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THE SPACE OF HARMONIC MAPS FROM THE 2-SPHERE TO THE COMPLEX PROJECTIVE PLANE

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ABSTRACT. In this paper we study the topology of the space of harmonic maps from S^2 to $\mathbb{CP}^2$. We prove that the subspaces consisting of maps of a fixed degree and energy are path connected. By a result of Guest and Ohnita it follows that the same is true for the space of harmonic maps to $\mathbb{CP}^n$ for $n \geq 2$. We show that the components of maps to $\mathbb{CP}^2$ are complex manifolds.

1. **Introduction.** Harmonic maps from the Riemann sphere to complex projective space are critical points of the energy functional

$$E: C^\infty(S^2, \mathbb{CP}^n) \longrightarrow [0, \infty)$$

defined on the space of smooth maps. As solutions of a classical variational problem harmonic maps have been studied extensively, especially with regard to questions of existence, uniqueness, regularity, etc. It is now well known that all of the harmonic maps from the Riemann sphere to complex projective space can be constructed from holomorphic maps. In this paper we study some global topological properties of the solution space.

Holomorphic (and anti-holomorphic) maps are the absolute minima of E in each path component of $C^\infty(S^2, \mathbb{CP}^n)$. A holomorphic map $f: S^2 \longrightarrow \mathbb{CP}^2$ is called *full* if its image is not contained in any projective line. The following theorem is the starting point for this paper:

THEOREM 1.1 [EW2]. *The set of all non-minimal harmonic maps $\phi: S^2 \longrightarrow \mathbb{CP}^2$ is in 1–1 correspondence with the set of full holomorphic maps $f: S^2 \longrightarrow \mathbb{CP}^2$.*

This is, in fact, a specific case of a more general theorem which describes how to construct all harmonic maps to $\mathbb{CP}^n$ set theoretically. It is due originally to Din and Zakrzewski [DZ] and Glaser and Stora [GS]. The paper [EW2] of Eells and Wood gives an excellent description for a mathematical audience and we refer to the construction as the Eells-Wood construction. The reader is also directed to [Bu], [G], [La] for other descriptions. With the Eells-Wood construction in hand it is possible to study the global, topological properties of the space of harmonic maps. Much is, in fact already known. Holomorphic and anti-holomorphic maps are local minima of E and the topology of these

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components is very well understood (see [S], [CCMM], [CS], [H]). In [C] we studied the subspaces of full holomorphic maps. Some results have also been obtained for spaces of harmonic maps of S^2 into other Riemannian manifolds (see [A], [FGKO], [K], [V], [Lo]).

Let $\mathrm{Harm}(\mathbb{CP}^n)$ be the space of harmonic maps $S^2 \longrightarrow \mathbb{CP}^n$ with the topology it inherits as a subset of the space $\mathrm{Map}(S^2, \mathbb{CP}^n)$ of all continuous maps. $\mathrm{Map}(S^2, \mathbb{CP}^n)$ has connected components, $\mathrm{Map}_k(S^2, \mathbb{CP}^n)$, indexed by the degree k of the individual maps. The critical values of E are discrete so $\mathrm{Harm}(\mathbb{CP}^n)$ is a disjoint union of the sets $\mathrm{Harm}_{k,E}(\mathbb{CP}^n)$ consisting of maps of degree k and energy E. Let $\mathrm{Hol}_k(\mathbb{CP}^n)$ denote the space of holomorphic maps of degree k when $k \geq 0$ and anti-holomorphic maps of degree k when $k < 0$. We normalize the energy functional so that $E(f) = k$ for a holomorphic map f of degree k.

Guest and Ohnita [GO] have conjectured that the spaces $\mathrm{Harm}_{k,E}(\mathbb{CP}^n)$ are connected. They show that, for $n \geq 3$, any harmonic map $\phi\colon S^2 \longrightarrow \mathbb{CP}^n$ can be continuously deformed through harmonic maps to a map whose image lies in $\mathbb{CP}^{n-1} \subset \mathbb{CP}^n$. Thus to prove the conjecture it suffices to prove it in the case where $n = 2$. The main result of this paper is the following theorem, which affirms this conjecture:

THEOREM 1.2. *The path components of* $\mathrm{Harm}(\mathbb{CP}^2)$ *are the minimal sets,* $\mathrm{Hol}_k(\mathbb{CP}^2)$, *and the non-minimal critical sets,* $\mathrm{Harm}_{k,E_r}(\mathbb{CP}^2)$, *where the critical energy values are* $E_r = 3|k| + 2r + 4$ *indexed by a non-negative integer* r. *Moreover these non-minimal components can be given the structure of complex manifolds of complex dimension* $3|k| + r + 8$.

REMARK. In a recent preprint [LW] Lemaire and Wood show that $\mathrm{Harm}_{k,E_r}(\mathbb{CP}^2)$ is, in fact, a smooth submanifold of the space of smooth maps.

The proof follows from an examination of the Eells-Wood construction. The set-theoretic assignment of Theorem 1.1 cannot be continuous. To see this recall that a map $f\colon S^2 \longrightarrow \mathbb{CP}^2$ is ramified at $x \in S^2$ if $df_x = 0$. If f is holomorphic and non-constant the set of points at which f is ramified is finite and the ramification index of f is the number of ramification points, counting multiplicities. Let r be the ramification index of a map f, then $r \leq 2k - 2$ and, unless the image of f lies in a projective line, $r \leq \frac{1}{2}(3k - 6)$. The harmonic map corresponding to f, in Theorem 1.1, has degree $k - 2 - r$ and energy $3k - 2 - r$. However there are plenty of maps with the same degree and different ramification indices so the connected space of all full maps of degree k is mapped by this correspondence to a number of different, disjoint pieces of $\mathrm{Harm}(\mathbb{CP}^2)$.

The first result of this paper is the following lemma which says that these are the only discontinuities. Let $\mathrm{Hol}_{k,r}(\mathbb{CP}^2) \subset \mathrm{Hol}_k(\mathbb{CP}^2)$ be the subspace of maps with ramification index r. Using the Eells-Wood construction we prove the following:

LEMMA 1.3. *For* $0 \leq r \leq k - 2$ *there is a homeomorphism*

$$\Phi_{k,r}\colon \mathrm{Hol}_{k,r}(\mathbb{CP}^2) \cong \mathrm{Harm}_{k-2-r,3k-2-r}(\mathbb{CP}^2).$$

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Note that for $r < \frac{1}{2}(3k-6)$ $\mathrm{Hol}_{k,r}(\mathbb{CP}^2)$ consists of full maps and that for $E > |k|$ $\mathrm{Harm}_{k,E}(\mathbb{CP}^2)$ consists of full maps so that Theorem 1.1 applies in the range of k and r considered in the lemma.

To get the results about connectedness and smoothness we prove:

THEOREM 1.4. *For $r \leq k-2$, $\mathrm{Hol}_{k,r}(\mathbb{CP}^2)$ is smooth connected complex submanifold of $\mathrm{Hol}_k(\mathbb{CP}^2)$ of complex dimension $3k-2r+2$.*

For the non-negative degree components Theorem 1.2 follows from Theorem 1.4 and Lemma 1.3. Using the fact that the involution $z \longmapsto \overline{z}$ induces a homeomorphism

$$\mathrm{Harm}_k(\mathbb{CP}^2) \cong \mathrm{Harm}_{-k}(\mathbb{CP}^2)$$

which preserves energy we obtain the result for all cases.

The paper is organized as follows: in section two we recall the construction of harmonic maps from holomorphic maps which lies at the heart of the proof of Theorem 1.1. We prove that, when restricted to $\mathrm{Hol}_{k,r}(\mathbb{CP}^2)$, this construction produces a continuous map and prove Lemma 1.3. In section three we describe the geometry of the spaces $\mathrm{Hol}_{k,r}(\mathbb{CP}^2)$ and prove Theorem 1.4. In the final section we give concrete descriptions of some of the simpler components and, where possible, compute their cohomology groups.
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2. **The Construction of harmonic maps.** In this section we describe the Eells-Wood construction and show that it restricts to a continuous, proper map on the subsets of holomorphic maps with fixed degree and ramification index. A number of descriptions of this construction exist in the literature. The one we use below is closest in spirit to [Bu] or [EW2]. For brevity let $\mathrm{Hol}_k = \mathrm{Hol}_k(\mathbb{CP}^2)$ and $\mathrm{Hol}_{k,r} = \mathrm{Hol}_{k,r}(\mathbb{CP}^2)$.

A holomorphic map $f\colon S^2 \longrightarrow \mathbb{CP}^2$ may be defined by letting

$$f(z) = [p_0(z), p_1(z), p_2(z)],$$

where z is a complex coordinate on $\mathbb{C} \cong S^2 \setminus \{\infty\}$ and $[u_0, u_1, u_2]$ are homogeneous coordinates on $\mathbb{CP}^2$. The p_i are polynomials which share no common zero. The topological degree of f is the maximum of the degrees of the p_i. Taking coefficients of the p_i

as homogeneous coordinates gives an embedding $\mathrm{Hol}_k \subset \mathbb{CP}^N$ as an open submanifold, where $N = 3k + 2$. Let $p\colon \mathbb{C} \longrightarrow \mathbb{C}^3$ be the polynomial function

$$p(z) = \big(p_0(z), p_1(z), p_2(z)\big).$$

This is just a lift of f to $\mathbb{C}^3$ over the coordinate patch. We will often write $[p_0, p_1, p_2]$ or even $[p]$ for f.

If $f \in \mathrm{Hol}_k$ then $p(z)$ and $p'(z)$ will be linearly independent for all but a finite number of points. The map $h = p \wedge p'$, given by

$$h(z) = p(z) \wedge p'(z) \in \bigwedge^2 \mathbb{C}^3,$$

is also polynomial. That is, identifying $\bigwedge^2 \mathbb{C}^3 \cong \mathbb{C}^3$, we can write

$$h(z) = \big(h_0(z), h_1(z), h_2(z)\big),$$

where the h_i are polynomials of degree less than or equal to $2k - 2$. If f is unramified then the h_i have no common zeros and $[h]$ is holomorphic map to $\mathbb{P}(\bigwedge^2 \mathbb{C}^3) \cong G_2(\mathbb{C}^3)$ of degree $2k-2$. Here $G_2(\mathbb{C}^3)$ denotes the Grassmanian of 2-planes in $\mathbb{C}^3$. If f is ramified at z then $h(z) = 0$, and if f is ramified at ∞ then the h_i will have degree strictly less than $2k-2$. We may write $h_i = bq_i$, for $i = 1, 2, 3$, where b is a greatest common divisor of the h_i. Let $2k-2-r$ be the maximum of the degrees of the q_i and let $q(z) = \big(q_0(z), q_1(z), q_2(z)\big)$, then $f_1 = [q]$ is a well-defined holomorphic map to $G_2(\mathbb{C}^3)$ of degree $2k - 2 - r$. The integer r is the ramification index of f. The map f_1 is called the *first associated curve* of f.

The line $f(z)$ is contained in the plane $f_1(z)$ with complex codimension 1. Thus we can define a map

$$\phi_1\colon S^2 \longrightarrow \mathbb{CP}^2$$

by taking

$$\phi_1(z) = f_1(z) \cap f^{\perp}(z).$$

The map ϕ_1 is harmonic and the assignment $f \longmapsto \phi_1$ is the correspondence of Theorem 1.1. This assignment, restricted to a fixed degree k and ramification index r, defines the map

$$\Phi_{k,r}\colon \mathrm{Hol}_{k,r} \longrightarrow \mathrm{Harm}_{k-2-r,3k-2-r}(\mathbb{CP}^2).$$

We will prove Lemma 1.3 by showing that $\Phi_{k,r}$ is continuous and proper.

Let $V_d \subset \mathbb{C}[z]$ be the subspace of polynomials of degree less than or equal to d. We can stratify the projective space $\mathbb{P}V_k^3$ by taking the subsets S_r of points $[p_0, p_1, p_2] \in \mathbb{P}V_k^3$ such that if b is a greatest common divisor of the p_i, and we write $p_i = bq_i$, then $k - r$ is the maximum of the degrees of the q_i. Note that $S_0 \cong \mathrm{Hol}_k$. In fact, the assignment

$$([b], [q_0, \dots, q_n]) \longmapsto [bq_0, \dots, bq_n]$$

defines an embedding

$$\xi\colon \mathbb{P}V_r \times \mathrm{Hol}_{k-r} \longrightarrow \mathbb{P}V_k^3$$

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which shows that for $0 \le r < k$, S_r is a submanifold of $\mathbb{P}V_k^3$. Note also that the closure of S_r is contained in the union of the strata $S_{r'}$ for $r' \ge r$.

Note that $\mathrm{Hol}_{k,r}$ is just the inverse image of S_r under the map

$$\Psi\colon \mathrm{Hol}_k \longrightarrow \mathbb{P}V_{2k-2}^3$$

given by $[p] \longmapsto [p \wedge p']$ where p is a 3-tuple of polynomials. It follows that the first associated curve $f_1 \in \mathrm{Hol}_{2k-2-r}\big(G_2(\mathbb{C}^3)\big)$ depends continuously on $f = [p]$. The remainder of the construction is manifestly continuous and it follows that $\Phi_{k,r}$ is continuous. We also remark that the first factor, $[b] \in \mathbb{P}V_r$, of $\xi^{-1}\big(\Psi(f)\big)$ also depends continuously on f. This is the ramification divisor of f and we denote it by $R\,(f)$.

LEMMA 2.2. *$\Phi_{k,r}$ is proper.*

PROOF. The proof follows the proof of Lemma 3.3 in [FGKO]. Suppose we have a sequence $\{\phi_n\}$ converging to ϕ in $\Phi_{k,r}(\mathrm{Hol}_{k,r}) \subset \mathrm{Harm}_{k-2-r,3k-2-r}(\mathbb{CP}^2)$ and suppose $\{f_n\} \subset \mathrm{Hol}_{k,r}$ is such that $\phi_n = \Phi_{k,r}(f_n)$ for each $n \ge 0$. It will suffice to find a convergent subsequence of $\{f_n\}$. We have $\mathrm{Hol}_{k,r} \subset \mathbb{P}V_k^3$. Since the latter space is compact there is a subsequence which converges to a point in $[p] \in \mathbb{P}V_k^3$. We must show that $[p] \in \mathrm{Hol}_{k,r}$. Suppose $[p]$ is in the stratum S_m for $m \ge 0$. Then we can write $[p] = [bq_0, bq_1, bq_2]$ where the q_i have no common zero. Thus $[q] = [q_0, q_1, q_2] \in \mathrm{Hol}_{(k-m)}$.

Similarly, by choosing a further subsequence if necessary, we can assume that $\Psi(f_n)$ converges to some point $[s] \in \mathbb{P}V_{2k-2}^3$. Since $\Psi(f_n) \in S_r$ we must have $[s] \in S_{r'}$ with $r' \ge r$. Write $[s] = [dt_0, dt_1, dt_2]$ with the t_i coprime. So $[t] = [t_0, t_1, t_2]$ is in $\mathrm{Hol}_{2k-2-r'}\big(G_2(\mathbb{C}^{n+1})\big)$.

Let $Z \subset S^2$ be the set of zeros of b and d and include the point at infinity if either $\deg b < m$ or $\deg d < r'$. The line $[q(z)]$ is contained in the plane represented by $[t(z)]$ for all z so let

$$\psi(z) = [t(z)] \cap [q(z)]^{\perp}.$$

ψ is a harmonic map which agrees with ϕ on $S^2 \setminus Z$. So, by the unique continuation property of harmonic maps, we must have $\psi = \phi$. But

$$\begin{aligned} E(\psi) &= (2k-2-r') + (k-m) = 3k-2-r'-m \\ \deg \psi &= (2k-2-r') - (k-m) = k-2-r'+m. \end{aligned}$$

Requiring $E(\psi) = E(\phi)$ and $\deg \psi = \deg \phi$ we must have $m = 0$ and $r' = r$. Thus $[p] \in \mathrm{Hol}_{k,r}$. ∎

3. **The desingularizing variety.** In this section we study the geometry of the strata $\mathrm{Hol}_{k,r}$. We start by constructing a filtration

$$\mathrm{Hol}_k = F_0 \supset F_1 \supset \cdots \supset \emptyset$$

by closed subsets. The strata are the differences of successive elements in this filtration. We construct varieties which sit over the F_r and show that these varieties are smooth. This is sufficient to show that $\mathrm{Hol}_{k,r}$ is smooth and connected.

To define the filtration let

$$F_r = \{f \in \mathrm{Hol}_k \mid f \text{ has ramification index} \geq r\}.$$

Then $\mathrm{Hol}_{k,r} = F_r \setminus F_{r+1}$. It is useful to think of the ramification divisor $R(f) \in \mathbb{P}V_r$ as being in the symmetric product $SP^r(S^2) = (S^2)^r/S_r$ where S_r is the symmetric group on r letters. An explicit homeomorphism $SP^r(S^2) \cong \mathbb{P}V_r$ is given by mapping an unordered r-tuple $\langle x_1, \ldots, x_r\rangle$ to the equivalence class of a polynomial whose zeros are precisely those points x_i which are not equal to ∞. We will say that $\langle x_1, \ldots, x_s\rangle \in SP^s(S^2)$ divides $R(f)$ if $R(f) = \langle x_1, \ldots, x_s, y_{s+1}, \ldots, y_r\rangle$ for some $y_{r+1}, \ldots, y_r$. If all the points x_i are finite then this is just the usual notion of polynomial division.

Let

$$X_r = \{([a], f) \in \mathbb{P}V_r \times \mathrm{Hol}_k \mid [a] \text{ divides } R(f)\}.$$

By projecting onto the second factor we get a quotient map $p_r: X_r \longrightarrow F_r$. The inverse image $p_r^{-1}(f)$ counts the (finite) number of elements $[a]$ which divide $R(f)$. For maps with ramification index exactly r there is only one point in the inverse image and p_r restricts to a homeomorphism $X_r \setminus p_r^{-1}(F_{r+1}) \cong F_r \setminus F_{r+1} = \mathrm{Hol}_{k,r}$. We will prove the following:

LEMMA 3.1. *For $r \leq k-2$ the spaces X_r are path-connected complex manifolds of complex dimension $3k - 2r + 2$.*

This will imply Theorem 1.4. First of all, it identifies $\mathrm{Hol}_{k,r}$ with an open submanifold of a complex manifold of the correct dimension. Second, $\mathrm{Hol}_{k,r}$ is connected since $p_r^{-1}(F_{r+1})$ is a proper algebraic subset and cannot disconnect a smooth variety.

In order to study the geometry of X_r we need to characterize the condition that $[a]$ divide $R(f)$. Let $f = [p_0, p_1, p_2]$. Recall that in Section 2 we saw that we could write $\Psi(f) = [bq_0, bq_1, bq_2]$ where $[b] = R(f)$. The polynomial factors of $\Psi(f)$ are of the form $p_ip_j' - p_i'p_j$, for $i < j$. If $\deg a = r$ then $[a]$ divides $R(f)$ if a divides $p_ip_j' - p_i'p_j$ for all $0 \leq i < j \leq 2$. These conditions are not independent: Suppose p_0 and a are coprime and that a divides $p_0p_i' - p_0'p_i$, for $i = 1, 2$. Now

$$p_1(p_0p_2' - p_0'p_2) - p_2(p_0p_1' - p_0'p_1) = p_0(p_1p_2' - p_1'p_2)$$

and if a divides both terms on the lefthand side it must also divide $p_1p_2' - p_1'p_2$.

Let $X_r' \subset X_r$ be the subset of pairs $([a], [p_0, p_1, p_2])$ such that $\deg a = r$, and a and p_0 are coprime. Lemma 3.1 will follow from the next two lemmas.

LEMMA 3.2. *Every point in X_r is contained in a neighbourhood biholomorphically equivalent to X_r'.*

PROOF. By a change of complex coordinate on S^2 we may assume that the configuration associated to $[a]$ does not include the point at infinity, so $\deg a = r$.

Now $\mathbb{P}\,\mathrm{GL}(3, \mathbb{C})$ acts on $\mathbb{CP}^2$ by complex, linear biholomorphisms. Thus it acts by composition on Hol_k leaving the subspaces $\mathrm{Hol}_{k,r}$ invariant. In fact, for $A \in \mathbb{P}\,\mathrm{GL}(3, \mathbb{C})$, $R(A \cdot f) = R(f)$. Write $f = [p_0, p_1, p_2]$. It suffices to find $\epsilon_0, \epsilon_1, \epsilon_2 \in \mathbb{C}$ so that $\epsilon_0 p_0 + \epsilon_1 p_1 + \epsilon_2 p_2$ is prime to a. Since no zero of a can be a zero of all the p_i this condition is satisfied by a generic choice of ϵ_i. ∎

LEMMA 3.3. *For $r \le k-2$ the spaces X_r' are complex manifolds of complex dimension* $3k-2r+2$.

PROOF. We will prove the lemma by giving an explicit description of the space as a smooth pullback. Let V_d^+ denote the set of monic polynomials in V_d. Let Z be the set of pairs $(a,p) \in V_r^+ \times V_k$ such that a and p are coprime. Let $\mathrm{Mat}(s,t)$ be the space of $s \times t$ complex matrices. For $s \le t$, let $\mathrm{Mat}^*(s,t)$ be the open subset of matrices with rank s. We may identify $\mathrm{Mat}^*(s,t)$ with the Stiefel manifold $V_s(\mathbb{C}^t)$ of s-frames in $\mathbb{C}^t$ by thinking of the s linearly independent rows of a matrix as a frame. Now define a map $L\colon Z \longrightarrow \mathrm{Mat}(r,k+1)$ as follows: Given $(a,p) \in Z$ we can construct a linear map

$$L(a,p) \in \mathrm{Hom}_{\mathbb{C}}(V_k, V_{r-1}) \cong \mathrm{Mat}(r,k+1)$$

by $L(a,p)\cdot u = [pu' - p'u]_a$ where, for any polynomial q, we write $[q]_a$ for the congruence class of q mod a.

Next consider the space

$$E = \{(A; u_1, u_2, u_3)) \in \mathrm{Mat}(r,k+1) \times V_3(\mathbb{C}^{k+1}) \mid u_i \in \ker A,\ i = 1,2,3\}.$$

Projection onto the second factor $E \longrightarrow V_3(\mathbb{C}^{k+1})$ makes E a vector bundle. To see this first note that the condition that each of the u_i be in $\ker A$ is equivalent to requiring that each of the rows of A be in the kernel of the matrix with rows u_1, u_2 and u_3. So, if we map

$$V_3(\mathbb{C}^{k+1}) \longrightarrow G_{k-2}(\mathbb{C}^{k+1})$$

by associating to each 3-frame a $3 \times (k+1)$ matrix which we map to its kernel, then E is the pullback of the r-fold Whitney sum of the canonical $\mathbb{C}^{k-2}$-bundle over $G_{k-2}(\mathbb{C}^{k+1})$.

We are now in a position to describe X_r'. Consider the diagram

$$\begin{array}{ccc} & & E \\ & & \downarrow{\scriptstyle \psi} \\ Z & \xrightarrow{\phi} & \mathrm{Mat}(r,k+1) \times \mathbb{C}^{k+1} \end{array}$$

where ψ is the projection $(A; u_1, u_2, u_3) \longmapsto (A, u_1)$ and ϕ sends $(a,p) \longmapsto (L(a,p), p)$. The pullback of this diagram can be described as the set of points $(a, p_0, p_1, p_2) \in V_r^+ \times V_k^3$ with $(a, p_0) \in Z$ and $p_i \in \ker L(a, p_0)$ for $i = 1, 2$. It is clear that $d\psi$ maps onto $T\,\mathrm{Mat}(r, k+1)$ and $d\phi$ maps onto $T\mathbb{C}^{k+1}$ so ψ and ϕ are transversal and the pullback is a manifold. Finally we projectivise by identifying $(a, p_0, p_1, p_2) \sim (a, \lambda p_0, \lambda p_1, \lambda p_2)$. Then we can equate X_r' with the open submanifold comprised of equivalence classes for which p_0, p_1, p_2 have no common zeros so that they define a holomorphic map. To compute the dimension of X_r' note that the dimension of the pullback is $\dim Z + \dim E - \dim\bigl(\mathrm{Mat}(r,k+1) \times \mathbb{C}^{k+1}\bigr) = 3k - 2r + 3$. After projectivising we get the stated dimension for X_r'. ∎

4. **Appendix: some examples.** With the description of the strata $\mathrm{Hol}_{k,r}$, for $r \leq k-2$, developed in the previous section it is possible to obtain explicit geometrical models for the first few strata. In this section we describe some examples. Let $G = \mathrm{GL}(3,\mathbb{C})/Z$ where $Z \cong \mathbb{C}^*$ is the centre. This is the automorphism group of $\mathbb{CP}^3$ and so it acts on Hol_k. The condition that the image not lie in any $\mathbb{CP}^1$ means that G acts freely on on the subspace of full maps in Hol_k. Moreover it fixes the strata $\mathrm{Hol}_{k,r}$ which, for $r \leq k-2$ consist of full maps. Thus the components $\mathrm{Harm}_{k,E}(\mathbb{CP}^2)$ have free G-actions. The construction in the last section can be modified slightly to describe these components as pullbacks of canonical principal G-bundles. We use the machinery of algebraic topology to make calculations of some of the cohomology groups. This section assumes some background in algebraic topology.

The first non-trivial case is degree 2. Consider a triple of degree 2 polynomials (p_0, p_1, p_2). In order to define a full map the p_i must be linearly independent in the space of polynomials, V_2. This condition would be violated if the p_i had a common zero. For the same reason the map they define must be unramified. We may use the coefficients of the three polynomials to form a matrix in $\mathrm{GL}(3,\mathbb{C})$, defined up to multiplication by a non-zero scalar. Thus $\mathrm{Hol}_{2,0} \cong G$. This result, for based maps, appears in [C].

The group G has the homotopy type of $\mathbb{P}U(3) = U(3)/S^1$. The cohomology of $\mathbb{P}U(3)$ is known [BB]. For $p \neq 3$

$$H^*\big(\mathbb{P}U(3);\mathbb{Z}/p\big) = H^*\big(SU(3);\mathbb{Z}/p\big) = \Lambda[e_3, e_5].$$

For $p = 3$

$$H^*\big(\mathbb{P}U(3);\mathbb{Z}/3\big) = \Lambda[e_1, e_3] \otimes \mathbb{Z}/3[x_2]/x_2^3 = 0.$$

Where $\Lambda[a_i, \ldots]$ and $\mathbb{Z}/3[b_i, \ldots]$ denote an exterior algebra and a polynomial algebra on the given generators. The subscripts denote the dimensions of the generators.

In degree $k = 3$ we can have $r = 0$, or 1 and we may again describe a full map by three linearly independent polynomials. Consider the condition that

$$\mu p_i(z) = \lambda p_i'(z), \quad i = 1, 2, 3 \tag{4.1}$$

for some $z \in \mathbb{C}$. If $\lambda \neq 0$ this corresponds to ramification at z. If $\lambda = 0$ then the p_i all vanish at z. In this way we see the possible ramifications at z being parameterized by $\mu \in \mathbb{C}$ with the extreme case, $\mu = \infty$, corresponding to the simultaneous vanishing of all three polynomials.

To see what happens as $z \longrightarrow \infty$ we change coordinates to $\xi = z^{-1}$. We look at new polynomials q_i defined by requiring $q_i(\xi) = \xi^3 p_i(\xi^{-1})$ for $\xi \neq 0$. And a new condition for ramification corresponding to 4.1: $\nu q_i(\xi) = \gamma q_i'(\xi)$. Gluing these two pictures together along the overlap $\mathbb{C} \setminus \{0\}$ we obtain an bundle over S^2 with fibre S^2. We denote the total space of this bundle by X. It is useful to think of X as a line bundle, Y, compactified by adding a section at infinity. The finite part of the fibre over z gives the data for ramifications at z, and the extra point at infinity corresponds to the condition that all three polynomials vanish at z. An explicit calculation of the transition functions shows that $c_1(Y) = 2$.

To obtain a description of the strata $\mathrm{Hol}_{3,1}$ we use the ramification data to construct a pullback bundle. This is essentially the same construction used in the last section. Matters are considerably simplified by the fact that, in degree 3, L maps Z into $\mathrm{Mat}^*(r,4)$ the full-rank matrices. So the kernel of $L(a,p)$ has constant dimension and E is the pull back over $\ker\colon \mathrm{Mat}^*(r,4) \longrightarrow G_{4-r}(\mathbb{C}^4)$ of a principal $\mathrm{GL}(3,\mathbb{C})$-bundle. Another way of putting this is that condition 4.1 is always non-degenerate and defines a 3-dimensional subspace of the space of polynomials $V_3 \cong \mathbb{C}^4$. This extends over the fibre at infinity to give a map $\phi\colon X \hookrightarrow G_3(V_3) \cong G_3(\mathbb{C}^4) \cong \mathbb{CP}^3$.

Since Y parameterizes the ramification data we can write

$$\mathrm{Hol}_{3,1} = \{(y,[p_0,p_1,p_2]) \in Y \times \mathbb{P}V_3^3 \mid p_i \text{ are linearly independent and span } \phi(y)\}.$$

Let $V_3(\mathbb{C}^4)$ be the Stiefel manifold of 3 frames in $\mathbb{C}^4$ and let $\mathbb{P}V_3(\mathbb{C}^4) = V_3(\mathbb{C}^4)/\mathbb{C}^*$. Then the canonical $\mathrm{GL}(3,\mathbb{C})$-bundle projection is C^*-invariant and we obtain a principal G-bundle

$$\pi\colon \mathbb{P}V_3(\mathbb{C}^4) \longrightarrow G_3(\mathbb{C}^4) \cong \mathbb{CP}^3$$

And so $\mathrm{Hol}_{3,1}$ is the total space of the pullback of π over

$$Y \subset X \xrightarrow{\phi} \mathbb{CP}^3.$$

By Corollary 1 this gives us a description of $\mathrm{Harm}_{0,6}(\mathbb{CP}^2)$. We can compute the cohomology of this space. A straightforward calculation shows that ϕ restricted to the zero section in Y has degree 3 and so the first Chern class pulls back to three times a generator in $H^2(Y)$. We need to know the cohomology of $\mathbb{P}V_3(\mathbb{C}^4)$ and the differentials in the Serre spectral sequence for the bundle π. The cohomology calculation is a straightforward application of Baum's results regarding the Eilenberg-Moore spectral sequence for the cohomology of a homogeneous space [Ba] and from this we can deduce the necessary differentials. The E_2 term is

$$E_2(\pi)^{*,*} = H^*(\mathbb{CP}^3;\mathbb{Z}/p) \otimes H^*\big(\mathbb{P}U(3);\mathbb{Z}/p\big).$$

We can describe the differentials in terms of the generators for the cohomology of the fibre given above. Let b be the mod p reduction of the first Chern class. There are three cases: For $p = 2$ there are no non-trivial differentials. For $p = 3$ there is only one non-trivial differential generated by $d_2(e_1) = b$. For $p > 3$ the only non-trivial differentials are generated by $d_4(e_3) = b^2$. This is sufficient to determine the differentials in the spectral sequence for cohomology with $\mathbb{Z}/p$ coefficients of the pullback bundle

$$\mathbb{P}U(3) \longrightarrow \mathrm{Hol}_{3,1} \longrightarrow Y.$$

Since the base has the homotopy type of S^2 the only possible differentials are at E_2. For $p \neq 3$ there are no non-trivial differentials in $E_2(\pi)$. For $p = 3$, b pulls back to zero so the only possible differential is zero. In either case the spectral sequence collapses at E_2 and

$$H^*\big(\mathrm{Harm}_{0,6}(\mathbb{CP}^2);\mathbb{Z}/p\big)\cong H^*(S^2;\mathbb{Z}/p)\otimes H^*\big(\mathbb{P}U(3);\mathbb{Z}/p\big).$$

The spectral sequence does not completely determine the cup products so this need not be an algebra isomorphism.

We can also describe $\mathrm{Hol}_{3,0}$. It is the restriction of the bundle π to $\mathbb{CP}^3\setminus\phi(X)$. This gives a geometric description of $\mathrm{Harm}_{1,7}(\mathbb{CP}^2)$, the first non-minimal critical level in the degree 1 component, as a principal G-bundle over the compliment of X in $\mathbb{CP}^3$. This compliment is not simply connected and so using the Serre spectral sequence to compute cohomology groups will involve understanding the system of local coefficients.

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