THE IDEMPOTENT-SEPARATING CONGRUENCES ON A REGULAR 0-BISIMPLE SEMIGROUP

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A congruence ρ on a semigroup is said to be idempotent-separating if each ρ-class contains at most one idempotent. For any idempotent e of a semigroup S the set eSe is a subsemigroup of S with identity e and group of units H_e, the maximal subgroup of S containing e. The purpose of the present note is to show that if S is a regular 0-bisimple semigroup and e is a non-zero idempotent of S then there is a one-to-one correspondence between the idempotent-separating congruences on S and the subgroups N of H_e with the property that $aN \subseteq Na$ for all right units a of eSe and $Nb \subseteq bN$ for all left units b of eSe. Some special cases of this result are discussed and, in the final section, an application is made to the principal factors of the full transformation semigroup $\mathscr{T}_X$ on a set X.

1. The notation of (**1**) will be used throughout. In particular, if ρ is an equivalence on a set X then $x\rho$ denotes the ρ-class containing the element x of X. As in (**1**), an exception is made in the case of Green's equivalences $\mathscr{R}$, $\mathscr{L}$ and $\mathscr{H}$ on a semigroup S: the corresponding classes containing the element a of S are denoted by R_a, L_a and H_a.

Let S contain an idempotent e. Then eSe is a subsemigroup of S and we write

$$S_e = eSe,\ P_e = R_e \cap S_e,\ Q_e = L_e \cap S_e.$$

Since e is a left identity for R_e and a right identity for L_e we see that

$$P_e = \{x \in R_e\colon xe = x\},\ Q_e = \{x \in L_e\colon ex = x\}.$$

In the first two lemmas we establish some basic properties of these sets.

Lemma 1. *Let e and f be $\mathscr{D}$-equivalent idempotents of a semigroup S. Then* $S_e \cong S_f$.

Proof. Since $(e, f) \in \mathscr{D}$ there exists an element a in $R_e \cap L_f$. By ((**1**), Theorem 2.18), there is a unique element a' in $R_f \cap L_e$ such that $aa' = e$ and $a'a = f$. Let $x \in S_e$. Then

$$a'xa = a'exea = fa'xaf \in S_f.$$

Similarly, if $y \in S_f$ then $aya' \in S_e$. Now let $\theta\colon\ S_e \to S_f$ and $\phi\colon\ S_f \to S_e$ be defined by $x\theta = a'xa\,(x \in S_e)$ and $y\phi = aya'\,(y \in S_f)$. Then $x\theta\phi = exe = x$ for all $x \in S_e$. Similarly, $y\phi\theta = y$ for all $y \in S_f$. Hence θ and ϕ are mutually inverse bijections. Finally, $a'xya = (a'xa)(a'ya)$ for all $x, y \in S_e$ and so θ is an isomorphism.

In particular, if S is a regular 0-bisimple semigroup then $S_e \cong S_f$ for any non-zero idempotents e, f. In this case it can also be shown that S_e is 0-bisimple.

Let T be a semigroup with an identity e. By a *right unit* of T we mean an element a of T such that $ax = e$ for some x in T. The set of all right units of T is readily seen to be a right cancellative subsemigroup of T (called the *right unit subsemigroup of* T). Left units are defined in a similar way; the set of all such elements is a left cancellative subsemigroup of T (the left unit subsemigroup of T). The elements of T that are both right units and left units are called *units.*

Lemma 2. *Let e be an idempotent of a semigroup S. Then P_e is the right unit subsemigroup of S_e and Q_e is the left unit subsemigroup of S_e. The group of units of S_e is H_e.*

Proof. Let $a \in P_e$. Then there exists $x \in S^1$ such that $ax = e$. Write $y = exe$. Since $ae = a$ we have that $ay = axe = e^2 = e$. Hence a is a right unit of S_e. Conversely, let a be a right unit of S_e. Then there exists $y \in S_e$ such that $ay = e$. Also $ea = a$ and so $a \in R_e \cap S_e = P_e$. Thus P_e is the right unit subsemigroup of S_e. Similarly, Q_e is the left unit subsemigroup of S_e.

Finally, $P_e \cap Q_e = R_e \cap L_e \cap S_e = H_e$.

2. By a *left normal divisor of* P_e we shall mean a subgroup N of H_e such that $aN \subseteq Na$ for all $a \in P_e$. Since $H_e \subseteq P_e$ it is clear that, in particular, a left normal divisor of P_e is a normal subgroup of H_e. Similarly, by a *right normal divisor of* Q_e we mean a subgroup N of H_e such that $Nb \subseteq bN$ for all $b \in Q_e$. This terminology is due to Rees (**5**). In the next lemma we establish a connection between such subgroups of H_e and the congruences on S contained in $\mathscr{H}$.

Lemma 3. *Let e be an idempotent of a semigroup S. Let ρ be any congruence on S contained in $\mathscr{H}$ and let $N = e\rho$. Then N is a left normal divisor of P_e and a right normal divisor of Q_e.*

Proof. Since $\rho \cap (H_e \times H_e)$ is a congruence on H_e and $\rho \subseteq \mathscr{H}$ it follows that N is a normal subgroup of H_e. Let $a \in P_e$ and let $b \in a\rho$. Then since $\rho \subseteq \mathscr{H}$ there exists $x' \in S^1$ such that $x'a = b$. Write $x = x'e$; then $xa = x'ea = x'a = b$ and $xe = x$. Also since $a\rho \subseteq H_a \subseteq R_e$ there exists $y \in S^1$ such that $by = e$. Now $(xby, xay) \in \rho$ since ρ is a congruence on S; that is, $(x, e) \in \rho$. Thus $x \in N$. Hence, since $b = xa$, we have that $a\rho \subseteq Na$.

Next let $z \in N$. Then $(az, ae) \in \rho$. But $ae = a$; hence $aN \subseteq a\rho$. Combining these results we see that $aN \subseteq Na$.

In the same way we can show that $Nb \subseteq bN$ for all $b \in Q_e$.

Corollary. *If $\mathscr{H}$ is a congruence on S then H_e is a left normal divisor of P_e and a right normal divisor of Q_e.*

Does *every* subgroup of H_e that is both a left normal divisor of P_e and a right normal divisor of Q_e arise, as in Lemma 3, from a congruence contained in $\mathscr{H}$? This is answered for regular 0-bisimple semigroups in Lemma 6. As a first step we establish

Lemma 4. *Let e be an idempotent of a semigroup S.*

(i) *Let* $a, b \in R_e$. *Then*

$$(a, b) \in \mathscr{H} \Leftrightarrow xa = b \text{ for some } x \in H_e.$$

(ii) *Let N be a left normal divisor of* P_e. *Define a relation* ρ_R *on* R_e *by the rule that*

$$(a, b) \in \rho_R \Leftrightarrow xa = b \text{ for some } x \in N.$$

Then ρ_R *is an equivalence on* R_e *contained in* $\mathscr{H}$. *Further, if* $(a, b) \in \rho_R$ *then* $(ca, cb) \in \rho_R$ *for all* $c \in P_e$.

Proof. (i) Since $(e, a) \in \mathscr{R}$ we have $ea = a$ and so $H_e a = H_a$ ((**1**), Lemma 2.2). This gives the required result.

(ii) That ρ_R is an equivalence on R_e follows from the fact that N is a group whose identity e is a left identity for R_e. From (i) we see that $\rho_R \subseteq \mathscr{H}$.

Let $(a, b) \in \rho_R$ and let $c \in P_e$. Since $a \in R_e$ there exists $z \in S^1$ such that $az = e$. Hence $caz = ce = c$ and so $(ca, c) \in \mathscr{R}$; that is, $ca \in R_e$. Similarly, $cb \in R_e$. Now $xa = b$ for some $x \in N$ and $cN \subseteq Nc$, by hypothesis. Hence

$$cb = cxa = yca$$

for some $y \in N$, which shows that $(ca, cb) \in \rho_R$.

Dually, for any right normal divisor N of Q_e we define a relation ρ_L on L_e by the rule that

$$(a, b) \in \rho_L \Leftrightarrow ax = b \text{ for some } x \in N.$$

Then ρ_L is an equivalence on L_e contained in $\mathscr{H}$ and if $(a, b) \in \rho_L$ then $(ac, bc) \in \rho_L$ for all $c \in Q_e$.

3. In this section we restrict our attention to 0-bisimple semigroups. By ((**1**), Theorem 2.11) such a semigroup is regular if and only if it contains a non-zero idempotent.

Lemma 5. *Let S be a* 0-*bisimple semigroup and let a be an arbitrary but fixed non-zero element of S. Let* ρ, τ *be congruences on S contained in* $\mathscr{H}$. *Then*

(i) $\rho = \mathscr{H}$ *if and only if* $a\rho = H_a$;

(ii) $\rho \subseteq \tau$ *if and only if* $a\rho \subseteq a\tau$.

Proof. (i) Let $a\rho = H_a$. To establish (i) we need only prove that $\mathscr{H} \subseteq \rho$. First, $H_0 = 0\rho = 0$. Now let $(b, c) \in \mathscr{H}$, where $b \neq 0$, $c \neq 0$. Since S is 0-bisimple there exist elements s, s', t, t' in S^1 such that $b = sat$, $a = s'bt'$ and the mappings

$$x \rightarrow sxt \ (x \in H_a), \ y \rightarrow s'yt' \ (y \in H_b)$$

are mutually inverse bijections from H_a to H_b and from H_b to H_a respectively ((**1**), Theorem 2.3). Thus, since $(b, c) \in \mathscr{H}$, we see that $(s'bt', s'ct') \in \mathscr{H}$. Since $a = s'bt'$ and $a\rho = H_a$ it follows that $(a, s'ct') \in \rho$. But ρ is a congruence on S and so $(sat, ss'ct't) \in \rho$; that is, $(b, c) \in \rho$. Thus $\mathscr{H} \subseteq \rho$.

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(ii) It is clear that if $\rho \subseteq \tau$ then $a\rho \subseteq a\tau$. Suppose, conversely, that $a\rho \subseteq a\tau$. Let $(b, c) \in \rho$, where $b \neq 0$, $c \neq 0$. To prove (ii) it suffices to show that $(b, c) \in \tau$. As above, since $(a, b) \in \mathscr{D}$ there exist elements s, s', t, t' in S^1 such that $b = sat$, $a = s'bt'$ and the mappings $x \rightarrow sxt\,(x \in H_a)$, $y \rightarrow s'yt'\,(y \in H_b)$ are mutually inverse bijections from H_a to H_b and from H_b to H_a respectively. Since ρ is a congruence, $(s'bt', s'ct') \in \rho$; that is, $(a, s'ct') \in \rho$. But $a\rho \subseteq a\tau$ and so $(a, s'ct') \in \tau$. Hence $(sat, ss'ct't) \in \tau$ since τ is a congruence. But $(b, c) \in \mathscr{H}$ since $\rho \subseteq \mathscr{H}$. Thus $ss'ct't = c$. It follows that $(b, c) \in \tau$, as required.

In particular, from (ii) above, $\rho = \tau$ if and only if $a\rho = a\tau$ for any non-zero element a of S; that is, a congruence contained in $\mathscr{H}$ on a 0-bisimple semigroup is uniquely determined by any one of its non-zero classes.

We now come to the key result.

Lemma 6. *Let S be a regular 0-bisimple semigroup and let e be a non-zero idempotent of S. Let N be a subgroup of H_e that is both a left normal divisor of P_e and a right normal divisor of Q_e. Then there exists a congruence ρ on S contained in $\mathscr{H}$ and such that $e\rho = N$.*

Proof. We construct ρ by defining $\rho \cap (H \times H)$ for each non-zero $\mathscr{H}$-class H in terms of the equivalence ρ_R on R_e described in Lemma 4 (ii). The argument depends on several applications of the dual of ((**1**), Lemma 2.2) which we shall refer to below as Green's lemma.

Let $(a, b) \in \mathscr{H}$, where $a \neq 0$, $b \neq 0$. Since S is 0-bisimple there exists an element $s \in S^1$ such that $sa \in R_e \cap L_a$. Then $sb \in R_e \cap L_a$ by Green's lemma. Now let $(sa, sb) \in \rho_R$. We prove first that $(za, zb) \in \rho_R$ for *any* $z \in S^1$ such that $za \in R_e \cap L_a$. By Lemma 4 (i), since $(sa, za) \in \mathscr{H}$, there exists $x \in H_e$ such that $xsa = za$. Also since $(a, b) \in \mathscr{H}$, there exists $y \in S^1$ such that $ay = b$. Now $(xsa, xsb) \in \rho_R$ by Lemma 4 (ii). But

$$xsb = xsay = zay = zb.$$

Hence $(za, zb) \in \rho_R$.

Let H be a non-zero $\mathscr{H}$-class of S. We define a relation ρ_H on H by the rule that

$$(a, b) \in \rho_H \Leftrightarrow (sa, sb) \in \rho_R, \quad (a, b \in H) \tag{1}$$

where s is any element of S^1 such that $sa \in R_e \cap L_a$. Clearly ρ_H is reflexive and symmetric. To see that it is transitive, let $(a, b) \in \rho_H$ and $(b, c) \in \rho_H$. Then there exists $s \in S^1$ such that $sa \in R_e \cap L_a$ and $(sa, sb) \in \rho_R$. Since $sb \in R_e \cap L_a$ it follows that $(sb, sc) \in \rho_R$ and so $(sa, sc) \in \rho_R$ since ρ_R is transitive. Thus ρ_H is an equivalence on H.

The definition in (1) lacks left-right symmetry. We shall now show that we would arrive at the same equivalence on H by using the congruence ρ_L on L_e defined in the dual form of Lemma 4 (ii).

Again, let $(a, b) \in \mathscr{H}$, where $a \neq 0$ and $b \neq 0$, let $s \in S^1$ be such that $sa \in R_e \cap L_a$ and let $(sa, sb) \in \rho_R$. Since S is 0-bisimple there exists $t \in S^1$ such that $at \in R_a \cap L_e$. Then $bt \in R_a \cap L_e$. It will be sufficient to show that $(at, bt) \in \rho_L$. By the definition of ρ_R there exists $g \in N$ such that $gsa = sb$.

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Hence $gsat = sbt$. Now, by Green's lemma, the mapping

$$x \to sx \ (x \in R_a)$$

is an $\mathscr{L}$-class-preserving bijection from R_a to R_e and so, since $at \in R_a \cap L_e$, it follows that $sat \in R_e \cap L_e = H_e$. Hence, since N is normal in H_e, there exists $h \in N$ such that $g(sat) = (sat)h$. Thus

$$sath = sbt. \qquad (2)$$

Now since $(a, sa) \in \mathscr{L}$ there exists $s' \in S^1$ such that $s'sa = a$. But $(a, b) \in \mathscr{H}$ and so $s'sb = b$, by Green's lemma. Premultiplying both sides of (2) by s' we find that $ath = bt$. Hence $(at, bt) \in \rho_L$.

Next we define $\rho \subseteq S \times S$ to be $\rho^* \cup \{(0, 0)\}$, where ρ^* is the union of all the subsets ρ_H of $S \times S$ as H runs through all the non-zero $\mathscr{H}$-classes of S. Since ρ_H is an equivalence on H for each H, it follows that ρ is an equivalence on S. Moreover, $\rho \subseteq \mathscr{H}$. We prove that ρ is a congruence on S.

Let $(a, b) \in \rho$ and let $c \in S$. It will be shown that $(ca, cb) \in \rho$. First suppose that $ca = 0$. Since $\rho \subseteq \mathscr{H}$ there exists $x \in S^1$ such that $ax = b$. Then $cb = cax = 0$. Hence $(ca, cb) \in \rho$. We therefore assume that $ca \neq 0$ and $cb \neq 0$. Since S is 0-bisimple and $a \neq 0$ there exist $s, s' \in S^1$ such that $sa \in R_e \cap L_a$ and $s'sa = a$. Then, as before, $s'sb = b$. From the definition of ρ we have that $(sa, sb) \in \rho_R$. Now $esa = sa$ since $sa \in R_e$; therefore

$$ca = cs'sa = cs'esa.$$

But $ca \neq 0$. Hence $cs'e \neq 0$ and so there exist elements $u, u' \in S^1$ such that

$$ucs'e \in R_e \cap L_{cs'e}, \ u'ucs'e = cs'e.$$

Further, $ucs'e \in P_e$ since $(ucs'e)e = ucs'e$. Then, applying Lemma 4 (ii), we find that

$$(ucs'e \, . \, sa, ucs'e \, . \, sb) \in \rho_R.$$

But $ucs'esa = uca$ and $ucs'esb = ucb$; thus

$$(uca, ucb) \in \rho_R. \qquad (3)$$

Now

$$u'uca = u'ucs'esa = cs'esa = ca \qquad (4)$$

and, similarly, $u'ucb = cb$. From (4), $(uca, ca) \in \mathscr{L}$ and so, by Green's lemma, the mapping

$$x \to u'x \ (x \in H_{uca})$$

is a bijection from H_{uca} to H_{ca}. Since $\rho_R \subseteq \mathscr{H}$ we deduce from (3) that

$$(ca, cb) \in \mathscr{H}.$$

But $u(ca) \in R_e \cap L_{ca}$. It then follows from (3) that $(ca, cb) \in \rho$.

In the same way, using the alternative definition of ρ in terms of the equivalence ρ_L on L_e, we can show that $(ac, bc) \in \rho$. Thus ρ is a congruence on S.

Finally, let $y \in H_e$. Then $(e, y) \in \rho$ if and only if $(se, sy) \in \rho_R$ for any $s \in S^1$ such that $se \in R_e \cap L_e$. In particular, taking $s = e$, we see that $(e, y) \in \rho$ if and only if $(e, y) \in \rho_R$. But $(e, y) \in \rho_R$ if and only if $xe = y$ for some $x \in N$. Hence $e\rho = N$. This completes the proof.

Corollary. *Let S be a regular 0-bisimple semigroup and let e be any non-zero idempotent of S. Let H_e be a left normal divisor of P_e and a right normal divisor of Q_e. Then $\mathscr{H}$ is a congruence on S.*

Proof. Take $N = H_e$ in Lemma 6. Then there exists a congruence ρ on S contained in $\mathscr{H}$ and such that $e\rho = H_e$. Then $\rho = \mathscr{H}$ by Lemma 5 (i).

4. Lallement ((**2**), Theorem 2.3) has shown that the idempotent-separating congruences on a regular semigroup can be characterised as the congruences contained in $\mathscr{H}$. From Lemmas 3, 5 and 6 and the corollaries to Lemmas 3 and 6 we then obtain the following theorem concerning the idempotent-separating congruences on a regular 0-bisimple semigroup.

Theorem. *Let S be a regular 0-bisimple semigroup and let e be a non-zero idempotent of S. Let Λ denote the set of all idempotent-separating congruences on S and let Δ denote the set of all subgroups of H_e that are left normal divisors of P_e and right normal divisors of Q_e. Then*

(i) *$e\rho \in \Delta$ for all $\rho \in \Lambda$;*

(ii) *$\rho \subseteq \tau$ if and only if $e\rho \subseteq e\tau$ $(\rho, \tau \in \Lambda)$;*

(iii) *to each N in Δ there corresponds ρ in Λ such that $e\rho = N$.*

Furthermore, $\mathscr{H}$ is a congruence on S if and only if $H_e \in \Delta$.

From ((**2**), Corollary 3.3) we see that Λ is a complete modular lattice. The greatest element μ of Λ is the greatest congruence contained in $\mathscr{H}$ and is characterised thus ((**4**), Lemma 1):

$$(a, b) \in \mu \Leftrightarrow (sat, sbt) \in \mathscr{H} \text{ for all } s, t \in S^1.$$

Let Δ be partially ordered by inclusion. Then the theorem shows that

$$\rho \to e\rho$$

is an order-preserving bijection from Λ to Δ whose inverse is also order-preserving. Hence Δ is a complete modular lattice and $\Delta \cong \Lambda$. A direct calculation establishes that Δ is a sublattice of the lattice of all normal subgroups of H_e.

It should also be noted that the theorem provides a description of the idempotent-separating congruences on a regular bisimple semigroup T; for $\rho \to \rho \cup \{(0, 0)\}$ is a bijection from the set of all such congruences on T to the set of all idempotent-separating congruences on the regular 0-bisimple semigroup T^0.

5. We now discuss two important classes of regular 0-bisimple semigroups.

First let S be a completely 0-simple semigroup. By ((**1**), Theorem 2.51), S is both regular and 0-bisimple. Let e be a non-zero idempotent of S. Then e is primitive and so $S_e = H_e^0$ ((**1**), Lemma 2.47). Thus $P_e = Q_e = H_e$. The set Δ in the theorem therefore consists of *all* normal subgroups of H_e and so there is a natural one-to-one correspondence between the idempotent-separating congruences on S and the normal subgroups of H_e. In particular, $\mathscr{H}$ is a congruence on S. These well-known results also follow immediately from the structure theorem for completely 0-simple semigroups ((1), Theorem 3.5).

Next, let S be a 0-bisimple inverse semigroup and let e be any non-zero idempotent in S. Let $x \in S_e$. Then, $x = exe$ and so

$$x^{-1} = e^{-1}x^{-1}e^{-1} = ex^{-1}e \in S_e.$$

This shows that S_e is an inverse subsemigroup of S. Hence, by Lemma 2, Q_e consists of the inverses of the elements of P_e. Let N be a left normal divisor of P_e; that is, $aN \subseteq Na$ for all $a \in P_e$. Then $Na^{-1} \subseteq a^{-1}N$ for all $a \in P_e$ and so $Nb \subseteq bN$ for all $b \in Q_e$. The set Δ can therefore be taken as the set of all left normal divisors of P_e. For a bisimple inverse semigroup the theorem has been given in this form by Reilly and Clifford ((6), Theorem 2.4).

We deduce, in particular, that the idempotent-separating congruences on a bisimple inverse semigroup S with an identity are in one-to-one correspondence with the left normal divisors of the right unit subsemigroup of S. This result is due to Warne (7).

6. To conclude, we give an application of the theorem to the principal factors of the full transformation semigroup $\mathcal{T}_X$ on a set X. It is easy to see that $\mathcal{T}_X$ is regular ((1), p. 33, Exercise 1). The further properties required for our discussion—and outlined below—are established in ((1), § 2.2). We remark that Mal'cev (3) has determined a set of generators for the lattice of congruences on $\mathcal{T}_X$.

For $\alpha \in \mathcal{T}_X$ the equivalence $\alpha \circ \alpha^{-1}$ on X will be denoted by π_α; the cardinal of a set A will be denoted by $|A|$. Then the relations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{D}$ on $\mathcal{T}_X$ are characterised as follows:

$$(\alpha, \beta) \in \mathcal{R} \Leftrightarrow \pi_\alpha = \pi_\beta,$$
$$(\alpha, \beta) \in \mathcal{L} \Leftrightarrow X\alpha = X\beta,$$
$$(\alpha, \beta) \in \mathcal{D} \Leftrightarrow |X\alpha| = |X\beta|.$$

It is also easily verified that if $\alpha, \varepsilon \in \mathcal{T}_X$ and $\varepsilon^2 = \varepsilon$ then

$$\alpha\varepsilon = \alpha \Leftrightarrow X\alpha \subseteq X\varepsilon. \tag{1}$$

Now let $|X| > 1$. The principal factors of $\mathcal{T}_X$ other than the kernel are of the form U_c/V_c where c is any cardinal such that $|X| \geqslant c > 1$ and U_c, V_c are the ideals of $\mathcal{T}_X$ defined by

$$U_c = \{\alpha \in \mathcal{T}_X\colon |X\alpha| \leqq c\},\ V_c = \{\alpha \in \mathcal{T}_X\colon |X\alpha| < c\}.$$

We write $T_c = U_c/V_c$. Let α be any element of $\mathcal{T}_X$ of rank c. Then it can readily be shown that, since $\mathcal{T}_X$ is regular, the $\mathcal{R}$-class R_α of $\mathcal{T}_X$ is also an $\mathcal{R}$-class of T_c; similarly, the $\mathcal{L}$-class L_α of $\mathcal{T}_X$ is an $\mathcal{L}$-class of T_c. Hence T_c is a regular 0-bisimple semigroup. Moreover, by (1), for any non-zero idempotent ε of T_c we have that

$$P_\varepsilon = \{\alpha \in T_c \backslash 0\colon \pi_\alpha = \pi_\varepsilon \textit{ and } X\alpha \subseteq X\varepsilon\}. \tag{2}$$

Consider first the case where c is finite. Let ε, η be non-zero idempotents of

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T_c such that $\varepsilon\eta = \eta = \eta\varepsilon$. Since $\eta = \eta\varepsilon$ it follows from (1) that $X\eta \subseteq X\varepsilon$. Thus $X\eta = X\varepsilon$ since $| X\eta | = | X\varepsilon | = c$. Hence $\varepsilon = \varepsilon\eta$ and so $\varepsilon = \eta$. This shows that T_c is completely 0-simple. Since H_ε is isomorphic to the symmetric group of degree c, we see that, for $c \geqq 5$, T_c has exactly three distinct idempotent-separating congruences (corresponding to the three distinct normal subgroups of H_ε). Note that one of these congruences is $\mathscr{H}$.

Next let X be infinite and let c be an infinite cardinal such that $| X | \geqq c$. We shall show that the only idempotent-separating congruence on T_c is the identity congruence. Let ε be a non-zero idempotent of T_c and let γ be an element of H_ε distinct from ε. Then there exists $y \in X$ such that $y\gamma \neq y\varepsilon$. Now $\gamma \in L_\varepsilon$ and so $y\gamma \in X\varepsilon$. Since $X\varepsilon$ is infinite there exists an element θ in $\mathscr{T}_X$ that induces a one-to-one mapping of $X\varepsilon$ into $X\varepsilon$ and is such that $y\varepsilon \in (X\varepsilon)\theta$ and $y\gamma \notin (X\varepsilon)\theta$. Write $\alpha = \varepsilon\theta$. Then $\alpha \in T_c \backslash 0$; also $\pi_\alpha = \pi_\varepsilon$ and $X\alpha \subseteq X\varepsilon$. Hence, by (2), $\alpha \in P_\varepsilon$. But there exists $x \in X$ such that $y\varepsilon = x\alpha$. Therefore, since $\varepsilon\gamma = \gamma$, we have that

$$x\alpha\gamma = y\varepsilon\gamma = y\gamma \notin X\alpha.$$

In particular, this shows that γ cannot belong to a left normal divisor of P_ε. Hence the only left normal divisor of P_ε is the subgroup $\{\varepsilon\}$ of H_ε. It then follows from the theorem that the only idempotent-separating congruence on T_c is the identity congruence.

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