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THE THIRD HANKEL DETERMINANT FOR INVERSE COEFFICIENTS OF CONVEX FUNCTIONS

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Abstract

The sharp bound for the third Hankel determinant for the coefficients of the inverse function of convex functions is obtained, thus answering a recent conjecture concerning invariance of coefficient functionals for convex functions.

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1. Introduction

Let $\mathcal{A}$ denote the class of normalised analytic functions f in the open unit disc $\mathbb{D} := \{z : |z| < 1,\ z \in \mathbb{C}\}$ with Taylor expansion

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D} \tag{1.1}$$

and let $\mathcal{S}$ denote the subclass of functions in $\mathcal{A}$ which are univalent in $\mathbb{D}$. The classes $\mathcal{S}^*$ of starlike functions and $\mathcal{C}$ of convex functions are subclasses of $\mathcal{S}$ and are analytically defined respectively as

$$\mathcal{S}^* = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(\frac{zf'(z)}{f(z)}\right) > 0,\ z \in \mathbb{D} \right\},$$
$$\mathcal{C} = \left\{ f \in \mathcal{A} : \mathrm{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0,\ z \in \mathbb{D} \right\}. \tag{1.2}$$

For any univalent function f, there exists an inverse function f^{-1} defined on some disc $|w| \le r_0(f)$, with Taylor series expansion

$$f^{-1}(w) = w + A_2 w^2 + A_3 w^3 + \cdots . \tag{1.3}$$

The qth Hankel determinant for analytic functions $f \in \mathcal{A}$ was introduced by Pommerenke [11] and is given by

$$H_q(n)(f) := \begin{vmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \dots & a_{n+q} \\ \vdots & \vdots & \dots & \vdots \\ a_{n+q-1} & a_{n+q} & \dots & a_{n+2q-2} \end{vmatrix},$$

where $n \geq 1$ and $q \geq 1$. Many authors have studied the Hankel determinants $H_2(2)(f) = a_2a_4 - a_3^2$ and $H_2(3)(f) = a_3a_5 - a_4^2$ of order 2 (see, for example, [2, 3, 9, 12, 14, 15]), whereas $H_2(1)(f) = a_2^2 - a_3$ is classical. Sharp bounds for

$$|H_3(1)(f)| = |2a_2a_3a_4 - a_3^3 - a_4^2 + a_3a_5 - a_2^2a_5| \tag{1.4}$$

are more difficult to find, but some notable recent results have been found, for example, [1, 4, 5, 7, 8, 13]. In particular, Kowalczyk *et al.* [5] obtained the sharp bound for $|H_3(1)(f)|$ for $f \in C$ by showing that

$$|H_3(1)(f)| \leq \tfrac{4}{135}.$$

In this paper, we find the sharp bound for $|H_3(1)(f^{-1})|$ for the inverse function when $f \in C$. Our result demonstrates a noninvariance property for $f \in C$ between corresponding functionals discussed in [5, 16], thus settling a conjecture made in [5].

Let $\mathcal{P}$ denote the class of analytic functions p defined for $z \in \mathbb{D}$ by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \tag{1.5}$$

with positive real part in $\mathbb{D}$. We will use the following lemma concerning the coefficients of functions in $\mathcal{P}$.

LEMMA 1.1 [6, 10]. *Let $p \in \mathcal{P}$ be given by (1.5) with $c_1 > 0$. Then,*

$$2c_2 = c_1^2 + \delta(4 - c_1^2), \tag{1.6}$$

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1\delta - (4 - c_1^2)c_1\delta^2 + 2(4 - c_1^2)(1 - |\delta|^2)\eta, \tag{1.7}$$

$$\begin{aligned} 8c_4 = c_1^4 &+ (4 - c_1^2)\delta(c_1^2(\delta^2 - 3\delta + 3) + 4\delta) - 4(4 - c_1^2)(1 - |\delta|^2)(c_1(\delta - 1)\eta \\ &+ \bar{\delta}\eta^2 - (1 - |\eta|^2)\rho), \end{aligned} \tag{1.8}$$

for some δ, η and ρ such that $|\delta| \leq 1$, $|\eta| \leq 1$ and $|\rho| \leq 1$.

2. Main result

THEOREM 2.1. *Let $f \in C$ be given by (1.1). Then,*

$$|H_3(1)(f^{-1})| \leq \tfrac{1}{36}.$$

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The inequality is sharp for the function $f_0 \in C$ given by

$$f_0(z) = \int_0^z \frac{dx}{(1-x^3)^{2/3}} = {}_2F_1\Big(\frac{1}{3}, \frac{2}{3}; \frac{4}{3}; z^3\Big) = z + \frac{1}{6}z^4 + \frac{5}{63}z^7 + \cdots . \tag{2.1}$$

PROOF. Let $f \in C$. Then from (1.2), there exists a function $p \in \mathcal{P}$ such that

$$1 + \frac{zf''(z)}{f'(z)} = p(z). \tag{2.2}$$

Since

$$\begin{aligned} 1 + \frac{zf''(z)}{f'(z)} &= 1 + 2a_2z + (6a_3 - 4a_2^2)z^2 + (12a_4 - 18a_2a_3 + 8a_2^3)z^3 \\ &\quad + (20a_5 - 32a_2a_4 - 18a_3^2 + 48a_3a_2^2 - 16a_2^4)z^4 + \cdots , \end{aligned} \tag{2.3}$$

we obtain by comparing coefficients from (1.5), (2.2) and (2.3),

$$a_2 = \tfrac{1}{2}c_1, \tag{2.4}$$

$$a_3 = -\tfrac{1}{6}c_2 + \tfrac{1}{6}c_1^2, \tag{2.5}$$

$$a_4 = \tfrac{1}{24}c_1^3 + \tfrac{1}{8}c_1c_2 + \tfrac{1}{12}c_3, \tag{2.6}$$

$$a_5 = \tfrac{1}{120}c_1^4 + \tfrac{1}{20}c_2c_1^2 + \tfrac{1}{15}c_1c_3 + \tfrac{1}{40}c_2^2 + \tfrac{1}{20}c_4. \tag{2.7}$$

Next note that since $f(f^{-1}(w)) = w$, using (1.3), it follows that

$$A_2 = -a_2, \tag{2.8}$$

$$A_3 = 2a_2^2 - a_3, \tag{2.9}$$

$$A_4 = 5a_2a_3 - 5a_2^3 - a_4, \tag{2.10}$$

$$A_5 = 14a_2^4 - 21a_3a_2^2 + 6a_2a_4 + 3a_3^2 - a_5. \tag{2.11}$$

Substituting (2.4)–(2.7) into (2.8)–(2.11), respectively, we obtain

$$A_2 = -\tfrac{1}{2}c_1, \tag{2.12}$$

$$A_3 = \tfrac{1}{3}c_1^2 - \tfrac{1}{6}c_2, \tag{2.13}$$

$$A_4 = -\tfrac{1}{4}c^3 + \tfrac{7}{24}c_1c_2 - \tfrac{1}{12}c_3, \tag{2.14}$$

$$A_5 = \tfrac{1}{5}c^4 - \tfrac{23}{60}c_1^2c_2 + \tfrac{11}{60}c_1c_3 + \tfrac{7}{120}c_2^2 - \tfrac{1}{20}c_4. \tag{2.15}$$

Using (2.12)–(2.15) in (1.4),

$$\begin{aligned} H_3(1)(f^{-1}) = \tfrac{1}{34560}(&16c_1^6 - 96c_1^4c_2 + 48c_1^3c_3 + 156c_1^2c_2^2 - 144c_1^2c_4 \\ &+ 144c_1c_2c_3 - 176c_2^3 + 288c_2c_4 - 240c_3^2), \end{aligned}$$

which after simplification, noting that since both the class C and the functional $H_3(1)(f^{-1})$ are rotationally invariant (so that $c \in [0, 2]$), and using (1.6)–(1.8) gives

$$H_3(1)(f^{-1}) = \tfrac{1}{34560}(v_1(c, \delta) + v_2(c, \delta)\eta + v_3(c, \delta)\eta^2 + \psi(c, \delta, \eta)\rho),$$

where $\delta,\ \eta, \rho \in \overline{\mathbb{D}}$ and

$$\begin{aligned} v_1(c,\delta) &:= \delta^2(4-c^2)^2(-16\delta + 10c^2\delta + 3c^2 + 3c^2\delta^2),\\ v_2(c,\delta) &:= -12\delta(4-c^2)^2(1-|\delta|^2)c(\delta+1),\\ v_3(c,\delta) &:= -12(4-c^2)^2(1-|\delta|^2)(5+\delta^2),\\ \psi(c,\delta,\eta) &:= 72\delta(4-c^2)^2(1-|\delta|^2)(1-|\eta|^2). \end{aligned}$$

Next, using $|\delta| = x, |\eta| = y$ and the fact $|\rho| \le 1$, we obtain

$$\begin{aligned} H_3(1)(f^{-1}) &\le \tfrac{1}{34560}(|v_1(c,x)| + |v_2(c,x)|y + |v_3(c,x)|y^2 + |\psi(c,x,y)|)\\ &\le G(c,x,y), \end{aligned}$$

where

$$G(c,x,y) := \tfrac{1}{34560}(g_1(c,x) + g_2(c,x)y + g_3(c,x)y^2 + g_4(c,x)(1-y^2)),$$

with

$$\begin{aligned} g_1(c,x) &:= x^2(4-c^2)^2(16x + 10c^2x + 3c^2 + 3c^2x^2),\\ g_2(c,x) &:= 12x(4-c^2)^2(1-x^2)c(x+1),\\ g_3(c,x) &:= 12(4-c^2)^2(1-x^2)(5+x^2),\\ g_4(c,x) &:= 72x(4-c^2)^2(1-x^2). \end{aligned}$$

Thus, we need to maximise $G(c,x,y)$ over the closed cuboid $\Lambda : [0,2]\times[0,1]\times[0,1]$. To do this, we find the maximum values in the interior of Λ, in the interior of the six faces, at the vertices and on the 12 edges.

I. The interior of Λ. Let $(c,x,y) \in (0,2)\times(0,1)\times(0,1)$. We can write

$$34560\, G(c,x,y) = g_1(c,x) + g_4(c,x) + g_2(c,x)y + (g_3(c,x) - g_4(c,x))y^2.$$

Since $g_2(c,x) > 0$ and

$$(g_3(c,x) - g_4(c,x)) = 12(4-c^2)(1-x^2) > 0,$$

for $c \in (0,2), x \in (0,1)$,

$$G(c,x,y) < G(c,x,1) := \tfrac{1}{34560}(g_1(c,x) + g_2(c,x) + g_3(c,x))$$

for $c \in (0,2), x \in (0,1)$ and $y < 1$. Thus, $G(c,x,y)$ does not have a maximum for $(c,x,y) \in (0,2)\times(0,1)\times[0,1)$.

II. The interiors of the faces of Λ. We deal with each face in turn, noting that we do not need to consider the face $y = 0$ after Step I.

On the face $c = 0$, $G(c,x,y)$ reduces to

$$k_1(x,y) := G(0,x,y) = \frac{3(1-x^2)(x-1)(x-5)y^2 - 2x(7x^2-9)}{540}, \quad x,y \in (0,1),$$

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and we see that k_1 has no critical points in $(0,1)\times(0,1)$ since

$$\frac{\partial k_1}{\partial y} = \frac{(1-x^2)(x-1)(x-5)y}{90} \neq 0, \quad x, y \in (0,1).$$

On the face $c = 2$, $G(c, x, y)$ reduces to

$$G(2, x, y) = 0, \quad x, y \in (0,1).$$

On the face $x = 0$, $G(c, x, y)$ reduces to

$$k_2(c, y) := G(c, 0, y) = \frac{(4-c^2)^2 y^2}{576}, \quad c \in (0,2), y \in (0,1),$$

and we see that k_2 has no critical point in $(0,2)\times(0,1)$ since

$$\frac{\partial k_2}{\partial y} = \frac{(4-c^2)^2 y}{288} \neq 0, \quad c \in (0,2), y \in (0,1).$$

On the face $x = 1$, $G(c, x, y)$ reduces to

$$k_3(c, y) := G(c, 1, y) = \frac{c^6 - 7c^4 + 8c^2 + 16}{2160}, \quad c \in (0,2).$$

Solving $\partial k_3/\partial c = 0$, we obtain a maximum value $25/2916 < 1/36$ at $c := c_0 = \sqrt{6}/3$.

Finally, on the face $y = 1$, $G(c, x, y)$ reduces to

$$k_4(c, x) := G(c, x, 1) = \frac{(4-c^2)^2}{34560}\begin{pmatrix}16x^3 + 10x^3c^2 + 3x^2c^2 + 3x^4c^2 + 12x^2c \\ -12x^4c + 12xc - 12x^3c + 60 - 48x^2 - 12x^4\end{pmatrix}.$$

Differentiating $k_4(c, x)$ with respect to c and x, we obtain

$$\frac{\partial k_4}{\partial x} = \frac{(4-c^2)^2}{34560}\begin{pmatrix}(3c^2 - 12c - 12)x^4 + (10c^2 - 12c + 16)x^3 \\ +(12c - 48 + 3c^2)x^2 + 12xc + 60\end{pmatrix}$$

and

$$\frac{\partial k_4}{\partial c} = \frac{-4(4-c^2)}{34560}\begin{pmatrix}x^2(6x^2 + 20x + 6)c^5 + 12x(1-x^2)(x+1)c^4 \\ -15x^2(x+3)(3x+1)c^3 - 84x(1-x^2)(x+1)c^2 \\ +(336x^3 + 60 + 48x^2 + 84x^4)c + 192x(1-x^2)(x+1)\end{pmatrix}.$$

Now equating both partial derivatives to zero, a numerical calculation shows that the only real roots are at $(c, x) = (7.03237\ldots, -5.85791\ldots), (-0.95398, 1.04715\ldots)$ and $(c, x) = (0, 0)$. We see that $c = x = 0$ gives a maximum value $1/36$ and all other critical points are saddle points so there does not exist a maximum inside $(0,2)\times(0,1)$.

III. The vertices of Λ. We have

$$G(0,0,0) = 0, \quad G(0,0,1) = \tfrac{1}{36}, \quad G(0,1,0) = \tfrac{1}{135}, \quad G(0,1,1) = \tfrac{1}{135},$$
$$G(2,0,0) = G(2,0,1) = G(2,1,0) = G(2,1,1) = 0.$$

IV. The interiors of the edges of Λ. Finally, we find the points of the maxima of $G(c, x, y)$ on the 12 edges of Λ:

$$G(c,0,0) = 0;$$

$$G(c,0,1) = \frac{(4-c^2)^2}{576} \le G(0,0,1) = \frac{1}{36}, \quad c \in (0,2);$$

$$G(c,1,0) = \frac{c^6 - 7c^4 + 8c^2 + 16}{2160} \le G(\lambda_1,1,0) = \frac{25}{2916} < \frac{1}{36}, \quad c \in (0,2),$$

where

$$c := \lambda_1 = \frac{\sqrt{6}}{3};$$

$$G(0,x,0) = \frac{-x(7x^2-9)}{270} \le G\left(0, \frac{\sqrt{21}}{7}, 0\right) = \frac{\sqrt{21}}{315} < \frac{1}{36}, \quad x \in (0,1);$$

$$G(0,x,1) = \frac{-3x^4 + 4x^3 - 12x^2 + 15}{540} \le G(0,0,1) = \frac{1}{36}, \quad x \in (0,1);$$

$$G(2,x,0) = 0, \quad x \in (0,1);$$

$$G(2,x,1) = 0, \quad x \in (0,1);$$

$$G(0,0,y) = \tfrac{1}{36}y^2 \le \tfrac{1}{36}, \quad y \in (0,1);$$

$$G(0,1,y) = \tfrac{1}{135}, \quad y \in (0,1);$$

$$G(2,0,y) = 0, \quad y \in (0,1);$$

$$G(2,1,y) = 0, \quad y \in (0,1).$$

Thus, there is only one maximum point at $(0,0,1)$ where $G(0,0,1) = 1/36$. We therefore conclude that

$$|H_3(1)(f^{-1})| \le \tfrac{1}{36}.$$

The result is sharp for f_0 given in (2.1), which is equivalent to choosing $a_2 = a_3 = a_5 = 0$ and $a_4 = \frac{1}{6}$. From (2.8)–(2.11), we see that $A_2 = A_3 = A_5 = 0$ and $A_4 = \frac{1}{6}$, which from (1.4) gives $|H_3(1)(f^{-1})| = 1/36$. This completes the proof. □

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