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A COMPLETE CONVERGENCE THEOREM FOR ATTRACTIVE REVERSIBLE NEAREST PARTICLE SYSTEMS

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ABSTRACT. In this paper we prove a complete convergence theorem for attractive, reversible, super-critical nearest particle systems satisfying a natural regularity condition. In particular this implies that under these conditions there exist precisely two extremal invariant measures. The result we prove is relevant to question seven of Liggett (1985), Chapter VII.

Introduction. In this paper we study a class of Nearest Particle Systems (NPS) on the state space $\{0,1\}^Z$. As a prelude to providing our main result, we introduce preliminary definitions and the necessary results of earlier work.

A point $\gamma \in \{0,1\}^Z$ is called a *configuration* and may be viewed as a function from Z to $\{0,1\}$. We say a site x is *occupied* if $\gamma(x)$ equals 1; otherwise we say it is *vacant*. A configuration γ may also be identified with the Z-subset of occupied sites. In this paper, we will use both interpretations of γ. We say γ is *finite* if it has only finitely many occupied sites; otherwise it will be termed *infinite*.

A NPS is a spin system $\{\gamma : t \geq 0\}$ on $\{0,1\}^Z$ with flip rates of the following form:

$$c(x,\gamma) = \begin{cases} 1 & \text{if } \gamma(x) = 1; \\ \beta(l_x, r_x) & \text{if } \gamma(x) = 0, \end{cases}$$

where $l_x = x - \sup\{y < x : \gamma(y) = 1\}$ and $r_x = \inf\{y > x : \gamma(y) = 1\} - x$. It is easy to see that if $\sum_n \big(\beta(n,\infty) + \beta(\infty,n)\big) < \infty$, then a.s. γ_t will be a finite configuration for all t whenever γ_0 is finite. Equally, if γ_0 is infinite then so must γ_t be for all later t. Thus we may speak of finite and of infinite processes.

A finite process can be thought of as a continuous time Markov chain on the countable state space of finite subsets of Z. For such a chain, the state $\mathbf{0}$ (which can be thought of as the empty subset) is a trap. We say a finite NPS survives if there is some (finite) initial configuration γ_0 so that this trap state is not hit a.s.. Most usual systems considered (including all those treated by this paper) have finite systems which are irreducible Markov chains. Therefore, for this paper a finite NPS will survive if

$$P^\gamma[\tau < \infty] < 1,$$

where γ is a (any) configuration with only one occupied site and τ is the hitting time of $\mathbf{0}$.

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An infinite process is said to *survive* if there exists a nontrivial invariant measure. That is, there exists a probability measure ν on $\{0,1\}^Z$ so that for all continuous f and all positive t, $\int f(\omega)\nu(d\omega) = \int P_t f(\omega)\nu(d\omega)$, where $P_.$ is the semigroup for the process.

We write $\gamma \geq \gamma'$ for two configurations if $\gamma(x) \geq \gamma'(x)$ for each x. We also express this relation by $\gamma' \subset \gamma$. A function $f\colon \{0,1\}^Z \longrightarrow R$ is *increasing* if $\gamma^1 \geq \gamma^2 \Rightarrow f(\gamma^1) \geq f(\gamma^2)$. If exactly one of the inequalities is reversed, then f is *decreasing*. Given two probability measures ν_1, ν_2 on $\{0,1\}^Z$ we write $\nu_1 \geq \nu_2$ if $\int f(\omega)\nu_1(d\omega) \geq \int f(\omega)\nu_2(d\omega)$ for every increasing function f. Henceforth we will write $\int f(\omega)\mu(d\omega)$ as $\langle \mu, f\rangle$ for any function f and measure μ.

A process on $\{0,1\}^Z$ is said to be *attractive* if the function $c(x,\gamma)$ is increasing when restricted to configurations γ with $\gamma(x) = 0$ and decreasing when restricted to γ with $\gamma(x) = 1$, for each x in Z.

A useful consequence of attractiveness is that if $\gamma_0 \geq \gamma_0'$, then we can couple the processes γ_t and γ_t', so that for all t, $\gamma_t \geq \gamma_t'$. Thus if f is an increasing function, then $P_t^\gamma(f)$ is also an increasing function of γ. Another consequence of attractiveness is that there exists a (unique) upper invariant measure ν such that $\nu \geq \nu'$ for every invariant measure ν'. If γ_t^1 is the NPS with $\gamma_0^1 \equiv 1$, then γ_t^1 tends to ν in distribution as t tends to infinity. See Liggett (1985) for a more general and fuller account of this property.

It is easy to see that a NPS is attractive if and only if the function $\beta(l,r)$ is decreasing in both l and r. If the function $\beta(\cdot\,,\cdot)$ has this property, then it is clear that for any family of finite NPS γ_t^λ starting from non-zero initial configurations and with flip rate determined by function $\lambda\beta(\cdot\,,\cdot)$, $\lambda \geq 0$, there exists a critical λ_c^1 so that for $\lambda > \lambda_c^1, \gamma_t^\lambda$ has a positive chance of surviving while for $\lambda < \lambda_c^1$ γ_t^λ must die. Similarly, there exists a λ_c^2 corresponding to survival of infinite systems.

In this paper we will be considering reversible, attractive NPS. A NPS is *reversible* if the flip-rate determining function $\beta(\cdot\,,\cdot)$ is of the form

$$\beta(l,r) = \frac{\beta(l)\beta(r)}{\beta(l+r)}, \quad \beta(l,\infty) = \beta(\infty,l) = \beta(l)$$

for some strictly positive function $\beta\colon Z \longrightarrow R$, satisfying $\sum_n \beta(n) < \infty$. It is shown in Liggett (1985) that a finite NPS is a reversible countable state space Markov chain if and only if the flip functions are of this form. (See Theorem 1.2, p. 318 of Liggett (1985).)

Reversible processes are of intrinsic interest, but reversible NPS are also studied because so much can be said of them. A key result, essential for this paper, is due to Griffeath and Liggett (1982):

THEOREM A. *A finite reversible nearest particle system with flip rates determined by β survives if and only if $\sum_n \beta(n) > 1$.*

DEFINITION. A reversible NPS is called *supercritical* if the underlying β satisfies $\sum_n \beta(n) > 1$.

Given a reversible NPS with rate defining function $\beta(\cdot)$ satisfying either $\sum_n \beta(n) > 1$ or both $\sum_n \beta(n) = 1$ and $\sum_n n\beta(n) < \infty$, then we can find $\theta \in (0,1]$ so that $g(n) = \beta(n)\theta^n$

is a probability on Z^+. So under the survival condition of Theorem A on β, we define the measure $\mathrm{Ren}(\beta)$ to be the measure on $\{0,1\}^Z$ where 1's are distributed according to the stationary renewal process corresponding to the probability law on the integers, $\{g(n)\}$. This measure is reversible for the process. It is important to realize that if $\beta_\theta(l) = \theta^l \beta(l)$ for $\theta \in (0,1)$, then $\mathrm{Ren}(\beta_\theta) = \mathrm{Ren}(\beta)$. This is because for l and r finite

$$\beta(l,r) = \frac{\beta(l)\beta(r)}{\beta(l+r)} = \frac{\beta_\theta(l)\beta_\theta(r)}{\beta_\theta(l+r)} = \beta_\theta(l,r),$$

while the rates $\beta(l,\infty) \geq \beta_\theta(l,\infty)$ are irrelevant for configurations γ where $\sum_{x\leq 0}\gamma(x) = \sum_{x\geq 0}\gamma(x) = \infty$. In fact, the condition that either $\sum \beta(n) > 1$ or both $\sum \beta(n) = 1$ and $\sum \beta(n)n < \infty$ is necessary and sufficient for the survival of infinite systems. Therefore, the family of attractive NPS corresponding to $\lambda \frac{\beta(l)\beta(r)}{\beta(l+r)}$ has critical value for infinite survival, λ_c^2, equal to that for finite survival λ_c^l. For this paper another crucial result comes from Liggett (1983):

THEOREM B. *For an attractive, reversible NPS satisfying*

$$\sum_n \frac{\beta(n)\beta(n)}{\beta(2n)} = \sum_n \beta(n,n) < \infty,$$

and either $\sum_n \beta(n) > 1$ *or both* $\sum_n \beta(n) = 1$ *and* $\sum_n n\beta(n) < \infty$, *the renewal measure* $\mathrm{Ren}(\beta)$ *is the unique non-trivial, translation invariant, stationary probability measure. Here, a measure is non-trivial if it puts no mass on the configuration* $\mathbf{0}$. *A consequence is that under the conditions of Theorem B, the unique upper invariant measure* ν *is equal to* $\mathrm{Ren}(\beta)$.

This paper is devoted to proving

THEOREM. *Let* γ_t *be a finite or infinite reversible, attractive supercritical NPS such that* $\sum_n \beta(n,n) < \infty$, *and let* τ *be the stopping time* $\inf\{t > 0 : \gamma_t = \mathbf{0}\}$. *Then as t tends to infinity,* γ_t *tends in distribution to*

$$P^{\gamma_0}[\tau < \infty]\delta_{\mathbf{0}} + (1 - P^{\gamma_0}[\tau < \infty])\,\mathrm{Ren}(\beta),$$

for all initial γ_0.

If the initial configuration is infinite then $P[\tau < \infty]$ must be zero, so the Theorem shows that for any infinite γ_0, γ_t tends to $\mathrm{Ren}(\beta)$ in distribution as t tends to infinity. It also extends Theorem B by showing that $\mathrm{Ren}(\beta)$ is the unique non-trivial invariant measure under suitable conditions. The Theorem is similar to the complete convergence theorem for the contact process (see Liggett (1985) or Durrett (1988)), but the path of the proof is different. We are unable to make use of any kind of duality and the (difficult) Theorem B is essential to the proof. For the contact process the result analogous to Theorem B is a simple consequence of the contact process complete convergence theorem.

We describe out result as *a* complete convergence theorem rather than *the* complete convergence theorem as significant open questions remain: can the condition $\sum_n \beta(n,n) < \infty$ be loosened? What can be said of the infinite NPS where $\sum_n \beta(n) = 1$, $\sum_n n\beta(n) < \infty$? The latter question is essentially question 17, p. 360 of Liggett (1985).

For a reversible NPS, attractiveness is equivalent to

$$\frac{\beta(n)}{\beta(n+1)} \text{ is non-increasing in } n.$$

It should be noted here that as $\sum_n \beta(n) < \infty$, the limit of the above sequence must be greater than or equal to one.

Our proof can be broken down as follows:

1. We consider the case $\lim_n \frac{\beta(n)}{\beta(n+1)} > 1$ and show that if the process is supercritical, then the process can be renormalized and compared to supercritical oriented percolation.
2. We use the results of Step 1 to prove the Theorem for $\beta(\cdot)$ of Step 1.
3. We extend our result to all $\beta(\cdot)$ satisfying the conditions of the Theorem.

The case where $\lim_n \frac{\beta(n)}{\beta(n+1)} > 1$ is easier to deal with than the general because such NPS have the property that if the interval $[0,\infty)$ is vacant, then the distribution of the site of the first subsequent birth on this semi-infinite interval is tight over all configurations of the NPS prior to the birth. This is not true in general.

The paper is planned as follows. In the first section, we recall some results from oriented percolation. In Section 2 we use the ideas of Bezuidenhout and Grimmett (1990) to show that for processes with $\lim_{n\to\infty} \frac{\beta(n)}{\beta(n+1)} = C > 1$, survival implies "block" survival. This completes Step 1 above. This implies in particular that on the event that the system survives, the position of the occupied site nearest the origin will be tight over time. In the succeeding section this result will be used, in conjunction with an approach introduced in Mountford (1993) to show the complete convergence theorem for the class of NPS considered in Section 2. In Section 4 we employ the ideas mentioned in the above paragraph to finish the proof of the Theorem.

From now on, we assume that any $\beta(\cdot)$ we are dealing with satisfies $\sum_n \beta(n,n) < \infty$. This condition holds for most systems of interest. In particular, it holds if

$$\sup_n \frac{\beta(n)}{\beta(2n)} < \infty,$$

which holds for $\beta(n) = \lambda / n^p$ with $p > 1$.

Section 1. In this paper, a 1-dependent oriented percolation system (of probability $1-\varepsilon$) $\{\Psi^A\}_{A\subset 2Z}$ is as follows:

a) Let $G = (V,E)$ be the directed graph with

$$V = \{(m,n) \in Z \times Z_+ : n + m \equiv 0 \bmod(2)\},$$
$$E = \Big\{\big((m,n),(m+1,n+1)\big), \big((m,n),(m-1,n+1)\big) : (m,n) \in V\Big\}.$$

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b) For every $(m,n) \in V$, we have two random variables $I_{m,n,+}$ and $I_{m,n,-}$ which are 1 with probability $1-\varepsilon$, 0 with probability ε.

c) The random variable $I_{m,n,+}$ is independent of all $I_{i,j,\pm}$ except $I_{m,n,-}$ and $I_{m+2,n,-}$. $I_{m,n,-}$ is independent of all $I_{i,j,\pm}$ except $I_{m,n,+}$ and $I_{m-2,n,+}$.

We say a bond $\big((m,n),(m\pm 1, n+1)\big)$ is *open* if $I_{m,n,\pm} = 1$; otherwise it is *closed*. We define $\Psi^A(m,n)$ to be 1 if there is a directed path of open bonds from a point $(0,x)$ to (m,n) for some $x \in A$; otherwise $\Psi^A(m,n)$ is 0. We say Ψ^A survives if for every n, there exists some m with $\Psi^A(m,n) = 1$. For a full, detailed account of oriented percolation the reader is referred to Durrett (1984). We require the following result.

THEOREM C. *(i) Given $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for Ψ a 1-dependent oriented percolation system of probability $1-\varepsilon$, $\varepsilon < \varepsilon_0$, and any singleton $x \in 2Z$, $P[\Psi^{\{x\}}$ survives$] > 1-\eta$.*

(ii) Given A, $\eta > 0$ and R, there exists a positive integer K so that for all n sufficiently large,

$$P\Big[\sum_{j=0}^{K} \Psi^A(j,n) > R,\ \Psi^A \text{ dies}\Big] < \eta.$$

Similarly for $\sum_{j=-K}^{0} \Psi^A(j,n)$.

(iii) Given $\eta > 0$, there exists $\varepsilon_0 > 0$, so that if Ψ is a 1-dependent oriented percolation system of probability $1-\varepsilon$ (with $\varepsilon < \varepsilon_0$), then for all even y and all even n sufficiently large, $P[\Psi^{\{0\}}(y,n) = 1] \geq 1-\eta$.

(iv) For fixed y the events $\{\Psi^A$ survives$\}$ and the event $\{\Psi^A(y,n) = 1$ for infinitely many $n\}$ are identical a.s..

All parts of Theorem C follow from the contour arguments of Durrett (1984).

Section 2. In this section we wish to establish that under the conditions

A) The decreasing limit as n tends to infinity of $\frac{\beta(n)}{\beta(n+1)}$ is equal to C, a number strictly greater than 1, and

B) $\sum_n \beta(n,n) < \infty$,

survival of the finite particle system is equivalent to survival in a "block argument sense". We will follow closely the proof of Bezuidenhout and Grimmett (1990). We wish to show that survival implies "block survival", since survival in the sense of non-extinction does not preclude a limit measure of the NPS being the trivial null measure. On the other hand (see Theorem C above), block survival does ensure that any limit measure must be non-trivial.

It is important to realize that Condition B above implies

C) $M = \sup_n \sum_{l+r=n} \beta(l,r) < \infty$, and in particular that $\sum_l \beta(l) \leq M$.

This follows since by the symmetry of the function $\beta(\cdot\,,\cdot)$, $\sum_{l+r=n} \beta(l,r) \leq 2 \sum_{l\leq n/2} \beta(l,n-l)$, and by attractiveness, this last expression is less than $2 \sum_{l\leq n/2} \beta(l,l)$.

Our arguments will rely on various couplings. We will assume that all NPS we shall consider, with various starting configurations and various constraints, are derived from the same Harris system of independent Poisson processes. See Durrett (1988) for greater details. We assume we are given independent rate one Poisson processes D_x, $x \in Z$ and independent rate $\beta(1,1)$ Poisson processes λ_x, so that associated with points $t_l^x, t_2^x, \ldots$ in λ_x, there are uniform [0,1] random variables $U_1^x, U_2^x, \ldots$, where as x and i vary, the random variables U_i^x are independent. From these Poisson processes and i.i.d. uniform random variables we construct a NPS starting at $A \subset Z$ as follows:

a) A particle at x dies at time t, if $\gamma_{t-}(x) = 1$ and $t \in D_x$.
b) A particle is created at site x at time t if $\gamma_{t-}(x) = 0$, $t = t_j^x \in \lambda_x$ and $U_j^x \leq \frac{\beta(l,r)}{\beta(1,1)}$. Here l is the distance to the left from x to the nearest occupied site at time t, r the distance to the right.
c) $\gamma_0(x) = I_{x\in A}$.

If A is infinite, the process is defined as the increasing limit of this procedure for finite sets increasing to A. Given an interval I and a subset A of I, we define a process $\gamma^{I,A}$ by taking $\gamma_0^{I,A}(x) = I_{x\in A}$ and suppressing all births of particles outside I. If the interval is equal to $[-N, N]$, we write $\gamma^{N,A}$ for the process. γ^I will always denote a NPS with births outside I suppressed. If we are given an unrestricted NPS γ then γ^I will denote the NPS with all births and particles outside I suppressed and such that $\gamma_0^I(x) = \gamma_0(x)$ for $x \in I$. The time τ^I will always denote this hitting time for an unrestricted process.

Restrictions of a process may extend to time dependent regions as well. Given a region R in $Z \times R_+$, the process, γ^R, restricted to R, is the process such that at time 0 all points x for which $(x, 0)$ is outside R are killed and every birth point (x, t_i^x) outside R is suppressed.

Given the Harris construction of processes, events can be thought of as subsets of the space of Poisson process realizations Ω. We say an event A is *increasing* if $\omega \in A$ and ω' is obtained from ω by either

a) deleting some points of the Poisson processes D_x,
b) adding some points to the Poisson processes λ_x, or
c) decreasing some U_x^j,

then $\omega' \in A$ also. We similarly define *decreasing* events. As our process is attractive we have the FKG inequality: if A and B are both increasing (decreasing) events then $P[A \cap B] \geq P[A]P[B]$.

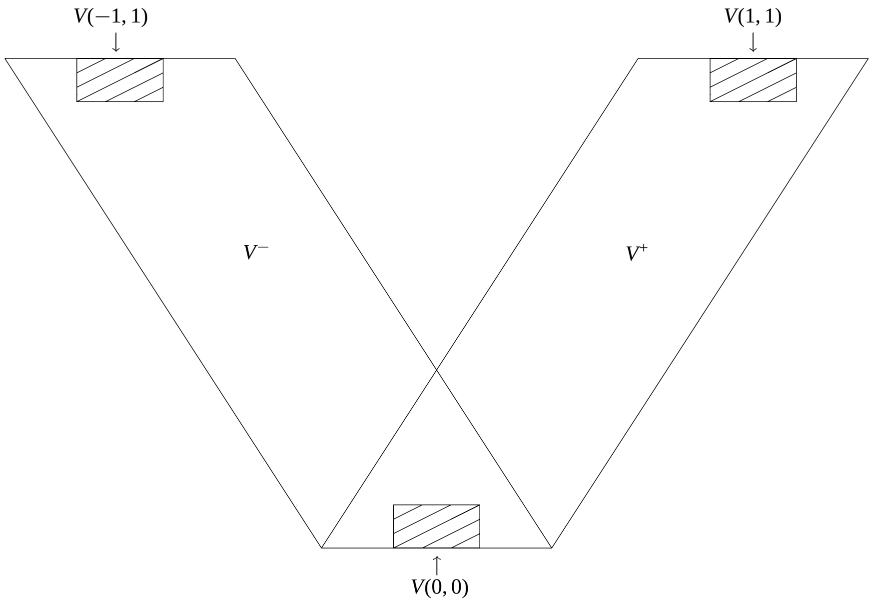
Given integers k, L, S, we define

(i) $w(p,q) = (pkL, 2qkS)$, for $(p,q) \in Z^1 \times Z_+$ $p + q \equiv 0 \pmod 2$,
(ii) $V^{\pm} = \{(x,t) \in Z \times R_+ : 0 \leq t \leq (2k+2)S, -5L \pm Lt/2S \leq x \leq 5L \pm Lt/2S\}$,
(iii) $V(p,q) = [-2L, 2L] \times [0, 2S] + (pkL, 2qkS)$, for $(p,q) \in Z^1 \times Z_+$ $p+q \equiv 0 \pmod 2$.

This section is devoted to proving

PROPOSITION 2.1. *Given a supercritical NPS whose corresponding β satisfies conditions A and B above, and $\varepsilon > 0$, we may choose integers r, k, L and S so that if T is a stopping time for the NPS γ_t and*

a) $T \in [0, 2S]$,
b) $\gamma_T \equiv 1$ *on* $[x - r, x + r]$ *for some* $x \in [-2L, 2L]$,



then there exists a stopping time $T' > T$, with respect to the natural filtration of the NPS restricted to $V^+(V^-)$, so that with probability at least $1-\varepsilon$, T' is in $[2kS, 2kS+2S]$ and there exists $y \in [-2L, 2L] + kL(-kL)$ so that $\gamma_{T'} \equiv 1$ on $[y-r, y+r]$. In particular the NPS may be coupled with a 1-dependent oriented percolation system Ψ of parameter $1-\varepsilon$ so that

> *$\Psi^A(p,q) = 1$ implies there exists (x,t) in $V(p,q)$ so that $\gamma_t \equiv 1$ on $[x-r, x+r]$, where $A = \{m$ even: there exists x in $[(m-2)L, (m+2)L]$ such that $\gamma_0 \equiv 1$ on $[x-r, x+r]\}$.*

As all processes (with or without suppressions) are attractive and are derived from the same Poisson process, the following lemma is immediate.

LEMMA 2.1. *If $I \subset J$, then $\gamma_0^I \subset \gamma_0^J$ implies that $\gamma_t^I \subset \gamma_t^J$ for all t. If I and J are disjoint intervals of Z, then conditional on the initial configuration γ_0, the processes γ^I and γ^J are independent.*

We state some simple preliminary lemmas before beginning our Proof of Proposition 2.1. The following is a simple consequence of attractiveness and supercriticality and echoes equation (6) of Bezuidenhout and Grimmett (1990).

LEMMA 2.2. *Given $\varepsilon > 0$, there exists an integer r so that $[x-r, x+r] \subset \gamma_0$ for some x implies that $P^{\gamma_0}[\tau = \infty] \geq 1 - \frac{1}{2}\varepsilon^{100}$.*

The lemma below is an analogue of equation (11) of Bezuidenhout and Grimmett (1990).

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LEMMA 2.3. *If γ_0, r and ε are connected as in Lemma 2.2, then for any N, there exists a T so that* $P^{\gamma_0}[|\gamma_t| < N$ for some $t > T] < \frac{3}{4}\varepsilon^{100}$.

DEFINITION. Given a process γ_t^I restricted to interval I, we define $R_t = \sup\{x \in I : \gamma_t^I(x) = 1\}$, $L_t = \inf\{x \in I : \gamma_t^I(x) = 1\}$. We say that γ^I *tries to give birth to the right* at time t if γ_t^I is non-empty and for some $x \in I^c$ and to the right of I, $t = t_i^x \in \lambda_x$ and $U_i^x \le \frac{\beta(x-R_t)}{\beta(1,1)}$; we similarly use the term γ^I *tries to give birth to the left*. If γ^I tries to give birth to either the left or right we simply say it *tries to give birth*.

Given a NPS γ_t^I, we define a blocked point process $W^+ = \{t_1^+, t_2^+, \ldots\}$ where $t_1^+ = \inf\{t : \gamma^I$ tries to give birth, to the right of R_t at time $t\}$, for $i > 1$, $t_i^+ = \inf\{t > t_{i-1}^+ + 1 : \gamma^I$ tries to give birth, to the right of R_t at time $t\}$. We define the blocked point process W^- similarly. We write W_t^+ for $W^+ \cap [0,t]$, W_t^- for $W^- \cap [0,t]$ and $W_t^\pm$ for $W_t^\pm \cup W_t^-$. Throughout the paper F_t will denote $\sigma\{\gamma_s : s \le t\}$. Given that the NPS $\gamma^{[-N,N]}$ tries to give birth to the right at time t, the conditional chance that the unrestricted process would have given birth at site $N+1$ at time t (from the same configuration) is equal to

$$\frac{\beta(N+1-R_t)}{\sum\limits_{l \ge N+1-R_t} \beta(l)}.$$

It is important to notice that under assumption A, made at the start of this section, this probability is at least $1 - 1/C$ irrespective of N and the random R_t. Without this assumption we cannot bound this conditional probability away from zero without the restricting N or R_t.

The lemma below is our equivalent of Equation 17 of Bezuidenhout and Grimmett (1990).

LEMMA 2.4. *Let γ be an unrestricted NPS with $\gamma_0 \subset [-L, L]$ and let γ^L be a restricted NPS such that $\gamma_0 = \gamma_0^L$. There is a constant $c > 0$, not depending on L or t so that $P[\tau < \infty \mid F_t^L] \ge c^{2+|W_t^\pm|+|\gamma_t^L|}$. Here $F_t^L = \sigma\{\gamma_s^L, s \le t, W_t^\pm\}$.*

PROOF. Given condition C on our NPS, it is clear that $P[\tau < \infty \mid F_t] > c^{|\gamma_t|}$, for some constant c, *e.g.*, we could take $c = 1/(2M+1)$ where M is the constant in Condition C of our restrictions on the function β. Therefore, the Lemma will be proven if we can show that

$$P[|\gamma_t| - |\gamma_t^L| \le 2 \mid F_t^L] > c^{|W_t^-|+|W_t^+|}$$

for some strictly positive c. In the following, we will treat the case where $\gamma_t^L \ne \mathbf{0}$. The case where $\gamma_t^L = \mathbf{0}$ is essentially the same but requires more notation.

In the following $r \wedge s$ denotes the minimum of r and s.

Let t_i^+ be an element of W^+, at which time γ^L tries to give birth to the right at site x (necessarily greater than L). We say that t_i^+ *did not influence* γ if all of the following events occur;

A(1,i): For $t_i^+ < t - 1$, the particle at x dies in the interval $(t_i^+, t_i^+ + 1]$.

A(2,i): At no time s in the time interval $\left(t_i^+, (t_i^+ + 1) \wedge t\right]$ does γ^I try to give birth to the right,

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A(3,i): At no time s in the time interval $\left(t_i^+, (t_i^+ + 1) \wedge t\right]$ is there a $y > x$, so that $s = t_j^y$ and $U_j^y \leq \frac{\beta(y-x)}{\beta(1,1)}$,

A(4,i): At no time s in the time interval $\left(t_i^+ (t_i^+ + 1) \wedge t\right]$ is there a $y \in (R_s, x)$ so that $s = t_j^y$ for some j and $\frac{\beta(y-R_s)}{\beta(1,1)} \leq U_j^y \leq \frac{\beta(y-R_s, x-y)}{\beta(1,1)}$.

The reason for the above definition is that we are interested in comparing an unrestricted NPS, γ, with a restricted NPS, γ^I, generated by the same system of Poisson processes. If $\gamma_{t_i^+ -} = \gamma^I_{t_i^+ -}$ and γ^I tries to give birth at site x at time t_i^+ then (neglecting births to the left of I) $\gamma^I_{t_i^+ + 1}$ will equal $\gamma_{t_i^+ + 1}$ provided

(i) in time interval $[t_i^+, t_i^+ + 1]$ the particle at x dies. This corresponds to event $A(1, i)$.

(ii) During the above interval the process γ^I does not try to give birth to the right. Obviously this corresponds to event A(2,i).

(iii) The NPS γ does not have a particle born to the right of x in time interval $[t_i^+, t_i^+ + 1]$. This event is contained in the event A(3,i).

(iv) No particles are born for the process γ but not for the process γ^I in the interval (R_s, x) because the site x is occupied (event A(4,i)). Note this includes extra births for γ both at sites within I and to the right of I.

We similarly define the event t_i^- did not influence γ for $t_i^- \in W_t^-$.

These events were introduced because of the following fact: On the event $\bigcap_{t_i^\pm \in W_t^\pm} \{t_i^\pm$ does not influence $\gamma\}$, we have for $s \in [t_i^\pm, (t_i^\pm + 1) \wedge t]$, some $t_i^\pm$, that $|\gamma_s^L| \geq |\gamma_s| - 2$; for other s in $[0, t]$, $\gamma_s = \gamma_s^L$. So certainly on the event $\bigcap_{t_i^\pm \in W_t^\pm} \{t_i^\pm$ does not influence $\gamma\}$, it is the case that $|\gamma_t^L| \geq |\gamma_t| - 2$ for all $s \in [0, t]$.

For fixed $t_i^\pm$, the events A(2,i) and A(3,i) are precisely the events that given Poisson processes of rate at most $\sum_n \beta(n)$ have no points in the interval $(t_i^\pm, t_i^\pm + 1]$. The event A(4,i) is equivalent to the event that a Poisson process of inhomogeneous rate depending on γ_t^L and at most M (as in Condition C) has no points in the interval $(t_i^\pm, t_i^\pm + 1]$. For $t_i^\pm < t-1$ the event A(1,i) is simply the event that a rate one Poisson process (independent of Poisson processes above) has a point in interval $[t_i^\pm, t_i^\pm + 1]$. Therefore the conditional probability of $\{t_i^\pm$ does not influence $\gamma\}$ is at least

$$(1 - e^{-1})\left(e^{-\sum_n \beta(n)}\right)\left(e^{-\sum_n \beta(n)}\right)e^{-M}.$$

The events $\{t_i^\pm$ does not influence $\gamma\}$ are all conditionally independent with respect to F_t^L on the event $\{\gamma_t \neq \mathbf{0}\}$ so we have

$$P\Big[\bigcap_{t_i^\pm \in W_t^\pm} \{t_i^\pm \text{ does not influence } \gamma\} \mid F_t^L\Big] > c^{|W_t^+| + |W_t^-|}$$

for c equal to $(1 - e^{-1})\left(e^{-\sum_n \beta(n)}\right)\left(e^{-\sum_n \beta(n)}\right)e^{-M}$. ∎

Let $\zeta = P[\gamma_1^r = 1 \text{ on } [-r, r] \mid \gamma_0^r(x) = I_{x=-r}]$. Here r is as in Lemma 2.2. Fix N and Q so that

$$(*) \qquad (1 - \zeta)^N \leq \varepsilon^{100}, \quad P[\mathrm{Bin}(Q, 1 - 1/C) \leq N] < \varepsilon^{100}.$$

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Here $\mathrm{Bin}(Q, 1 - 1/C)$ is a binomial random variable with parameters Q and $1 - 1/C$ and C is the value defined in Condition A at the start of this section.

When considering the blocked processes of times when a NPS γ^L tries to give birth to the right (left) and where L is not necessarily fixed, we will use superscripts and refer to the processes as $W^{+,L}(W^{-,L})$.

LEMMA 2.5. *Let T_i and L_i be increasing sequences of times and lengths. Then for any γ_0 with $\gamma_0(x) = 1$ for $|x| \leq$ r, (r as in Lemma 2.2)*

$$P^{\gamma^0}\Big[\big|W^{+,L_n}_{T_n}\big| + \big|W^{-,L_n}_{T_n}\big| + \big|\gamma^{L_n}_{T_n}\big| > 12Q \mid F^{L_n}_{T_n}\Big]$$

is greater than $1 - \varepsilon^{100}$ for n large.

PROOF. On the event $\tau < \infty$ (for the unrestricted NPS γ), $P\big[\tau = \infty \mid F^{L_n}_{T_n}\big]$ will be zero for n large enough. This must imply that for any positive constants k_1 and k_2, $P\Big[P\big[\tau = \infty \mid F^{L_n}_{T_n}\big] > 1 - k_1\Big] > P^{\gamma_0}[\tau = \infty] - k_2$ for n large enough. We chose r to ensure that $P^{\gamma_0}[\tau = \infty] > 1 - \frac{1}{2}\varepsilon^{100}$, so if we take $k_1 = c^{12Q+2}$ where c is the constant of Lemma 2.4 and $k_2 < \frac{1}{2}\varepsilon^{100}$, we obtain the conclusion: For n large enough, $P\Big[P\big[\tau = \infty \mid F^{L_n}_{t_n}\big] > 1 - c^{12Q+2}\Big] > 1 - \varepsilon^{100}$. The lemma follows from Lemma 2.4. ∎

The following lemma corresponds to Lemma 7 of Bezuidenhout and Grimmett (1990).

LEMMA 2.6. *There exist t and L such that if $\gamma_0 \equiv 1$ on $[-r, r]$, (again r is as in Lemma 2.5) then*

$$P^{\gamma_0}\Big[\big|\gamma^L_t\big|, \big|W^{+,L}_t\big|, \big|W^{-,L}_t\big| \text{ are all greater than } Q\Big] > 1 - \varepsilon^{24}.$$

PROOF. The event in question is an increasing event so we may take $\gamma_0(x) = I_{|x| \leq r}$ without loss of generality.

We first define increasing sequences T_n and L_n, in order to apply Lemma 2.5. Lemma 2.3 and our choice of r ensure that there exists T_1 such that

$$P^{\gamma_0}[|\gamma_{T_1}| > 12Q] > 1 - \frac{3}{4}\varepsilon^{100}.$$

As the unrestricted process γ is the limit of restricted NPS, we can choose $L_1\big(= L(T_1)\big)$, so that

$$P^{\gamma_0}\Big[\big|\gamma^{L_1}_{T_1}\big| > 12Q\Big] > 1 - \frac{3}{4}\varepsilon^{100}.$$

As the restricted process γ^{L_1} must die out, the function $P^{\gamma_0}\Big[\big|\gamma^{L_1}_t\big| > 6Q\Big]$ tends continuously to zero as t tends to infinity. Therefore there exists a time $s(T_1, L_1)$ $(> T_1)$, so that

$$P^{\gamma_0}\Big[\big|\gamma^{L_1}_{s(T_1,L_1)}\big| > 6Q\Big] = 1 - \varepsilon^{50}.$$

We now recursively choose $T_{n+1} = s(T_n, L_n) + 1$, $L_{n+1} = \max\{L(T_{n+1}), L_n + 1\}$.

It is clear that T_n and L_n are sequences tending to infinity, so we can apply Lemma 2.5 and conclude that for n large

$$P^{\gamma_0}\Big[\big|\gamma_{T_n}^{L_n}\big| + \big|W_{T_n}^{+,L_n}\big| + \big|W_{T_n}^{-,L_n}\big| > 12Q\Big] > 1-\varepsilon^{100}.$$

Therefore, by the FKG inequality,

$$\begin{aligned}\varepsilon^{100} &\geq P^{\gamma_0}\Big[\big|\gamma_{T_n}^{L_n}\big| + \big|W_{T_n}^{+,L_n}\big| + \big|W_{T_n}^{-,L_n}\big| \leq 12Q\Big]\\ &\geq P^{\gamma_0}\Big[\big|\gamma_{T_n}^{L_n}\big| \leq 6Q,\ \big|W_{T_n}^{+,L_n}\big| \leq 3Q,\ \big|W_{T_n}^{-,L_n}\big| \leq 3Q\Big]\\ &\geq \varepsilon^{50}P^{\gamma_0}\Big[\big|W_{T_n}^{+,L_n}\big| \leq 3Q\Big]^2\end{aligned}$$

since all three events are decreasing. This inequality implies that $P^{\gamma_0}\Big[\big|W_{T_n}^{+,L_n}\big| > 3Q\Big] > 1-\varepsilon^{25}$ and we are done. ∎

COROLLARY 2.1. *Let T, L and γ_0 be as in Lemma 2.6. There exists a stopping time $\nu \leq T$ so that with probability at least $1-\varepsilon^{10}$, $\gamma_\nu^{L+2r+1} \equiv 1$ on $[L+1, L+2r+1]$.*

PROOF. Consider γ_t^L. By the Proof of Lemma 2.6, we have that outside a set of probability ε^{25}, there are $3Q$ times at which γ^L tries to give birth at a point to the right of L. Since $\frac{\beta(n)}{\beta(n+1)}$ converges down to $C > 1$, at each of these times, independently of F_T^L, $\gamma_{\tau_i}^{L+2r}(L+1) = 1$ with probability at least $1 - 1/C$. By our choice of Q this implies that there exist with probability at least $(1-\varepsilon^{25})(1-\varepsilon^{100})$ stopping times (with respect to natural filtration of γ^{L+2r}) $s_1, s_2, \ldots s_N$, all less than T and more than one time unit apart, for which $\gamma_{s_i}^{L+2r}(L+1) = 1$. The (*) definition of N and ζ and the Strong Markov property ensure that with probability at least $(1-\varepsilon^{25})(1-\varepsilon^{100})^2$, there will be a stopping time ν for which $\gamma_\nu^{L+2r+1} \equiv 1$ on $[L+1, L+2r+1]$. ∎

We now take $K = L + 2r$ and T as before. Using exactly the same arguments as Bezuidenhout and Grimmett (1990), we conclude

LEMMA 2.8 (= LEMMA 18 OF BEZUIDENHOUT AND GRIMMETT (1990)). *If $\gamma_0(x) = I_{|x|\leq r}$, then*

$$P^{\gamma_0}\Big[\text{there exists } (x,t) \in [K, 2K] \times [T, 2T]\Big]$$

such that $\gamma_t^{2K}(y)$ is 1 if $|y-x| \leq r$] is greater than $1-\varepsilon^5$.

Recall that given a region $R \subset Z \times R_+$, we define a nearest particle system γ^R by suppressing all particles and births which occur at $(x,t) \in R^C$. Lemma 2.8 is used again precisely as in Bezuidenhout and Grimmett (1990) to show

LEMMA 2.9. *Let $R = \bigcup_{j=0}^{k-1} jL + [-3L, 4L] \times [2jT, (2j+4)T]$. Given that $\gamma_t^R(y) = 1$ for $|y-x| \leq R$, where $x \in [-2L, 2L]$ and $t \in [0, 2T]$. Then with probability greater than $(1-\varepsilon^5)^k$, there exists $(y,s) \in [(k-2)L, (k+2)L] \times [2kT, (2k+2)T]$ so that $\gamma_s^R(z) = 1$ if $|z-y| \leq r$.*

Given this lemma, Proposition 2.1 can be proved in a straightforward manner (see Bezuidenhout and Grimmett (1990) for full details).

COROLLARY 2.2. *Consider a supercritical NPS. Let γ_0 be an initial configuration. For each $\eta > 0$, there exists K so that*

$$\inf_t P^{\gamma^0}[\text{there exists } x \text{ with } |x| \le K \text{ and } \gamma_t(x) = 1] \ge P^{\gamma^0}[\tau = \infty] - \eta.$$

PROOF. Given η pick $\varepsilon > 0$, so that $\varepsilon < \eta/6$. Clearly it is sufficient to find a K that works for all t sufficiently large and $\varepsilon < \varepsilon_0$ for the ε_0 of Theorem C, Parts (i) and (iii) applied to $\eta/6$. Given this ε, pick r, k, S and L according to Proposition 2.1. Let V be the stopping time $\inf\{t > 0 : \text{there exists } X \text{ such that } \gamma_t \equiv 1 \text{ on } [X - r, X + r]\}$.

By the Markov property it is clear that the events $\{\tau = \infty\}$ and $\{V < \infty\}$ are a.s. equal. Let us define V^R to be the stopping time $\inf\{t > 0 : \text{ there exists } X^R \in [-R, R]$ such that $\gamma_t \equiv 1$ on $[X^R - r, X^R + r]\}$. We can find R and N so that $P[V^R < N] > P[\tau = \infty] - \eta/6$.

By the Strong Markov property and Proposition 2.1, there is an oriented 1-dependent percolation system Ψ of probability $1 - \varepsilon$ such that $\Psi^{\{0\}}(q, p) = 1$, implies that $\gamma_t \equiv 1$ on $[y - \mathrm{r}, y + \mathrm{r}]$ for some $y \in V(p, q) + (X^R, V^R)$ on the event $\{V^R < N\}$.

Theorem C, Part (iii) ensures that for all even n sufficiently large,

$$P\Big[\gamma_t \equiv 1 \text{ on } [y - \mathrm{r}, y + \mathrm{r}] \text{ for some } (y, t) \in V(0, n) + (X^R, V^R) \mid F_{V^R}\Big] > 1 - \eta/6$$

on the event $\{V^R < N\}$. Therefore by Proposition 2.1 for n large and even

$$\begin{aligned} P\Big[&\forall t \text{ in } [2nkS + 2S,\ 2(n + 3)kS],\\ &\qquad \exists x \in [X^R - (k + 7)L,\ X^R + (k + 7)L] \text{ s.t. } \gamma_t(x) = 1 \mid F_{V^R}\Big]\\ &> (1 - \eta/6)(1 - 3\varepsilon) > 1 - 2\eta/3 \end{aligned}$$

on the event $\{V^R < N\}$. Since $X^R \in$ [-R,R] on $\{V^R < \infty\}$, an event of probability at least $1 - \eta/6$, we have shown that for large t

$$P\Big[\gamma_t(x) = 1 \text{ for some } x \in [-(k+7)L - R, (k+7)L + R]\Big] \ge (1 - 2\eta/3)(P^{\gamma_0}[\tau = \infty] - \eta/6).$$

This completes the proof. ∎

We prove similarly, using Theorem C, Part (iv),

COROLLARY 2.3. *For any $y \in Z$ and any initial configuration, the event $\{\tau = \infty\}$ and the event $\{\gamma_t(y) = 1$ for unbounded $t\}$ are a.s. equal.*

We also see using Theorem C, Part (ii),

COROLLARY 2.4. *For each n and $\eta > 0$ and γ_0, there exists a K so that for all sufficiently large t*

$$P^{\gamma^0}\Big[\sum_{x=-2K}^{-(K+1)} \gamma_t(x) \text{ and } \sum_{x=K+1}^{2K} \gamma_t(x) \text{ are both greater than } n\Big] \ge P^{\gamma_0}[\tau = \infty] - \eta.$$

Section 3. This section is devoted to proving

PROPOSITION 3.1. *The complete convergence theorem holds for supercritical NPS which satisfy the condition of Section 2.*

Again, we will make use of the technique of coupling. Given a process γ and a stopping time T, such that $\gamma_T(x) = 1$, we define $\gamma^{x,T}$ to be the NPS derived from the underlying Poisson processes on $s \geq T$, defined for $s \geq T$ and satisfying $\gamma_T^{x,T}(y) = \delta_{xy}$. Such a process satisfies:

1) the process $\gamma'_t = \gamma^{x,T}_{t+T}$ is a NPS with $\gamma_0(y) = \delta_{xy}$ which is independent of F_T, and
2) for each $s \geq T$, $\gamma_s^{x,T} \subset \gamma_s$.

The Proof of Proposition 3.1 is deferred as we require some further results for the proof.

THEOREM D. *Let γ_t be a finite NPS corresponding to $\{\beta(l)\}$ where $\sum_l l\beta(l) < \infty$. If t_n is an increasing sequence of times converging to infinity, then for any s, the distance in absolute variation between the laws of γ_{t_n} and γ_{t_n-s} tends to zero.*

Theorem D was proven as Proposition 2.1 in Mountford (1993). In that paper, Proposition 2.1 was stated for processes of finite range, but the proof given requires no alteration to apply to the NPS of Theorem D, since what is really vital in the Proof of Proposition 2.1 is that the number of occupied sites of a finite particle system grows at most linearly with time.

We will be considering NPS where $\lim_{n\to\infty} \frac{\beta(n)}{\beta(n+1)} = C > 1$, so clearly Theorem D applies to such systems.

The following result is not a direct consequence of Theorem D as our processes are non-Feller.

LEMMA 3.1. *Let t_n be a sequence of times tending to infinity. If γ_{t_n} tends in distribution to ν then ν must be stationary for the NPS.*

PROOF. It is sufficient to show that for fixed $s > 0$ and fixed cylinder function f with $f(\mathbf{0}) = 0$, we have $\langle \nu, f\rangle = \langle \nu, P_s f\rangle$.

The function f is continuous so

$$\langle \nu, f\rangle = \lim_{n\to\infty} E[f(\gamma_{t_n})] = \lim_{n\to\infty} E[f(\gamma_{t_n-s})]$$

where the last equality follows from Theorem D. Now $E[f(\gamma_{t_n})] = E[P_s f(\gamma_{t_n-s})]$. If $P_s f$ were continuous, we could invoke Theorem D again to conclude that

$$\langle \nu, f\rangle = \lim_{n\to\infty} E[f(\gamma_{t_n})] = \lim_{n\to\infty} E[P_s f(\gamma_{t_n-s})] = \langle \nu, P_s f\rangle.$$

Unfortunately our process is non-Feller and we must modify this argument.

Define the function $P_s(K, f)$ by

$$\begin{aligned} P_s(K, f) = E^{\gamma}\Big[f(\gamma_s);\ \exists x \in (K, 2K], \exists y \in [-2K, -K) \\ \text{s.t. } \gamma_r(x) = \gamma_r(y) = 1\, \forall r \in [0, s]\Big]. \end{aligned}$$

For K so large that $[-K,K]$ contains the support of f, the function $P_s(K,f)$ is continuous and so for all K large enough

$$(**) \qquad \langle \nu, P_s(K,f)\rangle = \lim_{n\to\infty} E[P_s(K,f)(\gamma_{t_n-s})].$$

Note also that $P_s(K,f)(\mathbf{0}) = f(\mathbf{0}) = 0$. Also observe that occupied points remain occupied throughout $[0,s]$ independently with probability e^{-s}, we conclude that

$$|P_s(K,f)(\gamma) - P_s f(\gamma)| \le G_K(\gamma),$$

where $G_K(\mathbf{0}) = 0$ and for non-identically zero γ,

$$G(\gamma) = \left[(1-e^{-S})^{\sum_{x=K+1}^{2K}\gamma(x)} + (1-e^{-s})^{\sum_{x=-2K}^{-(K+1)}\gamma(x)}\right]\|f\|_\infty.$$

However, Corollary 2.4 guarantees that as K tends to infinity, $\langle \mu, G_K\rangle$ tends to zero uniformly over $\mu \in \{\nu, \gamma_{t_n-s},\ n = 1,2,\ldots\}$. (Here we have abused notation and taken γ_{t_n-s} to be the law of γ_{t_n-s}). Thus it follows that

$$\begin{aligned}
\langle \nu, P_s f\rangle &= \lim_{K\to\infty}\langle \nu, P_s(K,f)\rangle \\
&= \lim_{K\to\infty}\lim_{n\to\infty} E[P_s(K,f)(\gamma_{t_n-s})] \\
&= \lim_{n\to\infty}\lim_{K\to\infty} E[P_s(K,f)(\gamma_{t_n-s})] \\
&= \lim_{n\to\infty} E[P_s f(\gamma_{t_n-s})] \\
&= \lim_{n\to\infty} E[f(\gamma_{t_n})] \\
&= \langle \nu, f\rangle. \qquad \blacksquare
\end{aligned}$$

The following lemma will be used in this and the next section.

LEMMA 3.2. *Let f be a continuous, increasing function and γ_0 be any configuration, finite or infinite. Then*

$$\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] \le \langle \mathrm{Ren}(\beta), f\rangle.$$

PROOF. Theorem B implies that $\lim_{t\to\infty} E^{\mathbf{1}}[f(\gamma_t)] = \langle \mathrm{Ren}(\beta), f\rangle$. Attractiveness implies that

$$\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] \le \lim_{t\to\infty} E^{\mathbf{1}}[f(\gamma_t)] = \langle \mathrm{Ren}(\beta), f\rangle. \qquad \blacksquare$$

We first consider processes γ^x such that $\gamma_0^x(y) = \delta_{xy}$ and we first prove complete convergence for these processes.

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PROPOSITION 3.2. *As t tends to infinity, the process γ_t^x converges in distribution to the probability measure $P^x[\tau < \infty]\delta_{\mathbf{0}} + P^x[\tau = \infty]\,\mathrm{Ren}(\beta)$.*

PROOF. Suppose t_n increases to infinity and $\gamma_{t_n}^0$ converges in distribution to ν. Given Corollary 2.1 and Lemma 3.1, it is immediate that

$$\nu = P^0[\tau < \infty]\delta_{\mathbf{0}} + P^0[\tau = \infty]U$$

where U is an invariant measure which concentrates on infinite configurations. In order to apply Theorem B of Liggett we simply have to show that U is translation invariant. For any x, it is clear that $\gamma_{t_n}^x$ must tend in distribution to U^x, the translation of U by x. To show U is invariant we must show that, for arbitrary x, $\mathrm{U} = \mathrm{U}^{\mathrm{x}}$.

We prove that $U \geq U^x$. It is elementary that, for any $s > 0$, the distribution of $\gamma_{t_n - s}^0$ conditioned on $\tau > t_n - s$ converges to U. We introduce the sequence of stopping times $D_0, S_1, D_1, S_2, \ldots$ defined by

1. $D_0 = 0$,
2. $S_k = \inf\{t > D_{k-1} : \gamma_t^0(x) = 1\}$,
3. for $k > 0$, $D_k = \inf\{t > S_k : \gamma_t^{x,S_k} = \emptyset\}$.

It is important to realize that the events $\{\tau = \infty\}$ and $\{\exists r : S_r < \infty, D_r = \infty\}$, are a.s. equal by Corollary 2.2. Let f be an increasing function with $f(\emptyset) = 0$. Then, since $\gamma_t^0 \geq \gamma_t^{x,S_r}$ on $\{S_r < t < D_r\}$,

$$E[f(\gamma_{t_n}^0)] \geq \sum_r E[f(\gamma_{t_n}^{x,S_r}) I_{S_r < t_n < D_r}].$$

As n tends to infinity, the left hand side, by hypothesis, tends to $P[\tau = \infty]\langle \mathrm{U}, \mathrm{f}\rangle$, while the right hand side converges to $P[\tau = \infty]\langle U^x, f\rangle$. Thus $\mathrm{U} \geq \mathrm{U}^{\mathrm{x}}$, but the roles of U and U^x can be reversed in this argument and we conclude that $\mathrm{U} = \mathrm{U}^{\mathrm{x}}$. This implies (by the arbitrariness of x) that U is translation invariant. Theorem B implies that U must equal $\mathrm{Ren}(\beta)$. Because the space of measures on $\{0,1\}^Z$ is compact and the sequence $\{t_n\}$ is arbitrary we conclude that γ_t^0 converges in distribution to $P[\tau < \infty]\delta_{\mathbf{0}} + P[\tau = \infty]\,\mathrm{Ren}(\beta)$. This completes the Proof of Proposition 3.2.

We now return to the Proof of Proposition 3.1.

PROOF OF PROPOSITION 3.1. We first consider a NPS with γ_0 finite. It will be sufficient to prove that for any continuous increasing function f with $f(\mathbf{0}) = 0$, $E^{\gamma_0}[f(\gamma_t)]$ converges as t tends to infinity to $P[\tau = \infty]\langle \mathrm{Ren}(\beta), f\rangle$. First Lemma 3.2 states that for any γ (finite or not)

$$\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] \leq \overline{\lim_{t\to\infty}} E^{\mathbf{1}}[f(\gamma_t)] = \langle \mathrm{Ren}(\beta), f\rangle.$$

Therefore, for any n,

$$\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] = E^{\gamma_0}\Big[\overline{\lim_{t\to\infty}} E^{\gamma_n}[f(\gamma_t)]\Big] \leq P[\tau > n]\langle \mathrm{Ren}(\beta), f\rangle.$$

The time n can be arbitrarily large, so we can conclude that

$$\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] \leq P[\tau = \infty]\langle \mathrm{Ren}(\beta) f, \nu\rangle.$$

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It remains to show the reversed inequality. We do this by mimicking the proof of the translation invariance of U in Proposition 3.1 and then using this result. Redefine the stopping times

1. $D_0 = 0$,
2. $S_k = \inf\{t > D_{k-1} : \gamma_t(0) = 1\}$,
3. for $k > 0$, $D_k = \inf\{t > S_k : \gamma_t^{0,S_k} = \mathbf{0}\}$.

Again (by Corollary 2.2) we note that the events $\{\tau = \infty\}$ and the event $\{r : S_r < \infty, D_r = \infty\}$ are a.s. equal. As before

$$E^{\gamma_0}[f(\gamma_t)] \geq \sum_r E[f(\gamma_{t_n}^{0,S_r}) I_{S_r < t_n < D_r}].$$

This latter term converges to $P[\tau = \infty]\langle \mathrm{Ren}(\beta), f\rangle$ as t tends to infinity, so Proposition 3.1 is proven for NPS with γ_0 finite. To complete the proof for arbitrary γ_0 it remains to treat the case where γ_0 is infinite.

If γ_0 is infinite let γ_0^n be the finite configuration given by

$$\gamma_0^n(y) = \begin{cases} \gamma_0(y) & \text{if } |y| < n, \\ 0 & \text{otherwise.} \end{cases}$$

Given f, increasing, continuous and 0 on $\mathbf{0}$, attractiveness of our NPS yields

$$\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] \geq \overline{\lim_{t\to\infty}} E^{\gamma_0^n}[f(\gamma_t)] = P^{\gamma_0^n}[\tau = \infty]\langle \mathrm{Ren}(\beta), f\rangle.$$

But $P^{\gamma_0^n}[\tau = \infty]$ converges to one as n tends to infinity. We conclude that $\overline{\lim_{t\to\infty}} E^{\gamma_0}[f(\gamma_t)] \geq \langle \mathrm{Ren}(\beta), f\rangle$. That the converse inequality holds is guaranteed by Lemma 3.2, and the proof is complete. ∎

Section 4. We have so far dealt with attractive, reversible NPS with $\frac{\beta(n)}{\beta(n+1)}$ decreasing to some constant strictly greater than one. Throughout this section we assume that $\frac{\beta(n)}{\beta(n+1)}$ decreases to one and $\sum_l \beta(l) > 1$. Let γ be a corresponding NPS. We may choose $\theta < 1$, so that if $\bar{\beta}(l) = \theta^l \beta(l)$ for each l, then $\sum_l \bar{\beta}(l) > 1$. So the NPS, γ, corresponding to $\bar{\beta}$ satisfies the conclusions of Proposition 3.1. It should be noted that $\mathrm{Ren}(\bar{\beta}) = \mathrm{Ren}(\beta)$.

We define the process $\bar{\gamma}_t^n$ so that

1. $\bar{\gamma}_t^n = \gamma_t$ for $t \leq n$.
2. $\bar{\gamma}_{n+s}^n$ is a NPS corresponding to $\bar{\beta}$, which is conditionally independent of F_n given γ_n.
3. $\gamma_{n+s} \geq \bar{\gamma}_{n+s}^n$ for all n.

We may assert the existence of a $\bar{\gamma}^n$ since our processes are attractive and given a configuration, the flip rates of vacant sites will be larger for the γ process than the $\bar{\gamma}^n$ process.

As before it is sufficient to show that for any f increasing, continuous and zero on $\mathbf{0}$, we must have $\lim_{t\to\infty} E^{\gamma_0}[f(\gamma_t)] \geq P[\tau = \infty]\langle \mathrm{Ren}(\beta), f\rangle$. Let f be such a function. Then for any n

$$\lim_{t\to\infty} E^{\gamma_0}[f(\gamma_t)] \geq \lim_{t\to\infty} E^{\gamma_0}[f(\bar{\gamma}_t^n)].$$

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Proposition 3.1 may be applied to $\bar{\beta}$ NPS. Therefore the right hand side of the above inequality is equal to $\bar{P}^n[\tau = \infty]\langle \mathrm{Ren}(\beta), f\rangle$, where $\bar{P}^n$ refers to the probability for events defined by $\bar{\gamma}^n$ and so in this context τ is the hitting time of $\mathbf{0}$ by $\bar{\gamma}^n$. However, $\bar{P}^n[\tau = \infty]$ converges to $P[\tau = \infty]$ as n tends to infinity, since if the process γ_t never hits the empty set, then $|\gamma_n|$ tends to infinity as n tends to infinity. This completes the proof of our main Theorem. ∎

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