BULL. AUSTRAL. MATH. SOC. 60B05, 46B99
VOL. 12 (1975), 31-47.

Convergence of elements in random normed spaces

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For a random normed space of mappings into a separable normed linear space, convergence of identically distributed elements in the random norm (norm distribution) is shown to be equivalent to convergence in measure in the weak linear topology. Convergence in measure in each coordinate of a Schauder basis is also shown to be a necessary and sufficient condition for convergence in the random norm topology. These results have laws of large numbers for random elements in separable normed linear spaces as almost immediate corollaries and illustrate some of the recently obtained laws of large numbers for random elements. Similar results are also given for elements which need not have the same norm distributions, and the results are extended to linear metric spaces. Finally, applications of the results to stochastic processes are considered.

1. Introduction and preliminaries

The recent consideration of a stochastic process as a random element in a function space (a measurable function from a probability space to a function space) by Doob [5], Mann [7], Prokhorov [15], Billingsley [4], and others, has inspired the study of random elements and their properties. In particular, the laws of large numbers (the Cesàro convergence of a sequence or measurable functions) have been generalized to random elements by Mourier [9], [10], Beck [1], Beck and Warren [3], Beck and Giesy [2], Taylor [17], Taylor and Padgett [18], and others. These recent extensions

Received 24 September 1974.

of the laws of large numbers were summarized by Padgett, Taylor [14]. At the same time the concept of a random normed space of mappings and related properties were being developed by Šerstnev [16], Muštari, Šerstnev [12], and Mushtari [11]. In this paper results are obtained for convergence in the random norm topology of a linear space of random elements. In particular, necessary and sufficient conditions for the convergence are given in terms of the dual space of a separable normed linear space and in terms of coordinatewise conditions. These results illustrate some of the laws of large numbers referred to above and relate the concepts of random elements and random normed spaces of mappings.

A *random normed space of mappings* into a normed linear space X with norm $\| \ \|$ is defined [11, p. 337] to be a linear space L whose elements are functions from a probability space (Ω, F, P) into X such that $\{\omega \in \Omega : \|V(\omega)\| \geq \alpha\} \in F$ for any $V \in L$ and any real number α. The *random norm* (or norm distribution) on L is defined by $\rho(V)(\delta) = P[\|V\| \geq \delta]$, and the topology T of convergence in the random norm on L is given by the system $\{N_{\alpha,\delta} : \alpha > 0 \text{ and } \delta > 0\}$ of neighborhoods of θ (the zero element of L) where $N_{\alpha,\delta} = \{V \in L : \rho(V)(\delta) \leq \alpha\}$. When X is separable, one example of a random normed space of mappings is the set of all random elements from a probability space to X, that is, the set of functions which are measurable with respect to the sigma-field generated by the open subsets of X. However, when X is not separable, the set of all random elements may not be a linear space since the sum of two random elements need not be a random element [13]. For convenience, it will be assumed throughout this paper that the constant functions are in L.

The topology T defined by the random norm is the same as the topology of convergence in measure and is pseudometrizable. A pseudometric which generates the topology T is given by

$$d(V, Z) = \int \frac{\|V-Z\|}{1+\|V-Z\|} \, dP ,$$

where V and Z are in L, and it easily follows that $V_\lambda \to V$ in the random norm topology if and only if $d(V_\lambda, V) \to 0$.

Elements $\{V_\lambda : \lambda \in A\}$ in L are said to be identically distributed if $P[V_{\lambda_1} \in D] = P[V_{\lambda_2} \in D]$ for all $\lambda_1, \lambda_2 \in A$ and for each Borel subset D. Finally, $E\|V\|$ will denote the Lebesgue integral

$$\int \|V\| dP ,$$

and $E\|V\|^r < \infty$ denotes that $V \in L_r(\Omega, F, P)$ for $r > 0$.

2. Convergence results for random normed spaces of mappings

In this section it will be shown that for identically distributed elements in L convergence in the random norm topology is equivalent to convergence in measure in the weak linear topology of a separable normed linear space. If the normed linear space is a Banach space, then convergence of identically distributed elements in the random norm topology will be shown to be equivalent to convergence in measure coordinatewise for any Schauder basis. These results will also be extended to classes of elements which need not be identically distributed. Finally, an example will be given where the elements of L satisfy the hypothesis of the results and converges in measure but do not converge almost everywhere.

The first results will be for Banach spaces which have Schauder bases. These results are used in obtaining the results for separable normed linear spaces, but perhaps more importantly they provide convergence in the random norm topology with only coordinatewise conditions. In particular, they are used later to obtain results for linear metric spaces. Note that since T is pseudometrizable it suffices to state the result in terms of sequences.

THEOREM 1. *Let X be a Banach space which has a Schauder basis $\{b_k, f_k\}$, and let $\{V_n\}$ be a sequence of identically distributed elements in L, and let $V \in L$. For each k, $f_k(V_n)$ converges to $f_k(V)$ in measure if and only if V_n converges in L to V.*

Proof. The "if" part is obvious since convergence in L is equivalent to the convergence of $P[\|V_n - V\| > \varepsilon]$ to 0 for each $\varepsilon > 0$.

For the "only if" part, let $U_t(x) = \sum_{k=1}^{t} f_k(x) b_k$,

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$Q_t(x) = x - U_t(x)$, and $\varepsilon > 0$ be given. For each t and for all n ,

(2.1) $$d(V_n, V) = d(V_n - V, 0) \le d\big(U_t(V_n-V), 0\big) + d\big(Q_t(V_n-V), 0\big) .$$

Also, for each t and for all n ,

(2.2) $$\begin{aligned} d\big(Q_t(V_n-V), 0\big) &\le d\big(Q_t(V_n), 0\big) + d\big(Q_t(V), 0\big) \\ &= \int \frac{\|Q_t(V_n)\|}{1+\|Q_t(V_n)\|}\,dP + \int \frac{\|Q_t(V)\|}{1+\|Q_t(V)\|}\,dP \\ &= \int \frac{\|Q_t(V_1)\|}{1+\|Q_t(V_1)\|}\,dP + \int \frac{\|Q_t(V)\|}{1+\|Q_t(V)\|}\,dP , \end{aligned}$$

since the elements $\{Q_t(V_n) : n = 1, 2, \ldots\}$ are identically distributed for each t and $f(y) = \frac{y}{1+y}$ for $y \ge 0$ is a Borel function. Since $Q_t(x) \to 0$ for each $x \in X$ and $\frac{y}{1+y} \le 1$ for all $y \ge 0$, t can be chosen so that $d\big(Q_t(V_n-V), 0\big) < \varepsilon/2$ for all n . For this t ,

(2.3) $$\begin{aligned} d\big(U_t(V_n-V), 0\big) &= \int \frac{\|U_t(V_n-V)\|}{1+\|U_t(V_n-V)\|}\,dP \\ &\le \sum_{k=1}^{t} \int \frac{|f_k(V_n-V)|\,\|b_k\|}{1+|f_k(V_n-V)|\,\|b_k\|}\,dP . \end{aligned}$$

Since $f_k(V_n-V) \to 0$ in probability for each k , there exists N such that $d\big(U_t(V_n-V), 0\big) < \varepsilon/2$ for all $n \ge N$. Thus, from (2.1),

$$d(V_n, V) < \varepsilon/2 + \varepsilon/2 = \varepsilon$$

for all $n \ge N$. //

The condition of identically distributed elements was used to achieve the uniform truncation (in n) to finite-dimensional subspaces of the Banach spaces. It is easy to see that the condition of identical distributions can not be replaced by uniform boundedness of the elements even when X is assumed to be a separable Hilbert space. Let $U^{(n)}$ denote the sequence of zeros except for a one in the nth coordinate. The

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constant elements of L, $V_n \equiv U^{(n)}$, converge to the zero sequence in the weak linear topology of $X = l^2$, but $\|V_n\| \equiv 1$ for all n. Hence, $V_n = U^{(n)}$ does not converge to the zero sequence in L.

Other conditions can be used to obtain the uniform truncation to finite-dimensional subspaces, and hence the condition of identically distributed elements can be relaxed. For example, Corollaries 2 and 3 will show that convergence in each coordinate of a Schauder basis is equivalent to convergence in the random norm for classes of elements in L which need not be identically distributed. Later, it will be shown that the structure of the elements in Corollaries 2 and 3 is quite easy to obtain in application of these results to stochastic processes. Recall that random variables are measurable functions from a probability space into the real numbers.

COROLLARY 2. *Let X be a Banach space which has a Schauder basis $\{b_n, f_n\}$, and let $\{V_n\}$ be a sequence of identically distributed elements in L, and let $V \in L$. If $\{A_n\}$ is a corresponding sequence of random variables such that the essential suprémum of $|A_n|$ is uniformly bounded in n, then for each k, $f_k(A_nV_n)$ converges to V in measure if and only if A_nV_n converges in L to V.*

The proof of Corollary 2 will not be given here but follows by combining the proof of Theorem 1 with an appropriate modification of the proof of Corollary 3.

COROLLARY 3. *Let X be a Banach space which has a Schauder basis $\{b_n, f_n\}$, let $V \in L$, and let $\{V_n\}$ be a sequence of identically distributed elements in L such that*

$$E\|V_1\|^{(r/r-1)} < \infty \text{ for some } r > 1 .$$

If $\{A_n\}$ is a corresponding sequence of random variables such that $\left\{E\left[|A_n|^r\right]\right\}$ is a bounded sequence, then for each k, $f_k(A_nV_n)$ converges to V in measure if and only if A_nV_n converges in L to V.

Proof. The "if" part is again obvious. For the "only if" part, define U_t and Q_t as in the proof of Theorem 1. Let $\varepsilon > 0$ be given. For each t and for all n,

$$(2.4) \qquad d(A_nV_n, V) \le d\left(U_t(A_nV_n-V), 0\right) + d\left(Q_t(A_nV_n-V), 0\right) .$$

Also, for each t and for all n,

$$\begin{aligned}(2.5) \quad d\left(Q_t(A_nV_n-V), 0\right) &\le d\left(Q_t(A_nV_n), 0\right) + d\left(Q_t(V), 0\right) \\ &\le \int \frac{|A_n|\|Q_t(V_n)\|}{1+|A_n|\|Q_t(V_n)\|}\,dP + \int \frac{\|Q_t(V)\|}{1+\|Q_t(V)\|}\,dP \\ &\le E\left[|A_n|\|Q_t(V_n)\|\right] + E\left[\frac{\|Q_t(V)\|}{1+\|Q_t(V)\|}\right] \\ &\le \left(E|A_n|^r\right)^{1/r}\left(E\|Q_t(V_1)\|^{r/(r-1)}\right)^{(r-1)/r} \\ &\qquad + E\left[\frac{\|Q_t(V)\|}{1+\|Q_t(V)\|}\right] .\end{aligned}$$

Since $Q_t(x) \to 0$ for each $x \in X$, $\frac{y}{1+y} \le 1$ for all $y \ge 0$, and $\left\{E|A_n|^r\right\}$ is a bounded sequence, t can be chosen so that $d\left(Q_t(A_nV_n-V), 0\right) < \frac{\varepsilon}{2}$ for all n. Similarly to (2.3) it follows that for this t there exists N such that $d\left(U_t(A_nV_n-V), 0\right) < \frac{\varepsilon}{2}$ for all $n \ge N$. Hence, by (2.4), the proof is complete. //

These results obviously also hold for any normed linear space which has a Schauder basis so that the partial sum operators $\{U_t\}$ are uniformly bounded in norm (and hence are continuous and Borel measurable). For example, X could be a normed linear space which has a monotone basis [19]. Also, the structure of the product sequence $\{A_nV_n\}$ is often easily obtained in applications. Let $\{Z_n\}$ be a sequence of separable Wiener processes on $[0, 1]$. With probability one these processes can be regarded as random elements in the Banach space $C[0, 1]$ of real-valued continuous functions on $[0, 1]$ and hence are elements in L when

$X = C[0, 1]$. By letting A_n be the constant random variable $E\left[Z_n(1)^2\right]^{\frac{1}{2}}$ and $Z_n = A_n V_n$ for each n , the elements $\{V_n\}$ are identically distributed. An extension of this example will be given in the last section when random mappings in Fréchet spaces are introduced and applications of these results are considered. But first, the results for Banach spaces which have Schauder bases will be used to obtain the equivalence of convergence in measure in the weak linear topology of a separable normed linear space and convergence in the random normed space of mappings.

THEOREM 4. *Let X be a separable normed linear space, let $V \in L$, and let $\{V_n\}$ be a sequence of identically distributed elements in L . For each continuous linear functional f , $f(V_n)$ converges to $f(V)$ in measure if and only if V_n converges in L to V .*

Proof. Since X can be embedded isometrically in the Banach space $C[0, 1]$ with norm $\|x\| = \sup_t |x(t)|$ [8, p. 67], there exists a one-to-one, bicontinuous, linear function h from X into $C[0, 1]$. The elements $\{h(V_n)\}$ are identically distributed in a random normed space of mappings into $C[0, 1]$, and $C[0, 1]$ has a Schauder basis. For each continuous linear functional g on $C[0, 1]$ (and hence for each coordinate functional), $g(h(V_n)) = (h^*g)(V_n)$ converges to $(h^*g)(V) = g(h(V))$ in measure where h^* is the adjoint function from $C[0, 1]^*$ into X^* . Thus, by Theorem 1, $\|h(V_n)-h(V)\| \to 0$ in measure, and hence $\|V_n-V\| \to 0$ in measure.

COROLLARY 5. *Let X be a separable normed linear space, let $V \in L$, and let $\{V_n\}$ be a sequence of identically distributed elements in L . If $\{A_n\}$ is a corresponding net of random variables such that the essential supremum of $|A_n|$ is uniformly bounded in n , then for each continuous linear functional $f \in X^*$, $f(A_nV_n)$ converges to $f(V)$ in measure if and only if A_nV_n converges in L to V .*

COROLLARY 6. *Let X be a separable normed linear space, let*

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$V \in L$, and let $\{V_n\}$ be a sequence of identically distributed elements in L such that

$$E\|V_1\|^{r/(r-1)} < \infty \text{ for some } r > 1 .$$

If $\{A_n\}$ is a corresponding sequence of random variables such that $\left\{E\left[|A_n|^r\right]\right\}$ is a bounded sequence, then for each continuous linear functional $f \in X^*$, $f(A_nV_n)$ converges to $f(V)$ in measure if and only if A_nV_n converges in L to V.

3. Comparisons and laws of large numbers

In this section the condition of identical distributions in the results will again be examined, and the relationships of these results and the laws of large numbers will be considered. In both cases examples of spaces and elements will be given where the elements are identically distributed and where coordinatewise conditions or dual space conditions will be sufficient to provide convergence in the random norm and hence in measure but not convergence almost everywhere.

If the elements $\{V_n\}$ are identically distributed (or even have the same norm distribution, that is, $P[\|V_n\| \geq \alpha] = P[\|V_m\| \geq \alpha]$ for all $\alpha > 0$ and for all m and n), then $V_n \to 0$ in measure implies that $V_n = 0$ almost everywhere. This follows since $P[\|V_1\| \leq \varepsilon] = P[\|V_n\| \leq \varepsilon] \to 1$ for each $\varepsilon > 0$ as $n \to \infty$, and

$$P[\|V_1\| = 0] = P\left[\bigcap_{k=1}^{\infty} \left[\|V_1\| \leq \frac{1}{k}\right]\right]$$
$$= \lim_{k\to\infty} P\left[\|V_1\| \leq \frac{1}{k}\right] = 1 .$$

However, this does not detract from the results in the previous section, since $\{V_n\} \subset L$ being identically distributed and $V \in L$ need not imply that $\{V_n-V\}$ are identically distributed. This can be seen trivially by letting the probability structure be that of the coin-tossing problem and

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letting V_n be the indicator function for a head appearing on the nth toss. Note that $\{V_n\}$ are identically distributed but that $\{V_n - V\}$ are not when $V \equiv V_1$. To show that $\{V_n - V : n \geq m\}$ need not be identically distributed for any m, let

$$V = \sum_{k=1}^{\infty} \frac{1}{2^k} V_k$$

in the coin-tossing example.

A somewhat more involved example will give identically distributed elements in L and random variables which satisfy the conditions in Corollary 2, 3, 5, or 6, where the product sequence converges in measure but does not converge almost everywhere. Let $\Omega = \prod_{i=1}^{\infty} \Omega_i$, $F = \prod_{i=1}^{\infty} F_i$, and $P = \prod_{i=1}^{\infty} P_i$ where $\Omega_i = \{1, 2, 3, \ldots\}$, F_i = power set of Ω_i, and $P_i(\{n\}) = c/n^2$ for each i with $c = 6/\pi^2$. The space X may be taken to be l^p $(p \geq 1)$, c (convergent sequences), or c_0 (null convergent sequences). For $\omega = (\omega_1, \ldots, \omega_k, \ldots) \in \Omega$, define $V_k(\omega)$ to be $\left(\omega_k^{\frac{1}{2}}\right) u^{(\omega_k)}$ where $u^{(n)}$ is the sequence of zeros everywhere except for a one in the nth coordinate. Note that $\{V_k\}$ are identically distributed elements in L and that

$$P\left[V_k = \left(n^{\frac{1}{2}}\right) u^{(n)}\right] = P\left[\left\{\omega : V_k(\omega) = \left(n^{\frac{1}{2}}\right) u^{(n)}\right\}\right] = c/n^2$$

for every k. Define A_k to be the constant random variable $k^{-\frac{1}{2}}$ for each k. Then, $\|A_k V_k\| \to 0$ in measure since

$$\|A_k V_k\| = \left(\frac{n}{k}\right)^{\frac{1}{2}} \|u^{(n)}\| = \left(\frac{n}{k}\right)^{\frac{1}{2}}$$

with probability c/n^2 and

$$P[\|A_k V_k\| > \varepsilon] = \sum_{n>\varepsilon^2 k} \frac{c}{n^2} \to 0 \quad \text{as} \quad k \to \infty .$$

However,

$$\{\omega : \|A_k(\omega)V_k(\omega)\| \to 0\} = \bigcap_{m=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \left\{\omega : \|A_k(\omega)V_k(\omega)\| \le \frac{1}{m}\right\}$$
$$\subset \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\omega : \|A_k(\omega)V_k(\omega)\| \le 1\} .$$

But,

$$\bigcap_{k=n}^{\infty} \{\omega : \|A_k(\omega)V_k(\omega)\| \le 1\} = \bigcap_{k=n}^{\infty} \{\omega : \omega_k \le k\} .$$

Hence,

$$\begin{aligned} 0 &\le P[\{\omega : \|A_k(\omega)V_k(\omega)\| \to 0\}] \\ &\le P\left[\bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} \{\omega : \|A_k(\omega)V_k(\omega)\| \le 1\}\right] \\ &= \lim_{n\to\infty} P\left[\bigcap_{k=n}^{\infty} \{\omega : \|A_k(\omega)V_k(\omega)\| \le 1\}\right] \\ &= \lim_{n\to\infty} \prod_{k=n}^{\infty} P[\{\omega_k : \omega_k \le k\}] \\ &\le \lim_{n\to\infty} \prod_{k\ n}^{\infty} \left(1 - \frac{c}{k+1}\right) \le \lim_{n\to\infty} \prod_{k=n}^{\infty} e^{-(c/k+1)} \\ &= \lim_{n\to\infty} \lim_{m\to\infty} e^{-c\sum_{k=n}^{m} \frac{1}{k+1}} = 0 . \end{aligned}$$

Therefore, $\|A_k V_k\|$ does not converge to zero almost everywhere.

Weak laws of large numbers can be obtained from the structure of random normed spaces of mappings since convergence in the random norm topology is convergence in measure. First, let X be a Banach space which has a Schauder basis $\{b_k, f_k\}$, and let $\{V_n\}$ be a sequence of identically distributed elements from L such that $E\|V_1\| < \infty$. If the weak law of large numbers holds in each coordinate, that is, if for each

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k , $f_k\left(\frac{1}{n}\sum_{m=1}^{n} V_m\right) = \frac{1}{n}\sum_{m=1}^{n} f_k(V_m) \to E[f_k(V_1)]$ in measure as $n \to \infty$, then for each t ,

$$(3.1) \quad d\left\{U_t\left(\frac{1}{n}\sum_{m=1}^{n} V_m - EV_1\right), 0\right\} \le \sum_{k=1}^{t} \int \frac{\left|f_k\left(\frac{1}{n}\sum_{m=1}^{n} V_m\right) - f_k(EV_1)\right| \|b_k\|}{1+\left|f_k\left(\frac{1}{n}\sum_{m=1}^{n} V_m\right) - f_k(EV_1)\right| \|b_k\|} dP$$

$$\to 0 \text{ as } n \to \infty ,$$

where EV_1 denotes the Pettis integral of V_1 . Moreover,

$$(3.2) \quad d\left\{Q_t\left(\frac{1}{n}\sum_{m=1}^{n} V_m - EV_1\right), 0\right\} \le E\left[\frac{\left\|Q_t\left(\frac{1}{n}\sum_{m=1}^{n} V_m\right)\right\|}{1+\left\|Q_t\left(\frac{1}{n}\sum_{m=1}^{n} V_m\right)\right\|}\right] + \frac{\|Q_t(EV_1)\|}{1+\|Q_t(EV_1)\|}$$

$$\le \frac{1}{n}\sum_{m=1}^{n} E\|Q_t(V_m)\| + \|Q_t(EV_1)\|$$

$$= E\|Q_t(V_1)\| + \|Q_t(EV_1)\| .$$

As $t \to \infty$, the expression in (3.2) goes to zero (independent of n). Hence, from (3.1) and (3.2),

$$d\left(\frac{1}{n}\sum_{m=1}^{n} V_m, EV_1\right) \to 0$$

as $n \to \infty$, which implies that the weak law of large numbers holds; that is,

$$\left\|\frac{1}{n}\sum_{m=1}^{n} V_m - EV_1\right\| \to 0$$

in measure as $n \to \infty$. Similar to the proof of Theorem 4, it follows that for identically distributed elements in L the weak law of large numbers holding in the weak linear topology is necessary and sufficient for the weak law of large numbers to hold. Using the structure for the elements which is given in the corollaries, weak laws of large numbers are available for classes of elements which need not be identically distributed. By letting $V_n = u^{(n)}$ with probability $\frac{1}{2}$ and $V_n = -u^{(n)}$ with probability $\frac{1}{2}$ and letting $X = l^1$, it is seen that some condition of identical

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distribution is needed in general for the laws of large numbers to hold, since in this case

$$\left\|\frac{1}{n}\sum_{m=1}^{n} V_m\right\| \equiv 1 \neq 0 = EV_n \quad \text{for all } n .$$

Beck and Warren [3] constructed random elements $\{V_n\}$ in the separable Banach space of null convergent sequences which are identically distributed and uniformly bounded and where $\{f(V_n)\}$ satisfied the weak law of large numbers for each $f \in X^*$ but which did not satisfy the strong law of large numbers in the norm topology. Thus, from the development which was previously given for the random normed space of mappings and the weak laws of large numbers, their example provides identically distributed elements $\{V_n\}$ with $E\|V_1\| < \infty$ such that Cesàro convergence of $\{V_n\}$ to EV_1 occurs in measure but not almost everywhere.

4. Random metric space of mappings

In this section the results which were given in Sections 2 and 3 will be extended to certain linear metric spaces. The definition and structure of the random metric space of mappings into a linear metric space will be developed in a similar manner to that of the random normed space of mappings. It will be shown that for identically distributed elements convergence in the random metric topology is equivalent to convergence in measure in the weak linear topology of certain Fréchet spaces. These results also hold for classes of random elements which need not be identically distributed. In addition, if the linear metric space is complete, then convergence in measure coordinatewise for any Schauder basis is both necessary and sufficient for convergence of identically distributed elements in the random metric topology. Again, these results will illustrate laws of large numbers, and application of these results to stochastic processes will be considered.

A *random metric space of mappings* into a linear metric space M with metric m is defined to be a linear space L whose elements are functions from a probability space (Ω, F, P) into M such that $\{\omega \in \Omega : m(V(\omega), 0) \geq \delta\} \in F$ for any $V \in L$ and any real number δ. The *random metric* (or metric distribution) on L is defined by

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$\rho(V, Z)(\delta) = P[m(V, Z) \geq \delta]$, and the topology T of convergence in the random metric on L is given by the system $\{N_{\alpha,\delta} : \alpha > 0$ and $\delta > 0\}$ of neighborhoods of θ (the zero element of L) where $N_{\alpha,\delta} = \{V \in L : \rho(V, \theta)(\delta) \leq \alpha\}$.

To avoid trivial cases, it will again be assumed that the constant functions are in L . Hence, when M is separable, then L will be the set of random elements on M ; that is, $V \in L$ if and only if V is a measurable function with respect to the sigma-field generated by the open subsets of M . Also, the topology T defined in this manner is the same as the topology of convergence in measure and is again pseudometrizable. Identically distributed elements and integrals are defined similarly to the normed linear space case.

For the first results let M be a separable Fréchet space whose metric m is given by a sequence of seminorms $\{p_k\}$.

THEOREM 7. *Let M be a separable Fréchet space whose metric is given by the Fréchet combination of a sequence of seminorms $\{p_k\}$. Let $\{V_n\}$ be a sequence of identically distributed elements in L , and let $V \in L$. For each continuous linear functional f on M , $f(V_n)$ converges to $f(V)$ in measure if and only if V_n converges in L to V .*

Proof. Since f is a continuous linear functional on M , the "if" part is obvious. For the "only if" part, it suffices to show that for each k , $P[p_k(V_n-V) \geq \varepsilon] \to 0$ for each $\varepsilon > 0$.

Let the positive integer k be fixed, and let g be a continuous linear functional on the separable normed linear space M_k/N_k where M_k/N_k is the quotient space with $N_k = \{x \in M : p_k(x) = 0\}$ and with M_k denoting the seminormed space (M, p_k) . Let Q_k denote the canonical mapping of M_k onto M_k/N_k , and let $\| \|_k$ denote the norm on M_k/N_k defined by $\|Q_k(x)\|_k = p_k(x)$ for each $x \in M$. Thus, $\{Q_k(V_n)\}$ are identically distributed elements in a random normed space of mappings into M_k/N_k , and $Q_k(V)$ is an element in the same random normed space of mappings into M_k/N_k . Since $g \circ Q_k$ is a continuous linear functional on M_k , and hence on M , Theorem 4 implies that

$$p_k(V_n-V) = \|Q_k(V_n)-Q_k(V)\|_k \to 0$$

in measure. //

Again, the condition of identically distributed elements can not be dropped, but Corollaries 5 and 6 can similarly be stated and proved for the Fréchet space M . Also, it is important to note that stochastic processes are random elements in function spaces. In particular, discrete parameter stochastic processes are random elements in the Fréchet space s of all real-valued sequences, and hence are elements in the random metric space of mappings into the Fréchet space s . Before considering stochastic processes and the space $C[0, \infty)$, the last result for complete linear metric spaces which have Schauder bases will be given.

THEOREM 8. *Let M be a complete linear metric space which has a Schauder basis $\{b_k, f_k\}$, let $\{V_n\}$ be a sequence of identically distributed elements in L , and let $V \in L$. For each k , $f_k(V_n)$ converges to $f_k(V)$ in measure if and only if V_n converges in L to V .*

Proof. The proof will follow similarly to Theorem 1 if it can be shown that

$$\lim_{t\to\infty} \int \frac{m(Q_t(V_n-V),0)}{1+m(Q_t(V_n-V),0)}\, dP = 0$$

uniformly in n and that

$$\int \frac{m(U_t(V_n-V),0)}{1+m(U_t\ V_n-V),0)}\, dP$$

converges to zero in measure for each t .

First, since the elements $\{Q_t(V_n) : n = 1, 2, \ldots\}$ are identically distributed for each t and $Q_t(x) \to 0$ for each $x \in M$ as $t \to \infty$,

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$$\lim_{t\to\infty} \int \frac{m\left(Q_t\left(V_n-V\right),0\right)}{1+m\left(Q_t\left(V_n-V\right),0\right)}\,dP = 0$$

by the Lebesgue bounded convergence theorem. Also, since $f_k\left(V_n-V\right)$ converges to zero in measure for each k, every subsequence of

$$\left\{\int \frac{m\left(f_k\left(V_n-V\right)b_k,0\right)}{1+m\left(f_k\left(V_n-V\right)b_k,0\right)}\,dP : n = 1, 2, \ldots\right\}$$

has a further subsequence which converges to zero. Hence, for each t,

$$\lim_{n\to\infty} \int \frac{m\left(U_t\left(V_n-V\right),0\right)}{1+m\left(U_t\left(V_n-V\right),0\right)}\,dP \le \lim_{n\to\infty} \sum_{k=1}^{t} \int \frac{m\left(f_k\left(V_n-V\right)b_k,0\right)}{1+m\left(f_k\left(V_n-V\right)b_k,0\right)}\,dP$$
$$= 0 . \qquad //$$

While the condition of identically distributed elements can again not be dropped, Corollary 5 can be extended to complete linear metric spaces which have Schauder bases. However, Corollary 6 does not readily extend because of the possible non-homogeneity of the metric.

Let $\{Z_n\}$ be a sequence of separable Wiener processes on $C[0, \infty)$. With probability one the sample paths are continuous, and hence these processes can essentially be regarded as elements in the random metric space of mappings with $M = C[0, \infty)$. By letting A_n be the constant random variable $E\left[Z_n(1)^2\right]^{\frac{1}{2}}$ and $Z_n = A_nV_n$ for each n, the elements $\{V_n\}$ are identically distributed and the corollary to either Theorem 7 or Theorem 8 can be applied. In particular, $C[0, \infty)$ has a Schauder basis $\{b_n, f_n\}$ where

$$f_n(x) = x\left(r_1\right) - \frac{x\left(r_2\right)+x\left(r_3\right)}{2} ,$$

with r_1, r_2, and r_3 being dyadic rational numbers. Thus, Theorem 8 yields convergence in measure for the processes with only pointwise conditions, which also yield the weak convergence of Billingsley [4]. The Wiener processes can be replaced by any family of stochastic processes which have continuous sample paths on $[0, \infty)$, and the requirement of the same finite-dimensional distributions is sufficient to insure the condition of identical distributions.

References

[1] Anatole Beck, "On the strong law of large numbers", *Ergodic theory*, 21-53 (Proc. Internat. Sympos., Tulane University, New Orleans, October, 1961. Academic Press, New York, London, 1963).

[2] Anatole Beck and Daniel P. Giesy, "*P*-uniform convergence and a vector-valued strong law of large numbers", *Trans. Amer. Math. Soc.* 147 (1970), 541-559.

[3] Anatole Beck and Peter Warren, "A strong law of large numbers for weakly orthogonal sequences of Banach-spacè valued random variables", (MRC Technical Summary Report, #848. Mathematics Research Center, University of Wisconsin, Madison, Wisconsin, 1968).

[4] Patrick Billingsley, *Convergence of probability measures* (John Wiley & Sons, New York, London, Sydney, Toronto, 1968).

[5] J.L. Doob, "Probability in function space", *Bull. Amer. Math. Soc.* 53 (1947), 15-30.

[6] John Horváth, *Topological vector spaces and distributions*, Volume I (Addison-Wesley, Reading, Massachusetts; Palo Alto; London; Don Mills, Ontario; 1966).

[7] Henry B. Mann, "On the realization of stochastic processes by probability distributions in function spaces", *Sankhyā* 11 (1951), 3-8.

[8] Jürg T. Marti, *Introduction to the theory of bases* (Springer Tracts in Natural Philosophy, 18. Springer-Verlag, Berlin, Heidelberg, New York, 1969).

[9] Edith Mourier, "Eléments aléatoires dans un espace de Banach", *Ann. Inst. H. Poincaré* 13 (1953), 161-244.

https://doi.org/10.1017/S0004972700023625 Published online by Cambridge University Press

[10] Edith Mourier, "L-random elements and L^*-random elements in Banach spaces", *Proc. Third Berkeley Sympos. Math. Statist. Probability*, Volume II, 231-242 (University of California Press, Berkeley and Los Angeles, 1956).

[11] D.Kh. Mushtari, "On almost sure convergence in linear spaces of random variables", *Theor. Probability Appl.* 15 (1970), 337-342.

[12] Д.Х. Муштари, А.Н. Шерстнев [D.H. Muštari, A.N. Šerstnev], "О способах введения топологии в случайных метрических пространствах" [On methods of introducing a topology in random metric spaces], *Izv. Vysš. Učebn. Zaved. Matematika* 1966, no. 6 (55), 99-106.

[13] Jiří Nedoma, "Note on generalized random variables", *Proc. First Prague Conf. Information Theory, Statistical Decision Functions, Random Processes*, Liblice, 1956, 139-141 (Publishing House of the Czechoslovak Academy of Sciences, Prague, 1957).

[14] W.J. Padgett, R.L. Taylor, *Laws of large numbers for normed linear spaces and certain Fréchet spaces* (Lecture Notes in Mathematics, 360. Springer-Verlag, Berlin, Heidelberg, New York, 1973).

[15] Yu.V. Prokhorov, "Convergence of random processes and limit theorems in probability theory", *Theor. Probability Appl.* 1 (1956), 157-214.

[16] А.Н. Шерстнев [A.N. Šerstnev], "О понятии случайного нормированного пространства" [On the concept of a stochastic normalized space], *Dokl. Akad. Nauk SSSR* 149 (1963), 280-283.

[17] Robert Lee Taylor, "Weak laws of large numbers in normed linear spaces", *Ann. Math. Statist.* 43 (1972), 1267-1274.

[18] R.L. Taylor and W.J. Padgett, "Some laws of large numbers for normed linear spaces", *Sankhyā Ser. A* (to appear).

[19] Albert Wilansky, *Functional analysis* (Blaisdell [Ginn and Co.], New York, Toronto, London, 1964).

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