_1p_2/(p_1+p_2)} \\
&\qquad \times \exp\{-(1+\varepsilon^3)((1+\varepsilon/2)^{p_1/p_2}a^{-p_1} + (1+\varepsilon/2)^{-1}a^{p_2})x^{p_1p_2/(p_1+p_2)}\} \\
&\qquad + Cx^{p_1p_2/(p_1+p_2)} \\
&\qquad \times \exp\{-(1+\varepsilon^3)((1+\varepsilon/2)^{-p_1/p_2}a^{-p_1} + (1+\varepsilon/2)a^{p_2})x^{p_1p_2/(p_1+p_2)}\}.
\end{aligned}$$

To prove (3.6), use Taylor's expansion to expand both

$$(1+\varepsilon/2)^{p_1/p_2}a^{-p_1} + (1+\varepsilon/2)^{-1}a^{p_2} \quad \text{and} \quad (1+\varepsilon)^{p_1/p_2}a^{-p_1} + (1+\varepsilon)^{-1}a^{p_2}$$

in ε up to the ε^2 term. Then we find that the coefficients of the constant terms and the ε terms are equal, but the coefficient of the ε^2 term of the first is smaller than the corresponding coefficient of the second. Therefore, for all small $\varepsilon > 0$,

$$\begin{aligned}
&(1+\varepsilon^3)((1+\varepsilon/2)^{p_1/p_2}a^{-p_1} + (1+\varepsilon/2)^{-1}a^{p_2}) \\
&\qquad < (1-\varepsilon^3)((1+\varepsilon)^{p_1/p_2}a^{-p_1} + (1+\varepsilon)^{-1}a^{p_2}),
\end{aligned} \tag{3.7}$$

which implies that $I_1(x) = o(I_2(x))$. Similarly as above, for all small $\varepsilon > 0$,

$$\begin{aligned}
&(1+\varepsilon^3)((1+\varepsilon/2)^{-p_1/p_2}a^{-p_1} + (1+\varepsilon/2)a^{p_2}) \\
&\qquad < (1-\varepsilon^3)((1+\varepsilon)^{-p_1/p_2}a^{-p_1} + (1+\varepsilon)a^{p_2}),
\end{aligned} \tag{3.8}$$

which implies that $I_3(x) = o(I_2(x))$. Hence, relation (3.6) holds. This proves that relation (3.2) holds for all small $\varepsilon > 0$, hence, for all $\varepsilon > 0$. □

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PROPOSITION 3.2. *Let F_1 and F_2 be as given in (2.3) with $0 < p_2 \le p_1 < \infty$ satisfying (2.4). Further assume that $b_1(\cdot)$ is differentiable with $|b_1'(x)| = O(x^{-1})$. Then $G = F_1 \otimes F_2 \in \mathcal{L}(0)$.*

PROOF. It suffices to prove that $\overline{G}(x+1) \gtrsim \overline{G}(x)$. For any fixed small $\varepsilon > 0$, write

$$\Delta(\varepsilon, x) = ((2+\varepsilon)^{-1/p_2} a^{-1} x^{p_1/(p_1+p_2)}, (2+\varepsilon)^{1/p_2} a x^{p_1/(p_1+p_2)}],$$

where $a = (p_1/p_2)^{1/(p_1+p_2)}$ as before. By (3.3)–(3.5),

$$\overline{G}(x+1) \sim \int_{y\in\Delta(\varepsilon,x)} \frac{\overline{F_1}((x+1)/y)}{\overline{F_1}(x/y)} \overline{F_1}(x/y)\, dF_2(y) \gtrsim L(x)\overline{G}(x),$$

where

$$\begin{aligned} L(x) &= \inf_{y\in\Delta(\varepsilon,x)} \frac{\overline{F_1}((x+1)/y)}{\overline{F_1}(x/y)} \\ &= \inf_{y\in\Delta(\varepsilon,x)} \frac{c_1((x+1)/y)\exp\{-b_1((x+1)/y)((x+1)/y)^{p_1}\}}{c_1(x/y)\exp\{-b_1(x/y)(x/y)^{p_1}\}}. \end{aligned}$$

We need to prove that, uniformly for all $y \in \Delta(\varepsilon, x)$,

$$I(x,y) = b_1((x+1)/y)((x+1)/y)^{p_1} - b_1(x/y)(x/y)^{p_1} \to 0. \tag{3.9}$$

Since $b_1(x)$ is differentiable with $|b_1'(x)| = O(x^{-1})$ we have,

$$\begin{aligned} |I(x,y)| &= \left|(b_1((x+1)/y) - b_1(x/y))((x+1)/y)^{p_1} + b_1(x/y)\frac{(x+1)^{p_1} - x^{p_1}}{y^{p_1}}\right| \\ &\le ((x+1)/y)^{p_1} \int_{x/y}^{(x+1)/y} |b_1'(z)|\, dz + b_1(x/y)\frac{(x+1)^{p_1} - x^{p_1}}{y^{p_1}} \\ &\le C x^{p_1 p_2/(p_1+p_2)} \int_{x/y}^{(x+1)/y} z^{-1}\, dz + C x^{p_1 p_2/(p_1+p_2)-1} \\ &\le C x^{p_1 p_2/(p_1+p_2)-1} \end{aligned}$$

uniformly for all $y \in \Delta(\varepsilon, x)$. Hence, by (2.4), relation (3.9) holds. □

4. Proof of Theorem 2.1

If we have proven Theorem 2.1 for $b_1 = b_2 = 1$, then using the identity

$$Y = \frac{1}{b_1^{1/p_1} b_2^{1/p_2}} (b_1^{1/p_1} X_1)(b_2^{1/p_2} X_2),$$

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the extension to the general case is straightforward because $b_1^{1/p_1}X_1$ and $b_2^{1/p_2}X_2$ still have Weibull-type distributions and, by the definition of the class $\mathcal{S}^*$, the distribution of Y belonging to $\mathcal{S}^*$ is equivalent to the distribution of the product $(b_1^{1/p_1}X_1)(b_2^{1/p_2}X_2)$ belonging to $\mathcal{S}^*$. Therefore, we may assume without loss of generality that $b_1 = b_2 = 1$. We may also assume without loss of generality that $0 < p_2 \le p_1 < \infty$. Hence, relation (2.5) holds with $i = 1$. Recall that $a = (p_1/p_2)^{1/(p_1+p_2)}$.

In the proof below ε denotes a positive constant which can be arbitrarily small. Denote by $w(\cdot)$ the function on the right-hand side of (3.2); that is,

$$w(x) = \int_{(1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1} - b_2(y)y^{p_2}\}y^{p_2-1}\,dy \quad \forall x \ge 0.$$

We need to verify that $w(x)$ is eventually nonincreasing. By direct computation, shows that

$$w'(x) = J_1(x) - J_2(x) - J_3(x), \tag{4.1}$$

where

$$\begin{aligned} J_1(x) = {} & \frac{p_1}{p_1+p_2}(1+\varepsilon)a^{p_2}x^{p_1p_2/(p_1+p_2)-1} \\ & \times \exp\{-b_1((1+\varepsilon)^{-1/p_2}a^{-1}x^{p_2/(p_1+p_2)})(1+\varepsilon)^{-p_1/p_2}a^{-p_1}x^{p_1p_2/(p_1+p_2)}\} \\ & \times \exp\{-b_2((1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)})(1+\varepsilon)a^{p_2}x^{p_1p_2/(p_1+p_2)}\}, \end{aligned}$$

$$\begin{aligned} J_2(x) = {} & \frac{p_1}{p_1+p_2}(1+\varepsilon)^{-1}a^{p_2}x^{p_1p_2/(p_1+p_2)-1} \\ & \times \exp\{-b_1((1+\varepsilon)^{1/p_2}a^{-1}x^{p_2/(p_1+p_2)})(1+\varepsilon)^{p_1/p_2}a^{-p_1}x^{p_1p_2/(p_1+p_2)}\} \\ & \times \exp\{-b_2((1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)})(1+\varepsilon)^{-1}a^{p_2}x^{p_1p_2/(p_1+p_2)}\}, \end{aligned}$$

and

$$\begin{aligned} J_3(x) = {} & \int_{(1+\varepsilon)^{-1/p_2}ax^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{1/p_2}ax^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1} - b_2(y)y^{p_2}\} \\ & \times \left(b_1'(x/y)(x/y)^{p_1}\frac{1}{y} + b_1(x/y)p_1\frac{x^{p_1-1}}{y^{p_1}}\right)y^{p_2-1}\,dy. \end{aligned}$$

It is easy to see that

$$\begin{aligned} J_1(x) \lesssim {} & Cx^{p_1p_2/(p_1+p_2)-1} \\ & \times \exp\{-(1-\varepsilon^3)((1+\varepsilon)^{-p_1/p_2}a^{-p_1} + (1+\varepsilon)a^{p_2})x^{p_1p_2/(p_1+p_2)}\}. \end{aligned}$$

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By (2.5) and the monotonicity of the function $(x/y)^{p_1} + y^{p_2}$ in y,

$$\begin{aligned} J_3(x) &\gtrsim C \int_{(1+\varepsilon)^{-1/p_2} a x^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{1/p_2} a x^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1} - b_2(y) y^{p_2}\} \frac{x^{p_1-1}}{y^{p_1}} y^{p_2-1}\, dy \\ &\gtrsim C x^{2p_1p_2/(p_1+p_2)-1} \\ &\quad \times \int_{a x^{p_1/(p_1+p_2)}}^{(1+\varepsilon/2)^{1/p_2} a x^{p_1/(p_1+p_2)}} \exp\{-b_1(x/y)(x/y)^{p_1} - b_2(y) y^{p_2}\} y^{-1}\, dy \\ &\gtrsim C x^{2p_1p_2/(p_1+p_2)-1} \\ &\quad \times \exp\{-(1+\varepsilon^3)((1+\varepsilon/2)^{-p_1/p_2} a^{-p_1} + (1+\varepsilon/2) a^{p_2}) x^{p_1p_2/(p_1+p_2)}\}. \end{aligned}$$

Therefore for all small $\varepsilon > 0$ such that (3.7)–(3.8) hold,

$$\begin{aligned} \frac{J_1(x)}{J_3(x)} &\lesssim \frac{C}{x^{p_1p_2/(p_1+p_2)}} \\ &\quad \times \frac{\exp\{-(1-\varepsilon^3)((1+\varepsilon)^{-p_1/p_2} a^{-p_1} + (1+\varepsilon) a^{p_2}) x^{p_1p_2/(p_1+p_2)}\}}{\exp\{-(1+\varepsilon^3)((1+\varepsilon/2)^{-p_1/p_2} a^{-p_1} + (1+\varepsilon/2) a^{p_2}) x^{p_1p_2/(p_1+p_2)}\}}, \end{aligned}$$

so $J_1(x)/J_3(x) \to 0$. Likewise, $J_2(x)/J_3(x) \to 0$. It follows from (4.1) that

$$w'(x) \sim -J_3(x). \tag{4.2}$$

Hence, $w'(x)$ is negative for all large x and $w(x)$ is eventually monotone.

Introduce a distribution W on $[0, \infty)$ with tail

$$\overline{W}(x) = \begin{cases} 1 & \text{if } x \le x_0, \\ w(x) & \text{if } x > x_0, \end{cases}$$

where $x_0 > 0$ is some large number such that $w(x)$ is nonincreasing for $x \ge x_0$ and $w(x_0) \le 1$. By Propositions 3.1–3.2 above and [16, Theorem 2.1(b)], it suffices to verify that $W \in \mathcal{S}^*$, which amounts to

$$\lim_{x\to\infty} \int_0^x \frac{\overline{W}(x-y)\overline{W}(y)}{\overline{W}(x)}\, dy = 2 \int_0^\infty \overline{W}(y)\, dy. \tag{4.3}$$

By the definition of W, following the proof of Proposition 3.2, we can obtain that $W \in \mathcal{L}(0)$. Write $R(x) = -\log \overline{W}(x)$ and $r(x) = R'(x)$. Recall that $(x/y)^{p_1} + y^{p_2}$, as a function of y, attains its minimum at $y = a x^{p_1/(p_1+p_2)}$,

$$\begin{aligned} \overline{W}(x) &\lesssim \int_{(1+\varepsilon)^{-1/p_2} a x^{p_1/(p_1+p_2)}}^{(1+\varepsilon)^{1/p_2} a x^{p_1/(p_1+p_2)}} \exp\{-(1-\varepsilon)((x/y)^{p_1} + y^{p_2})\} y^{p_2-1}\, dy \\ &\le C x^{p_1p_2/(p_1+p_2)} \exp\{-(1-\varepsilon)(a^{-p_1} + a^{p_2}) x^{p_1p_2/(p_1+p_2)}\}, \end{aligned}$$

which implies that

$$R(x) \gtrsim (1-\varepsilon)(a^{-p_1} + a^{p_2}) x^{p_1p_2/(p_1+p_2)}. \tag{4.4}$$

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Write $l_1 = \limsup b_1^{'}(x)x$. It follows from (4.2) and (2.5) that, for all $x > x_0$,

$$r(x) = -\frac{(\overline{W}(x))'}{\overline{W}(x)} \sim \frac{-w'(x)}{w(x)} \sim \frac{J_3(x)}{w(x)} \lesssim (1+\varepsilon)^{p_1/p_2} a^{-p_1}(l_1 + p_1) x^{p_1 p_2/(p_1+p_2)-1}. \tag{4.5}$$

Furthermore, by (2.4), uniformly for all $0 \le y \le x/2$,

$$\int_{x-y}^{x} v^{p_1 p_2/(p_1+p_2)-1}\,dv \le y(x-y)^{p_1 p_2/(p_1+p_2)-1} \le y^{p_1 p_2/(p_1+p_2)}. \tag{4.6}$$

Using (4.4)–(4.6) we obtain that, for all $0 \le y \le x/2$ and all large $x > 0$,

$$\begin{aligned}\frac{\overline{W}(x-y)\overline{W}(y)}{\overline{W}(x)} &= \exp\bigg\{\int_{x-y}^{x} r(v)\,dv - R(y)\bigg\} \\ &\le \exp\bigg\{(1+2\varepsilon)^{p_1/p_2} a^{-p_1}(l_1+p_1)\int_{x-y}^{x} v^{p_1 p_2/(p_1+p_2)-1}\,dv \\ &\qquad - (1-2\varepsilon)(a^{-p_1} + a^{p_2}) y^{p_1 p_2/(p_1+p_2)}\bigg\} \\ &\le \exp\{((1+2\varepsilon)^{p_1/p_2} a^{-p_1}(l_1+p_1) \\ &\qquad - (1-2\varepsilon)(a^{-p_1} + a^{p_2})) y^{p_1 p_2/(p_1+p_2)}\}.\end{aligned} \tag{4.7}$$

By (2.5) and the definition of a, we obtain

$$a^{-p_1}(l_1+p_1) < a^{-p_1}\frac{p_1+p_2}{p_2} = a^{-p_1} + a^{p_2}.$$

Consequently, for all small $\varepsilon > 0$,

$$(1+2\varepsilon)^{p_1/p_2} a^{-p_1}(l_1+p_1) < (1-2\varepsilon)(a^{-p_1} + a^{p_2}).$$

Therefore, the right-hand side of (4.7) as a function of y is integrable over $[0, \infty)$. Applying the dominated convergence theorem, we obtain (4.3), as

$$\lim_{x\to\infty}\int_0^x \frac{\overline{W}(x-y)\overline{W}(y)}{\overline{W}(x)}\,dy = 2\lim_{x\to\infty}\int_0^{x/2} \frac{\overline{W}(x-y)\overline{W}(y)}{\overline{W}(x)}\,dy = 2\int_0^{\infty}\overline{W}(y)\,dy.$$

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YAN LIU, School of Mathematics and Statistics, Wuhan University, Wuhan, Hubei 430072, PR China
e-mail: yanliu@whu.edu.cn

QIHE TANG, Department of Statistics and Actuarial Science, University of Iowa, 241 Schaeffer Hall, Iowa City, IA 52242, USA
e-mail: qtang@stat.uiowa.edu

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