Proceedings of the Edinburgh Mathematical Society (1989) **32**, 255–259 ©

A COUNTEREXAMPLE OF HERMITIAN LIFTINGS

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(Received 12th January 1988)

1. Introduction

Let X be a complex Banach space, and let $\mathscr{B}(X)$ and $\mathscr{C}(X)$ denote respectively the algebras of bounded and compact operators on X. The quotient algebra $\mathscr{A}(X)=\mathscr{B}(X)/\mathscr{C}(X)$ is called the *Calkin algebra* associated with X. It is known that both $\mathscr{B}(X)$ and $\mathscr{C}(X)$ are complex Banach algebras with unit e. For such unital Banach algebras B, set

$$S=\{f\in B^*: f(e)=1=\|f\|\}$$

and define the *numerical range* of $x\in B$ as

$$W(x)=\{f(x): f\in S\}.$$

x is said to be *hermitian* if $W(x)\subseteq \mathbf{R}$. It is known that

Fact 1. ([**4** vol. I, p. 46]) *x is hermitian if and only if $\|e^{i\alpha x}\|=(or\ \leqq)1$ for all $\alpha\in\mathbf{R}$, where e^x is defined by*

$$e^x=\sum_{n=0}^{\infty}\frac{x_n}{n!}.$$

It also known that if $B=\mathscr{B}(X)$, then $T\in B$ is hermitian if and only if T satisfies one of the following conditions [**4**, vol. I, p. 84 and §9]:

(1) *For any x with $\|x\|=1$, $f(Tx)\in\mathbf{R}$ if $f\in X^*$ and if $f(x)=1=\|f\|$.*

(2) *Let $[\cdot,\cdot]$ be any semi-inner product compatible with the norm on X (for definition see* [**4** vol. I, §9]). *$[Tx,x]\in\mathbf{R}$ for any $x\neq 0$.*

An operator $T\in\mathscr{B}(X)$ is said to be *essentially hermitian* if $T+\mathscr{C}(X)$ is hermitian in $\mathscr{A}(X)$. Allen, Legg and Ward ([**1**, **2**, **3** and **7**]) have shown that in some classical Banach spaces,

*Research supported in part by NSF.

$$(*)\begin{cases}\text{every essentially hermitian operator is a compact perturbation}\\ \text{of hermitian operator i.e. if } T \text{ is essentially hermitian, then}\\ T+\mathscr{C}(X) \text{ contains a hermitian operator.}\end{cases}$$

On the other hand, Legg [6] has shown there is a Banach space (Orlicz sequence space) which is isomorphic to l_2, but it does not have the property $(*)$. Ward asked whether l_p, $1 \leqq p < \infty$, $p \neq 2$, can be renormed so that it does not have the property $(*)$. In this article, we show the answer is affirmative.

Part of this research was done when the author visited Texas A & M University. The author expresses his thanks to the Department of Mathematics for its kind hospitality. He also thanks Professor J. D. Ward for bringing his attention to this problem.

2. The example

Let E be a Banach space with a 1-unconditional basis $\{e_i\}$ such that for any $j \neq k$, $\operatorname{span}\{e_j, e_k\}$ is not isometrically isomorphic to l_2^2 (two dimensional Hilbert space). Let X_i be a sequence of Banach spaces. Then the direct sum $(\sum \oplus X_i)_E$ denotes the set

$$\{(x_i): x_i \in X_i \text{ and } \sum \|x_i\| e_i \in E\}$$

The norm of (x_i) is given by

$$\|(x_i)\| = \|\sum \|x_i\| e_i\|.$$

Fleming and Jamison proved that [5, Theorem 4.8(iii)] if T is a hermitian operator on $(\sum \oplus X_i)_E$, then $T(X_i) \subseteq X_i$ and $T|_{X_i}$ is hermitian for each i. On the other hand, if $T_i: X_i \to X_i$ is hermitian for each i, then for each $\alpha \in \mathbf{R}$ $\|e^{i\alpha T_i}\| = 1$. So for each $\alpha \in \mathbf{R}$, $\|\exp i\alpha(\sum \oplus T_i)\| = 1$ and $(\sum \oplus T_i)$ is hermitian. Let $p_i > 1$ be a strictly increasing sequence which converges to 2, and let X_i be $l_{p_i}^2$. Then every hermitian operator on $(\sum \oplus X_i)_E$ is diagonal. (Note: by the result of Flemming and Jamison, every hermitian operator on $l_p, p \neq 2$ is diagonal.) We claim that $(\sum \oplus X_i)_E$ does not have the property $(*)$. Indeed, let

$$T_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $T = (\sum \oplus T_i)$ is essentially hermitian, but it is not a compact perturbation of a hermitian (diagonal) operator. Before the proof, we need the following facts.

Fact 2. *Let $Y_i = l_2^2$. Then T is a hermitian operator on $(\sum \oplus Y_i)_E$. (Note: T_i is an hermitian operator on Y_i.)*

Fact 3. *Let X and Y be two Banach spaces. The* Banach–Mazur distance *between X and Y is defined as*

$$d(X,Y)=\inf\{\|S\|\cdot\|S^{-1}\|;\ S\colon X\to Y \text{ isomorphism}\}.$$

Let I_n denote the mapping

$$I_n(a,b)=(a,b)$$

from X_n onto Y_n. It is known that $\|I_n\|=1$. By the Hölder's inequality, $\|I_n^{-1}\|\leqq 2^{(2-p_n)/2p_n}$. So

$$d(X_n,Y_n)\leqq\|I_n\|\cdot\|I_n^{-1}\|2^{(2-p_n)/2p_n}$$

and

$$d\left(\left(\sum_{i=n}^{\infty}\oplus Y_i\right),\left(\sum_{i=n}^{\infty}\oplus X_i\right)\right)\leqq\left\|\left(\sum_{i=n}^{\infty}\oplus I_i\right)\right\|\cdot\left\|\left(\sum_{i=n}^{\infty}\oplus I_i\right)^{-1}\right\|=\|I_n\|\cdot\|(I_n)^{-1}\|\leqq 2^{(2-p_n)/2p_n}.$$

Fact 4. *Suppose that E is a Banach space with a 1-unconditional basis, and that $\{X_i\}$ is a sequence of finite dimensional Banach spaces. Let P_n denote the natural projection from $X=(\sum_{i=1}^{\infty}\oplus X_i)_E$ onto $(\sum_{i=n}^{\infty}\oplus X_i)$. We claim that for any operator S on X, the essential norm $\|S\|_e$ of S equals $\liminf_{n\to\infty}\|P_nSP_n\|(=\lim_{n\to\infty}\|P_nSP_n\|$ since $\|P_n\|\leqq 1)$.*

Since $I-P_n$ is a finite rank operator. $(I-P_n)S+P_nS(I-P_n)$ is a compact operator. So

$$\|S\|_e\leqq\liminf_{n\to\infty}\|S-(I-P_n)S-P_nS(I-P_n)\|=\lim_{n\to\infty}\|P_nSP_n\|.$$

Let K be a compact operator on X. It is known [9, p. 30, Proposition 1.e.2] that there is a sequence $\{x_n\}_{n=1}^{\infty}$ in X such that $\lim_{n\to\infty}\|x_n\|=0$ and $K(\{x\colon\|x\|\leqq 1\})\subseteq\overline{\text{conv}}\{x_n\}_{n=1}^{\infty}$. So for any $\varepsilon>0$, there exists M and N such that if $m>M\geqq k$ and $n>N$, then

$$\|x_m\|<\varepsilon \text{ and } \|P_n(x_k)\|<\varepsilon.$$

So if $a_i\geqq 0$ and $\sum_{i=1}^{k}a_i=1$, then for $n>N$,

$$\left\|P_n\left(\sum_{i=1}^{k}a_ix_i\right)\right\|\leqq\sum_{i=1}^{k}a_i\|P_n(x_i)\|\leqq\varepsilon.$$

This implies

$$\lim_{n\to\infty}\|P_n\circ K\|=0.$$

Therefore, for any compact operator K on X

$$\begin{aligned}\|S-K\|&\geqq\lim_{n\to\infty}\|P_n(S-K)P_n\|\\&\geqq\lim_{n\to\infty}\|P_nSP_n\|-\lim_{n\to\infty}\|P_nKP_n\|\end{aligned}$$

https://doi.org/10.1017/S0013091500028650 Published online by Cambridge University Press

$$= \lim_{n\to\infty} \|P_n S P_n\|.$$

Now, we show that T is essentially hermitian. By Fact 1, we only need to show $\|e^{i\alpha T}\|_e \leqq 1$ for any $\alpha \in \mathbf{R}$. Let $T^{(n)}$ be the operator on X which is defined by $T^{(n)}(x_i) = (z_i)$ where

$$z_i = \begin{cases} 0, & i < n \\ T_i(x_i), & i \geqq n. \end{cases}$$

It is easy to see that $P_n \circ T \circ P_n = T^{(n)}$ and $P_n \circ e^{i\alpha T} \circ P_n = e^{i\alpha T^{(n)}}$. By Fact 4, it is enough to show that for any $\varepsilon > 0$ and $N > 0$, there exists $n > N$ such that

$$\|P_n \circ e^{i\alpha T} \circ P_n\| = \|e^{i\alpha T^{(n)}}\| \leqq 1 + \varepsilon$$

for all $\alpha \in \mathbf{R}$.

It is known that $\|S\| = \sup \|S_n\|$ if $S = (\sum \oplus S_n)$. Since $T^{(n)}$ is a hermitian operator on $Y = (\sum_{i=1}^{\infty} \oplus Y_i)$, we have $\|T^{(n)}\|_Y = 1$ and

$$\begin{aligned}
\|e^{i\alpha T^{(n)}}\|_X &= \|e^{i\alpha T^{(n)}}|_{\sum_{i=n}^{\infty} X_i}\| \\
&= \left\| \left(\sum_{k=n}^{\infty} \oplus I_k\right)^{-1} (e^{i\alpha T^{(n)}}|_{\sum_{i=n}^{\infty} X_i}) \left(\sum_{k=n}^{\infty} \oplus I_k\right) \right\|_X \\
&\leqq \left\| \left(\sum_{k=n}^{\infty} \oplus I_k\right)^{-1} \right\| \cdot \|e^{i\alpha T^{(n)}}|_{\sum_{i=n}^{\infty} Y_i}\|_Y \cdot \left\| \left(\sum_{k=n}^{\infty} \oplus I_k\right) \right\| \\
&\leqq 2(2 - p_n)/2p_n.
\end{aligned}$$

So T is essentially hermitian.

Remark 1. If E is l_p, then $(\sum \oplus X_i)_E$ is isomorphic to l_p. Moreover, if $1 < p < \infty$, the $(\sum \oplus X_i)_E$ is uniformly convex and uniformly smooth.

Remark 2. Let X and Y be two complex Banach spaces with trivial L^2-*structure*, i.e. there do not exist two subspaces X_1 and X_2 (resp. Y_1 and Y_2) of X (resp. Y) such that X (resp. Y) is isometrically isomorphic to $(X_1 \oplus X_2)_2$ (resp. $(Y_1 \oplus Y_2)_2$). The author **[8]** show that if $\dim(X) > 1$ and $\dim(Y) > 1$, and if T is a hermitian operator on $(X \oplus Y)_2$, then $T(X) \subseteq X$ and $T(Y) \subseteq Y$. So the assumption that $\text{span}\{e_j, e_k\}$ is not isometrically isomorphic to l_2^2 is superfluous.

Remark 3. In our example and Legg's example **[6]**, for any $\varepsilon > 0$ the space X contains a two dimensional 1-complemented subspace Y such that $d(Y, l_2^2) < 1 + \varepsilon$. We do not know whether there is a uniformly convex complex Banach space without the property (*) and the above property.

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