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Linear free resolutions over non-commutative algebras

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Abstract

The main result of this paper is that, over a non-commutative Koszul algebra, high truncations of finitely generated graded modules have linear free resolutions.

Introduction

Eisenbud and Goto [EG84], Avramov and Eisenbud [AE92], and the present author [Jor99b] have all studied whether high truncations of finitely generated graded modules over graded algebras have linear free resolutions.

The original study of this took place over polynomial algebras. The main result [EG84, Proposition, p. 89] is: If M is a finitely generated graded module over $k[X_1, \ldots, X_t]$ with k a field, then for large s, the minimal free resolution of the degree shifted truncation $M_{\geqslant s}(s)$ is linear. That is, the mth module in the minimal free resolution has all its generators placed in degree m.

This was later extended in [AE92, Corollary 2] to commutative Koszul algebras, and in [Jor99b, Theorem 2.6] to non-commutative AS-regular algebras, which are algebras with good homological behaviour generalizing that of polynomial algebras.

This paper proves a common extension of [AE92, Corollary 2] and [Jor99b, Theorem 2.6]: If A is a non-commutative Koszul algebra satisfying a few weak conditions given in Setup 0.1, then for any finitely generated graded module M and large s, the minimal free resolution of $M_{\geqslant s}(s)$ is linear. This is Theorem 3.1 below.

Along the way, I prove Theorems 2.5 and 2.6 and Corollaries 2.8 and 2.9, which show that the two competing definitions of Castelnuovo–Mumford regularity of graded modules given in [AE92], respectively [EG84] and [Jor99b], are in fact closely related.

Setup 0.1. Throughout the paper, k is a field, and A is a noetherian $\mathbb{N}$-graded connected k-algebra which has a balanced dualizing complex.

See [Jor99a] and [Jor99b] for generalities on the theory of graded algebras, and [vdB97] and [Yek92] for information on dualizing complexes. My notation is mostly standard, but I do want to list a few items of terminology.

The opposite algebra of A is denoted A^{opp}, and A-right-modules are identified with A^{opp}-left-modules.

The abelian category of graded A-left-modules and graded homomorphisms of degree zero is denoted $\mathsf{Gr}\,A$. The derived category of $\mathsf{Gr}\,A$ is denoted $\mathsf{D}(\mathsf{Gr}\,A)$. If X is in $\mathsf{D}(\mathsf{Gr}\,A)$, then $\mathrm{h}^m X$ denotes the mth cohomology module of X. The derived category $\mathsf{D}(\mathsf{Gr}\,A)$ has full subcategories $\mathsf{D}^-(\mathsf{Gr}\,A)$ consisting of complexes X with $\mathrm{h}^m X = 0$ for m large positive, $\mathsf{D}^+(\mathsf{Gr}\,A)$ consisting of complexes X with $\mathrm{h}^m X = 0$ for m large negative, and $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\,A)$ consisting of complexes X with $\mathrm{h}^m X = 0$ for m large positive or negative and each $\mathrm{h}^m X$ finitely generated.

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The derived functors of Hom_A and $\otimes_A$ are denoted RHom_A and $\overset{\mathrm{L}}{\otimes}_A$. Section functors of graded A-left- and graded A-right-modules are denoted $\Gamma_{\mathfrak{m}}$ and $\Gamma_{\mathfrak{m}^{\mathrm{opp}}}$, and their derived functors are denoted $\mathrm{R}\Gamma_{\mathfrak{m}}$ and $\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}$. These give rise to local cohomology functors $\mathrm{H}^m_{\mathfrak{m}} = \mathrm{h}^m\,\mathrm{R}\Gamma_{\mathfrak{m}}$ and $\mathrm{H}^m_{\mathfrak{m}^{\mathrm{opp}}} = \mathrm{h}^m\,\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}$. The Matlis duality functor is denoted $(-)'$ and defined on graded A-modules by $(M')_p = \mathrm{Hom}_k(M_{-p}, k)$. Matlis duality exchanges graded A-left- and graded A-right-modules, and is exact and therefore well defined on derived categories.

1. Background results

Proposition 1.1. *For X and Y in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\,A)$ there is a natural isomorphism*

$$\mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}X, Y) \cong \mathrm{RHom}_A(X, Y).$$

Proof. First observe that there are natural isomorphisms

$$\begin{aligned}
\mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}A, Y) &\cong \mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}A, Y'') \\
&\overset{(a)}{\cong} \left(Y' \overset{\mathrm{L}}{\otimes}_A \mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}A\right)' \\
&\overset{(b)}{\cong} \mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(Y')' \\
&\overset{(c)}{\cong} Y'' \\
&\cong Y.
\end{aligned}$$

Here (a) is by [Jor99a, Theorem 1.5] and (b) is by [Jor99a, Theorem 1.6], while (c) can be seen as follows: Let F be a free resolution of Y consisting of finitely generated free modules. It is then easy to see that F' is an injective resolution of Y'. As F consists of finitely generated free modules, F' consists of torsion graded injective modules, so $\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(F') \cong F'$ whence $\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}(Y') \cong \Gamma_{\mathfrak{m}^{\mathrm{opp}}}(F') \cong F' \cong Y'$.

Now compute:

$$\begin{aligned}
\mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}X, Y) &\overset{(d)}{\cong} \mathrm{RHom}_A\left(\mathrm{R}\Gamma_{\mathfrak{m}}(A) \overset{\mathrm{L}}{\otimes}_A X, Y\right) \\
&\cong \mathrm{RHom}_A(X, \mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}}A, Y)) \\
&\overset{(e)}{\cong} \mathrm{RHom}_A(X, \mathrm{RHom}_A(\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}}A, Y)) \\
&\overset{(f)}{\cong} \mathrm{RHom}_A(X, Y).
\end{aligned}$$

Here (d) is by [Jor99a, Theorem 1.6] again, while (e) is by [vdB97, Corollary 4.8] and (f) is by the previous computation. □

Lemma 1.2. *For X in $\mathsf{D}^{-}(\mathsf{Gr}\,A)$ and Y in $\mathsf{D}^{+}(\mathsf{Gr}\,A)$ there is a convergent spectral sequence*

$$E_2^{mn} = \mathrm{Ext}^m_A(\mathrm{h}^{-n}X, Y) \Rightarrow \mathrm{Ext}^{m+n}_A(X, Y).$$

Proof. Let J be an injective resolution of Y. Consider the double complex given by

$$M^{mn} = \mathrm{Hom}_A(X^{-m}, J^n).$$

The spectral sequence arising from the second standard filtration of the total complex $\mathrm{Tot}\,M$ gives the lemma's spectral sequence. □

Lemma 1.3. *For X in $\mathsf{D}^{-}(\mathsf{Gr}\,A)$ there is a convergent spectral sequence*

$$E_2^{mn} = \mathrm{Tor}^A_{-m}(\mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}}A, X) \Rightarrow \mathrm{H}^{m+n}_{\mathfrak{m}}X.$$

Proof. Let F be a flat resolution of X. Consider the double complex given by

$$M^{mn} = (\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}} A)^m \otimes_A F^n.$$

The spectral sequence arising from the second standard filtration of Tot M gives the lemma's spectral sequence.

To see that the sequence has the indicated limit, one needs the computation

$$\mathrm{Tot}\, M \cong (\mathrm{R}\Gamma_{\mathfrak{m}^{\mathrm{opp}}} A) \overset{\mathrm{L}}{\otimes}_A X \overset{(a)}{\cong} (\mathrm{R}\Gamma_{\mathfrak{m}} A) \overset{\mathrm{L}}{\otimes}_A X \overset{(b)}{\cong} \mathrm{R}\Gamma_{\mathfrak{m}} X,$$

where (a) is by [vdB97, Corollary 4.8] and (b) is by [Jor99a, Theorem 1.6]. □

2. Two notions of regularity

The following is almost the classical definition of Castelnuovo–Mumford regularity of graded modules, given over polynomial algebras in [EG84, Definition, p. 95] and more generally in [Jor99b, Definition 2.1].

Definition 2.1 (Castelnuovo–Mumford regularity). The complex X in $\mathsf{D}(\mathsf{Gr}\, A)$ is called p-regular if

$$\mathrm{H}^m_{\mathfrak{m}}(X)_{\geqslant p+1-m} = 0$$

for all m.

If X is p-regular but not $(p-1)$-regular, then I define the *Castelnuovo–Mumford regularity of* X to be

$$\mathrm{CMreg}\, X = p.$$

If X is not p-regular for any p, then $\mathrm{CMreg}\, X = \infty$. If X is p-regular for every p (that is, if $\mathrm{H}_{\mathfrak{m}}(X) = 0$), then $\mathrm{CMreg}\, X = -\infty$.

The following is the competing definition of Castelnuovo–Mumford regularity given in [AE92]. In order not to confuse things, I have to use a different name.

Definition 2.2 (Ext-regularity). The complex X in $\mathsf{D}(\mathsf{Gr}\, A)$ is called r-Ext-regular if

$$\mathrm{Ext}^m_A(X,k)_{\leqslant -r-1-m} = 0$$

for all m.

If X is r-Ext-regular but not $(r-1)$-Ext-regular, then I define the *Ext-regularity of* X to be

$$\mathrm{Ext.reg}\, X = r.$$

If X is not r-Ext-regular for any r, then $\mathrm{Ext.reg}\, X = \infty$. If X is r-Ext-regular for every r (that is, if $\mathrm{Ext}_A(X,k) = 0$), then $\mathrm{Ext.reg}\, X = -\infty$.

Observation 2.3. Let X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\, A)$ have $X \ncong 0$. Since A has a balanced dualizing complex, the local duality theorem [Yek92, Theorem 4.18] holds, so $\mathrm{R}\Gamma_{\mathfrak{m}}(X)'$ is in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\, A^{\mathrm{opp}})$ and has $\mathrm{R}\Gamma_{\mathfrak{m}}(X)' \ncong 0$. Hence $\mathrm{CMreg}\, X \neq \pm\infty$.

By [vdB97, Corollary 4.8] I have $\mathrm{H}^n_{\mathfrak{m}} A \cong \mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}} A$ for each n, whence

$$\mathrm{CMreg}({}_A A) = \mathrm{CMreg}(A_A).$$

I denote this number by CMreg A.

Observation 2.4. Let X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\, A)$ have $X \ncong 0$. It is easy to see that $\mathrm{Ext}_A(X,k) \ncong 0$ whence $\mathrm{Ext.reg}\, X \neq -\infty$. However, $\mathrm{Ext.reg}\, X = \infty$ is possible.

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If F is a minimal free resolution of X, then X is r-Ext-regular exactly if the generators of F_m are placed in degrees less than or equal to $r+m$ for each m. This has a nice consequence: From considering $\mathrm{Tor}^A(k_A, {}_Ak)$ it follows that the minimal free resolutions of k_A and ${}_Ak$ have their generators placed in the same degrees. Hence

$$\mathrm{Ext.reg}(k_A) = \mathrm{Ext.reg}({}_Ak).$$

I denote this number by $\mathrm{Ext.reg}\, k$.

The following two theorems show that the notions of Castelnuovo–Mumford and Ext-regularity enjoy a close relationship. Note the structural similarity between the proofs.

Theorem 2.5. *Given X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\,A)$ with $X \not\cong 0$. Then*

$$\mathrm{Ext.reg}\, X \leqslant \mathrm{CMreg}\, X + \mathrm{Ext.reg}\, k.$$

Proof. Observation 2.3 gives $\mathrm{CMreg}\, X \neq -\infty$, so for $\mathrm{Ext.reg}\, k = \infty$ the theorem makes sense and holds trivially. So I can assume that $\mathrm{Ext.reg}\, k = r$ is finite. By Observation 2.4, the minimal free resolution F of k_A then has the generators of F_m placed in degrees less than or equal to $r+m$ for each m, so F_m can be written as a finite coproduct

$$F_m = \coprod_j A(-\sigma_{mj})$$

with $\sigma_{mj} \leqslant r+m$. Taking Matlis duals, $I = F'$ is a minimal injective resolution of ${}_Ak$ which has

$$I^m = \coprod_j A'(\sigma_{mj}),$$

still with

$$\sigma_{mj} \leqslant r+m. \tag{1}$$

Set $p = \mathrm{CMreg}\, X$ and $Z = \mathrm{R}\Gamma_{\mathfrak{m}} X$. Then

$$\mathrm{h}^{-n}(Z)_{\geqslant p+1+n} = \mathrm{h}^{-n}(\mathrm{R}\Gamma_{\mathfrak{m}} X)_{\geqslant p+1+n} = \mathrm{H}^{-n}_{\mathfrak{m}}(X)_{\geqslant p+1+n} = 0$$

for each n, whence

$$((\mathrm{h}^{-n}Z)')_{\leqslant -p-1-n} = 0. \tag{2}$$

Now, $\mathrm{Ext}^m_A(\mathrm{h}^{-n}Z, k)$ is a subquotient of

$$\mathrm{Hom}_A(\mathrm{h}^{-n}Z, I^m) = \mathrm{Hom}_A\left(\mathrm{h}^{-n}Z, \coprod_j A'(\sigma_{mj})\right) \cong \coprod_j (\mathrm{h}^{-n}Z)'(\sigma_{mj}),$$

and this vanishes in degrees less than or equal to $-p-1-n-r-m$ by Equations (1) and (2), so also

$$\mathrm{Ext}^m_A(\mathrm{h}^{-n}Z, k)_{\leqslant -p-1-n-r-m} = 0. \tag{3}$$

Lemma 1.2 provides a convergent spectral sequence

$$E_2^{mn} = \mathrm{Ext}^m_A(\mathrm{h}^{-n}Z, k) \Rightarrow \mathrm{Ext}^{m+n}_A(Z, k),$$

and since Equation (3) shows $(E_2^{mn})_{\leqslant -p-1-r-(m+n)} = 0$ it follows that

$$\mathrm{Ext}^q_A(Z, k)_{\leqslant -p-1-r-q} = 0 \tag{4}$$

for each q.

Finally, Proposition 1.1 gives

$$\mathrm{Ext}^q_A(Z, k) = \mathrm{Ext}^q_A(\mathrm{R}\Gamma_{\mathfrak{m}} X, k) \cong \mathrm{Ext}^q_A(X, k),$$

so Equation (4) implies

$$\operatorname{Ext}^q_A(X,k)_{\leqslant -p-1-r-q} = 0$$

for each q, showing

$$\operatorname{Ext.reg} X \leqslant p + r = \operatorname{CMreg} X + \operatorname{Ext.reg} k. \qquad \square$$

Theorem 2.6. *Given X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\, A)$ with $X \ncong 0$. Then*

$$\operatorname{CMreg} X \leqslant \operatorname{Ext.reg} X + \operatorname{CMreg} A.$$

Proof. Observation 2.3 gives $\operatorname{CMreg} A \neq -\infty$, so for $\operatorname{Ext.reg} X = \infty$ the theorem makes sense and holds trivially. So I can assume that $\operatorname{Ext.reg} X = r$ is finite. By Observation 2.4, the minimal free resolution F of X then has the generators of F_m placed in degrees less than or equal to $r+m$ for each m, so F_m can be written as a finite coproduct

$$F_m = \coprod_j A(-\sigma_{mj})$$

with

$$\sigma_{mj} \leqslant r + m. \tag{5}$$

Set $p = \operatorname{CMreg} A$. Observation 2.3 gives $\operatorname{CMreg}(A_A) = \operatorname{CMreg} A$, so I get

$$\mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}}(A)_{\geqslant p+1-n} = 0 \tag{6}$$

for each n.

Now, $\operatorname{Tor}^A_{-m}(\mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}} A, X)$ is a subquotient of

$$\mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}}(A) \otimes_A F_{-m} \cong \mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}}(A) \otimes_A \coprod_j A(-\sigma_{-m,j}) \cong \coprod_j \mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}}(A)(-\sigma_{-m,j}),$$

and this vanishes in degrees larger than or equal to $p+1-n+r-m$ by Equations (5) and (6), so also

$$\operatorname{Tor}^A_{-m}(\mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}} A, X)_{\geqslant p+1-n+r-m} = 0. \tag{7}$$

Lemma 1.3 provides a convergent spectral sequence

$$E_2^{mn} = \operatorname{Tor}^A_{-m}(\mathrm{H}^n_{\mathfrak{m}^{\mathrm{opp}}} A, X) \Rightarrow \mathrm{H}^{m+n}_{\mathfrak{m}} X,$$

and since Equation (7) shows $(E_2^{mn})_{\geqslant p+1+r-(m+n)} = 0$, it follows that

$$\mathrm{H}^q_{\mathfrak{m}}(X)_{\geqslant p+1+r-q} = 0 \tag{8}$$

for each q, showing

$$\operatorname{CMreg} X \leqslant p + r = \operatorname{Ext.reg} X + \operatorname{CMreg} A. \qquad \square$$

Let me end the section with some easy consequences. First recall the following definition.

Definition 2.7. The algebra A is called *Koszul* if $\operatorname{Ext.reg} k = 0$.

For A to be Koszul means exactly that the minimal free resolutions of ${}_A k$ and k_A are linear; cf. Observation 2.4.

The following corollary is immediate from Theorems 2.5 and 2.6.

Corollary 2.8. *Suppose that A is Koszul and has $\operatorname{CMreg} A = 0$. Then each X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\, A)$ has $\operatorname{Ext.reg} X = \operatorname{CMreg} X$.*

The following corollary is also immediate from Observation 2.3 and Theorem 2.5. It extends [AE92, Theorem 1] and [AP01, Theorem 1] to the non-commutative case.

Corollary 2.9. *Suppose that A has $\operatorname{Ext.reg} k < \infty$. Then each X in $\mathsf{D}^{\mathrm{b}}_{\mathrm{fg}}(\mathsf{Gr}\, A)$ has $\operatorname{Ext.reg} X < \infty$.*

3. Linear free resolutions

The following main result is a simultaneous extension of [AE92, Corollary 2] (to the non-commutative case) and [Jor99b, Theorem 2.6] (to the non-AS-regular case).

Recall that A is the algebra of Setup 0.1.

Theorem 3.1. *Suppose that A is Koszul, and let M in $\mathsf{Gr}\,A$ be finitely generated with $M \ncong 0$. Then for $s \geqslant \operatorname{CMreg} M$, the minimal free resolution of $M_{\geqslant s}(s)$ is linear. (Note that $\operatorname{CMreg} M$ is finite.)*

Proof. The result clearly holds if $M_{\geqslant s}(s)$ is 0, so I can assume $M_{\geqslant s}(s) \ncong 0$.

Let F be the minimal free resolution of $M_{\geqslant s}(s)$. As $M_{\geqslant s}(s)$ sits in non-negative degrees, it is clear for each m that F_m has no generators placed in degrees strictly smaller than m. Hence it is enough to prove for each m that F_m also has no generators placed in degrees strictly larger than m. By Observation 2.4 this is the same as proving

$$\operatorname{Ext.reg}(M_{\geqslant s}(s)) \leqslant 0. \tag{9}$$

Since A is Koszul, $\operatorname{Ext.reg} k = 0$ holds. By Theorem 2.5, the inequality (9) will therefore follow from $\operatorname{CMreg}(M_{\geqslant s}(s)) \leqslant 0$, which is again the same as $\operatorname{CMreg}(M_{\geqslant s}) \leqslant s$, that is

$$\mathrm{H}^m_{\mathfrak{m}}(M_{\geqslant s})_{\geqslant s+1-m} = 0$$

for each m. To show this is easy: there is a short exact sequence $0 \longrightarrow M_{\geqslant s} \longrightarrow M \longrightarrow M/M_{\geqslant s} \longrightarrow 0$ resulting in a long exact sequence consisting of pieces

$$\mathrm{H}^m_{\mathfrak{m}}(M_{\geqslant s}) \longrightarrow \mathrm{H}^m_{\mathfrak{m}}(M) \longrightarrow \mathrm{H}^m_{\mathfrak{m}}(M/M_{\geqslant s}).$$

Now combine this with $\mathrm{H}^m_{\mathfrak{m}}(M)_{\geqslant s+1-m} = 0$ for each m (because $s \geqslant \operatorname{CMreg} M$) and

$$\mathrm{H}^m_{\mathfrak{m}}(M/M_{\geqslant s}) \cong \begin{cases} M/M_{\geqslant s} & \text{for } m = 0, \\ 0 & \text{for } m \geqslant 1 \end{cases}$$

(because $M/M_{\geqslant s}$ is torsion). ☐

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