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ON THE ERGODIC HILBERT TRANSFORM FOR OPERATORS IN L_p, $1 < p < \infty$

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Dedicated to Professor Shigeki Yano on his retirement from Tokyo Metropolitan University

ABSTRACT. In this paper the ergodic Hilbert transform is investigated at the operator theoretic level. Let T be an invertible positive operator on $L_p = L_p(X, \mathcal{F}, \mu)$ for some fixed p, $1 < p < \infty$, such that $\sup\{\|T^n\|_p: -\infty < n < \infty\} < \infty$. It is proved that the limit

$$Hf(x) = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f(x)$$

exists almost everywhere and in the strong operator topology, where the prime denotes that the term with zero denominator is omitted. Related results are also proved.

1. **Introduction and the ergodic Hilbert transform**. In [7], Cotlar studied the ergodic Hilbert transform for invertible measure-preserving point transformations on a measure space $(X, \mathcal{F}, \mu)$. (See also Calderón [3] and Petersen [12].) Recently, Campbell [4] (see also [5]) gave a direct new proof of Cotlar's results and then generalized these to invertible positive isometries on L_p by using ideas of A. Ionescu-Tulcea [10]. In this paper, however, we shall apply ideas of de la Torre [13] and generalize Campbell's results to the power-bounded operator case. In the proof below we use a known fact from the theory of classical discrete Hilbert transform.

The following theorem is basic.

THEOREM 1. *Let T be an invertible positive operator on L_p, $1 < p < \infty$, such that $\sup\{\|T^n\|_p: -\infty < n < \infty\} = M < \infty$. Define the ergodic maximal Hilbert transform H^*, associated with T, as*

$$H^*f(x) = \sup_{n\geq 1} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f(x) \right|.$$

Then there exists a constant $C > 0$, depending only on M, such that $\|H^*f\|_p \leq C\|f\|_p$ for all $f \in L_p$.

PROOF. For an integer $K \geq 1$, define the truncated maximal operator H_K^* as

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$$H_K^* f(x) = \max_{1 \le n \le K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f(x) \right|.$$

If $i \geq 1$, then, since T^i is positive, it follows that

$$\left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^{k+i} f \right| \leq T^i \left(\left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f \right| \right) \leq T^i H_K^* f,$$

and therefore $H_K^* T^i f(x) \leq T^i H_K^* f(x)$ a.e. on X; replacing f by $T^{-i} f$ we get

$$H_K^* f(x) \leq T^i H_K^* T^{-i} f(x) \quad \text{a.e. on } X.$$

This implies

$$\begin{aligned} \int |H_K^* f|^p \, d\mu &\leq \frac{1}{L} \sum_{i=1}^{L} \int |T^i H_K^* T^{-i} f|^p \, d\mu \\ &\leq \frac{M^p}{L} \sum_{i=1}^{L} \int |H_K^* T^{-i} f|^p \, d\mu. \end{aligned}$$

On the other hand, by a known result about the classical discrete Hilbert transform (see e.g. [9] or [6]) we have for almost all $x \in X$

$$\begin{aligned} \sum_{i=1}^{L} [H_K^* T^{-i} f(x)]^p &= \sum_{i=1}^{L} \left[\max_{1 \le n \le K} \left| \sum_{k=-n}^{n}{}' \frac{1}{k} T^{k-i} f(x) \right| \right]^p \\ &\leq A^p \sum_{i=-L-K}^{K} |T^i f(x)|^p, \end{aligned}$$

where A is an absolute constant depending only on p. It follows that

$$\begin{aligned} \|H_K^* f\|_p^p &\leq \frac{M^p}{L} A^p \sum_{i=-L-K}^{K} \int |T^i f(x)|^p \, d\mu \\ &\leq \frac{2(L+K)}{L} M^{2p} A^p \|f\|_p^p; \end{aligned}$$

letting $L \to \infty$, we get $\|H_K^* f\|_p \leq 2M^2 A \|f\|_p$. This inequality establishes the theorem, because $\lim_{K\to\infty} H_K^* f(x) = H^* f(x)$ a.e. on X. (This proof was inspired by a paper [13] of de la Torre.)

THEOREM 2. *Let T be an invertible positive operator on L_p, $1 < p < \infty$, such that* $\sup \{\|T^n\|_p : -\infty < n < \infty\} < \infty$. *Then the limit*

$$Hf(x) = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{1}{k} T^k f(x)$$

exists a.e. on X and in the strong operator topology.

PROOF. If $Hf(x)$ exists a.e. on X, then, by Theorem 1 together with Lebesgue's convergence theorem, $\sum_{k=-n}^{\prime n} (1/k) T^k f$ converges strongly to Hf. And to prove the a.e. existence of $Hf(x)$, we first remark that $\{h + (g - Tg) : h = Th, g \in L_p\}$ is a dense subspace of L_p. This is due to a mean ergodic theorem (see e.g. [8], p. 662). By Theorem 1 and a standard approximation argument, it then suffices to prove that $H(g - Tg)(x)$ exists a.e. on X. To do so, we use the equation

$$\sum_{k=-n}^{n}{}' \frac{1}{k} T^k(g - Tg) = g + Tg - \frac{1}{n}(T^{n+1}g + T^{-n}g) - \sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1}\right)(T^{i+1}g + T^{-i}g).$$

Since

$$\sum_{i=1}^{n-1} \left(\frac{1}{i} - \frac{1}{i+1}\right) |T^{i+1}g + T^{-i}g| \leq \sum_{i=1}^{\infty} (1/i^2)(|T^{i+1}g| + |T^{-i}g|)$$

and

$$\left\| \sum_{i=1}^{\infty} (1/i^2)(|T^{i+1}g| + |T^{-i}g|) \right\|_p \leq 2M \|g\|_p \sum_{i=1}^{\infty} (1/i^2) < \infty,$$

it follows that

$$\sum_{i=1}^{\infty} \left(\frac{1}{i} - \frac{1}{i+1}\right) |T^{i+1}g(x) + T^{-i}g(x)| < \infty$$

for almost all $x \in X$. Similarly, from the inequalities

$$\left\| \sum_{n=1}^{\infty} (1/n^p)(|T^{n+1}g|^p + |T^{-n}g|^p) \right\|_1 \leq 2M^p \|g\|_p^p \sum_{n=1}^{\infty} (1/n^p) < \infty$$

it follows that

$$\sum_{n=1}^{\infty} (1/n^p)(|T^{n+1}g(x)|^p + |T^{-n}g(x)|^p) < \infty$$

for almost all $x \in X$; consequently

$$\lim_{n \to \infty} \frac{1}{n}[T^{n+1}g(x) + T^{-n}g(x)] = 0$$

for almost all $x \in X$. These observations establish the a.e. existence of $H(g - Tg)(x)$, completing the proof of Theorem 2.

As an immediate consequence of Theorem 2 we have the following corollary. We note that much of the corollary is included in Campbell [4]; in particular, it follows from [4] that the limit in the corollary exists in the strong operator topology without the assumption of positivity of T.

COROLLARY. *Let T be a positive operator on L_2 such that $\|T\|_2 \leq 1$. Then the limit*

$$\lim_{n \to \infty} \sum_{k=1}^{n} \frac{1}{k} [T^k f(x) - T^{*k} f(x)]$$

exists a.e. on X and in the strong operator topology, where T^* denotes the adjoint operator of T.

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PROOF. By a dilation theorem (see e.g. [1] or [2]) there is another measure space $(Y, \mathcal{B}, \lambda)$ and an invertible positive isometry U from $L_2(Y)$ onto itself so that

$$DT^n = PU^nD \quad \text{for all} \quad n \geq 0,$$

where D is a positive isometry from $L_2(X)$ into $L_2(Y)$ and P is a positive self-adjoint projection from $L_2(Y)$ onto $DL_2(X)$ $(\subset L_2(Y))$. It is directly seen that $DT^{*n} = PU^{-n}D$ for all $n \geq 0$. It follows that

$$D\Big[\sum_{k=1}^{n} \frac{1}{k}(T^k - T^{*k})\Big] = P\Big[\sum_{k=-n}^{n}{}' \frac{1}{k} U^k\Big]D.$$

Thus if we let, for any $f \in L_2(X)$,

$$H_U^*(Df)(y) = \sup_{n \geq 1}\Big|\sum_{k=-n}^{n}{}' \frac{1}{k} U^k(Df)(y)\Big| \quad (y \in Y)$$

and

$$H_U(Df)(y) = \lim_{n\to\infty} \sum_{k=-n}^{n}{}' \frac{1}{k} U^k(Df)(y) \quad (y \in Y),$$

then, by Theorems 1 and 2, we see that $\|H_U^*(Df)\|_2 \leq C\|Df\|_2 < \infty$ and that $H_U(Df)(y)$ exists for almost all $y \in Y$. Therefore if we put

$$F_n(y) = \sup_{i \geq n}\Big|H_U(Df)(y) - \sum_{k=-i}^{i}{}' \frac{1}{k} U^k(Df)(y)\Big|,$$

then $F_1 \geq F_2 \geq \cdots \geq 0$ and $\lim_{n\to\infty}\|F_n\|_2 = 0$, so that $PF_1 \geq PF_2 \geq \cdots \geq 0$ and $\lim_{n\to\infty}\|PF_n\|_2 = 0$. Since

$$\Big|PH_UDf - P\Big[\sum_{k=-n}^{n}{}' \frac{1}{k} U^k\Big]Df\Big| \leq PF_n,$$

it follows that

$$\Big|D^{-1}PH_UDf - \sum_{k=1}^{n} \frac{1}{k}(T^kf - T^{*k}f)\Big| \leq D^{-1}\Big|PH_UDf - P\Big[\sum_{k=-n}^{n}{}' \frac{1}{k} U^k\Big]Df\Big|$$
$$\leq D^{-1}PF_n;$$

this completes the proof, because $D^{-1}PF_1 \geq D^{-1}PF_2 \geq \cdots \geq 0$ a.e. on X and $\lim_{n\to\infty}\|D^{-1}PF_n\|_2 = 0$.

2. **The continuous parameter case**.

THEOREM 3. *Let $\{T_t: -\infty < t < \infty\}$ be a strongly continuous one-parameter group of operators on L_p, $1 < p < \infty$, such that* $\sup\{\|T_t\|_p: -\infty < t < \infty\} < \infty$ *and $T_t \geq 0$ for all $t \geq 0$. Then:*

(1) *if*

$$H^{\sim}f = \sup_{0<\epsilon<G}\Big|\int_{\epsilon<|t|<G} \frac{T_tf}{t}\,dt\Big|,$$

there is a constant $C > 0$ such that

$$\|H^{\sim}f\|_p \leq C\|f\|_p \quad \textit{for all} \quad f \in L_p;$$

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(2) *the limit*

$$\lim_{\substack{\epsilon \to +0 \\ G \to \infty}} \int_{\epsilon < |t| < G} \frac{T_t f}{t} \, dt$$

exists a.e. on X and in the strong operator topology.

PROOF. (1) follows from Theorem 1 by a standard approximation argument (see e.g. [8], p. 691), and (2) follows from (1) as in Theorem 2. We omit the details.

In conclusion we note that the assumption of $T_t \geq 0$ for all $t \geq 0$ does not necessarily imply $T_t \geq 0$ for all $-\infty < t < \infty$. In fact, let $L_p = L_p(dx)$ where dx denotes the Lebesgue measure on the real line $(-\infty, \infty)$ and let, for $t \geq 0$ and $f \in L_p$,

$$T_t f(x) = f(t + x) + \delta f(t + 1 + x)\chi_{[-t-1,-1)}(x)$$

where δ is any constant with $0 < \delta < 1$. It is easily seen that $T_t \geq 0$, $\|T_t\|_p \leq 1 + \delta$, $T_t T_s = T_{t+s}$ for all $t, s \geq 0$, and that T_t converges strongly to the identity operator $I = T_0$ as $t \to +0$. A simple computation shows that T_t^{-1} exists but it is not positive, and that $\|T_t^{-1}\|_p \leq (1 - \delta)^{-1}$. Therefore, by setting $T_{-t} = T_t^{-1}$ for $t \geq 0$, we have given an example. (This example is a modification of Example 3.1 in Kan [11].)

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