MOULTON PLANES

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1. Introduction.[1] In 1902, F. R. Moulton **(12)** gave an early example of a non-Desarguesian plane. Its "points" are ordered pairs (x, y) of real numbers. Its "lines" coincide with lines of the real affine plane *except that* lines of negative slope are "bent" on the x-axis, line $\{y = b + mx\}$, for negative m, being replaced by $\{y = b + mx \text{ if } y \leqq 0, y = [m/2]\cdot[x + (b/m)] \text{ if } y > 0\}$. A certain Desarguesian configuration in the classical plane is shifted just enough to vitiate Desargues' Theorem for Moulton's geometry. The plane is neither a translation plane ("Veblen–Wedderburn" in the sense of Hall **(7)**, p. 364) nor even the dual of one (Veblen and Wedderburn **(17)**. It is natural to ask if the same construction is feasible when real numbers are replaced by elements from an arbitrary field. If the construction *does* work—what geometric properties, what co-ordinate systems, and what collineation groups are obtained? Are the planes essentially "new"? In this paper and in a forthcoming sequel, I construct "Moulton planes" over a wide class of fields and answer relevant questions about their geometries. The classical ordering is replaced by a generalization of positives and negatives—the appropriate concept being that of "pseudo-order" used earlier by Dickson **(5)**, Kustaanheimo **(9)** and **(10)**, Pickert **(13)**, Sperner **(16)**, and others. The "positive" elements of F shall consist of a multiplicative subgroup P having index 2; and the "negatives," the other coset of non-zero elements. The product of two "positives," or of two "negatives," is still "positive"—while the product of a "negative" and a "positive" is "negative." (Write $x > 0$ or $x < 0$ according as $x \in P$ or $x \notin P \cup \{0\}$; say that non-zero elements x and y have the same or opposite "sign" according as $x/y > 0$ or < 0.) The field F is ordered in the usual sense if and only if P is closed under addition. The "pseudo-ordered" fields include ordered fields as special cases: rationals, reals, etc., under the standard order. A single field, F, may admit more than one "pseudo-order." For example, an unfamiliar definition of "positive" and "negative" exists on the rationals as follows. Given any (rational) prime p, a rational number, r, is uniquely expressible in the form $(p^i a)/b$ where i is integral, a and b denote (rational) integers prime to p, and a/b is reduced to lowest terms. For $a \neq 0$, one may call r "positive" or "negative" depending on whether i is even or odd. The rationals are *not ordered* under this definition of "pseudo-order." (For instance—given a prime p, and any non-zero intege

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b prime to p, $[(p-1)/b] + [1/b] = p/b$, showing that the sum of two "positives" can be "negative.")

Another non-trivial "pseudo-order" can be constructed as follows. Let $F(x)$ denote the field of rational forms over a field F. A quotient $f(x)/q(x)$, reduced to lowest terms with $f(x) \cdot q(x) \neq 0$, is > 0 or < 0 according as the difference of degrees, $\delta(f) - \delta(q)$, is even or odd.

A multiplicative subgroup P of index 2 must contain *all* non-zero squares. On the other hand, P may or may not consist *only* of squares. Under the usual ordering, positive real numbers are all squares—positive rationals are not. (Note that the "positive" rationals are not necessarily squares under the alternative "pseudo-order" described above.) What if F is finite? In the case of characteristic 2, $x \to x^2$ is an automorphism, F contains *only squares*, and *no* "pseudo-order" is possible. In finite fields of odd characteristic, however, the powers of a primitive element indicate that *at least half* of the non-zero elements are squares; since any equation $x^2 = a_0^2$, with $a_0 \neq 0$, has two distinct solutions, the map $x \to x^2$ cannot be "onto"; so the squares form a *proper* multiplicative subgroup—and that subgroup has index 2.

Dickson **(5)**, studying equations over finite fields, used non-zero squares as "positives" in his effective treatment of discriminants. Sperner **(16)** used the more general concept of "pseudo-order" to investigate relations between algebraic semiorder and geometric order. Recently, Kustaanheimo **(9)** has utilized the same concept to develop order and congruence relations for finite geometries. He has suggested the intriguing possibility that such ideas may be applied to problems of quantum physics—where some of the difficulties encountered are not necessarily intrinsic, but may stem from the imposition of infinite models on finite situations.

My own interest in Moulton's construction—especially over finite fields—has motivated Carlitz to prove two basic theorems. The statements of his results require a preliminary definition, which will also be needed later.

Definition 1. A single-valued function ϕ on a field F is called "order-preserving" if and only if $[\phi(u) - \phi(v)]/(u - v) > 0$, for all distinct $u, v \in F$; "monotonic" if $[\phi(u) - \phi(v)]/(u - v)$ retains the same "sign," for all $u \neq v \in F$.

(A). (Carlitz **(3)**). If F denotes a finite field of odd characteristic, the most general one-to-one monotonic function, ϕ, on F is given by $\phi(x) = a \cdot \sigma(x) + b$ (where σ is an automorphism; $b, a \in F$ with $a \neq 0$). According as $a > 0$ or $a < 0$, ϕ is order-preserving or reversing. If $\phi(0) = 0$, $\phi(1) = 1$, then ϕ is an automorphism.

(B). (A generalization of the above, unpublished, but contained in a written communication to me.) Assume that F has order p^n, where p is odd. Put

$$\psi(a) = a^{\frac{1}{2}(p^n - 1)},$$

and let $\lambda_1 = \pm 1, \ldots, \lambda_k = \pm 1$. Let $f(x_1, \ldots, x_k)$ be a polynomial with coefficients in $GF(p^n)$ such that $\psi\{f(x_1, \ldots, x_r, \ldots, x_k) - f(x_1, \ldots, y_r, \ldots, x_k)\} = \lambda_r \psi(x_r - y_r)$ for all $r = 1, \ldots, k$ and all x_j, y_j in $GF(p^n)$. Then

$$f(x_1, \ldots, x_k) = c_1 x_1^{p^{r_1}} + \ldots + c_k x_k^{p^{r_k}} + d,$$

where $\psi(c_j) = \lambda_j$ and $0 \leqslant r_j < n$.

2. Definition and construction of "Moulton planes." Throughout this paper, I shall assume that a "pseudo-order"—hence a multiplicative subgroup P of index 2—exists and has been specified on a given field F. Terms and symbols of "order," "inequality," etc. will refer to the designated "pseudo-order"; they will no longer be enclosed by quotation marks. It will be convenient to replace the x-axis by the y-axis as the line along which "bending" occurs.

Definition 2. Let ϕ denote a one-to-one function of a given field F onto itself. A *Moulton construction*, $C_\phi(F)$, consists of "points" and classes of "points"—called "lines"—in which:

(i) Each "point" is an ordered pair (x, y) of elements $\in F$.

(ii) Each "line" consists of all "points" (x, y) satisfying an equation of the form $\{x = c\}(c \in F)$, or $\{y = b + m \circ x\}(b, m \in F)$, where $m \circ x$ is defined from the field multiplication by $m \circ x = mx$ or $\phi(m) \cdot x$ according as $x \geqslant 0$ or $x < 0$ [a "line" of the latter type is said to have "slope" m].

Definition 3. A *Moulton plane* is a construction, $C_\phi(F)$, whose "points" and "lines" form an affine plane. If $C_\phi(F)$ is such a plane, it will be denoted by $M_\phi(F)$.

Remark. When convenient, $M_\phi(F)$ will also be regarded as the projective plane obtained by adding ideal elements to the affine Moulton plane.

3. Geometry of Moulton planes.

THEOREM 1. *A construction, $C_\phi(F)$ forms a Moulton plane if and only if:*

(a) *The function ϕ is order-preserving.*

(b) *Given any negative $n_0 \in F$, $x \to [\phi(x) - n_0 x]$ maps F onto F.*

Proof. Two distinct "points" determine a unique "line" except possibly in the case of non-zero abscissae having unlike signs. Given $u_0 < 0$, $p_0 > 0$, $v_0, q_0 \in F$, the existence of *at least* one "line" $(u_0, v_0) \cup (p_0, q_0)$ amounts to the existence of $m \in F$ such that (u_0, v_0) satisfies $y = \phi(m) \cdot x + (q_0 - mp_0)$, that is, of an m for which $\phi(m) - [p_0/u_0] \cdot m = (v_0 - q_0)/u_0$. Such an m exists for all (u_0, v_0) and (p_0, q_0), with $u_0 < 0$, $p_0 > 0$, if and only if condition (b) holds. Suppose (Fig. 1) that both (u_0, v_0) and (p_0, q_0) belong to lines of "slope" m and n, whence $v_0 = \phi(m) \cdot u_0 + (q_0 - mp_0)$ and

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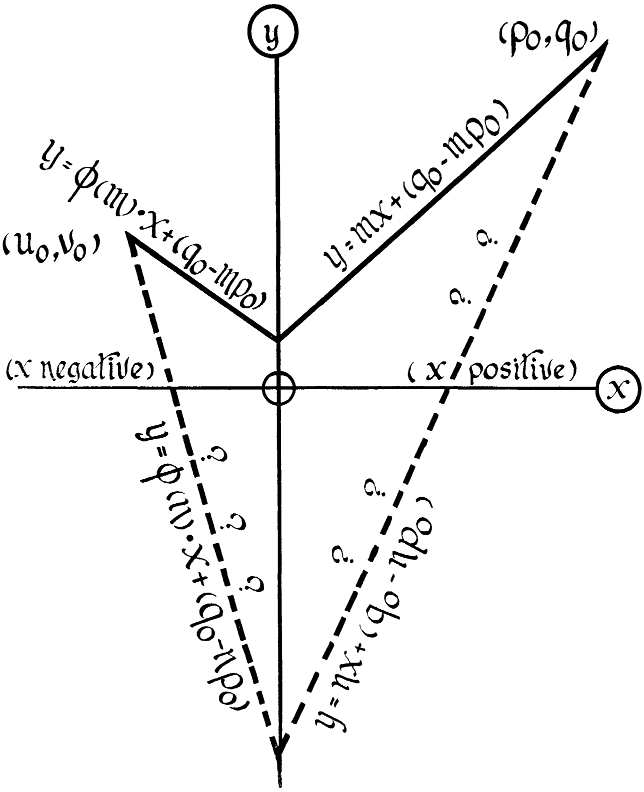

FIG. 1.

$v_0 = \phi(n) \cdot u_0 + (q_0 - np_0)$. Subtraction gives $0 = u_0 \cdot [\phi(m) - \phi(n)] - p_0 \cdot [m - n]$. Unless $m - n = 0$, we get $[\phi(m) - \phi(n)]/[m - n] = p_0/u_0 < 0$. Thus, order-preservation is sufficient to prove that *not more than* one "line" joins any two distinct "points." Conversely, the existence of $m \neq n$ such that $[\phi(m) - \phi(n)]/[m - n] = 1/r_0 < 0$ would permit us to put $v_0 = \phi(m) \cdot r_0 - m = \phi(n) \cdot r_0 - n$, forcing (r_0, v_0) to lie on distinct "lines" of "slope" m and n through $(1, 0)$. It follows that order-preservation is equivalent to the existence of *at most* one "line" through any two distinct "points."

Let us now verify Euclid's parallel postulate. Two "lines" are "parallel" if and only if they coincide or have no point in common. An ordinary "point" (x_0, y_0) must be shown to lie on exactly one "line" parallel to an ordinary "line" l. (i) If l is given by $\{x = c\}$, (x_0, y_0) lies on $\{x = a\}$ if and only if $a = x_0$. On the other hand $\{x = c\}$ intersects *every* "line" of the form $\{y = b + m \circ x\}$. (ii) If l is given by $\{y = b + m \circ x\}$, then (x_0, y_0) lies on $\{y = c + m \circ x\}$ if and only if $c = y_0 - m \circ x_0$. For $m \neq n$, $\{y = b + m \circ x\}$ meets $\{y = d + n \circ x\}$ in the point $(u_0, b + m \circ u_0)$, with $u_0 = (b - d)/(n - m)$ or $(b - d)/[\phi(n) - \phi(m)]$ according as $u_0 \geqslant 0$ or $u_0 < 0$. Such a u_0 exists since, according to (a), $(b - d)/(n - m)$ and $(b - d)/[\phi(n) - \phi(m)]$ have the same sign.

The presence of three non-collinear "points" is trivial—(0, 0), (1, 0), (0, 1) for example.

COROLLARY 1. *If F is finite (of odd characteristic), and if ϕ is one-to-one on F, then $C_\phi(F)$ is a plane if and only if ϕ preserves order.*

Proof. If ϕ fails to preserve order, the theorem shows that $C_\phi(F)$ cannot be a plane.

Assume, conversely, that ϕ *does* preserve order. Given

$$n_0 < 0, x \to [\phi(x) - n_0 x]$$

is one-to-one "into":

$$\phi(u) - n_0 u = \phi(v) - n_0 v \to \phi(u) - \phi(v) = n_0 \cdot (u - v),$$

which is impossible unless $u = v$. By finiteness, one-to-one "into" is "onto."

COROLLARY 2. *If F is a finite field (of odd characteristic), and if ϕ is a one-to-one function on F, then $C_\phi(F)$ is a Moulton plane if and only if $\phi(m) = a^2 \cdot \sigma(m) + b$, for some $b, a \neq 0 \in F$, and some automorphism σ. In case $\phi(0) = 0$ and $\phi(1) = 1$, a plane is obtained if and only if ϕ is an automorphism.*

Proof. This is a restatement of Corollary 1 in the presence of the Carlitz Theorem **(3)**.

Examples over the real field R (relative to the usual order). A construction $C_\phi(R)$ is a Moulton plane if and only if 0 is an increasing function of R onto itself—for instance:

(1) $\phi(m) = m^3$,

(2) $\phi(m) = m$ or $p_0 m (p_0 > 0)$ according as $m \geqslant 0$ or $m < 0$ (the example originally given by Moulton); see also Pickert **(13)**, p. 93, *et seq.*

(3)
$$\phi(m) = \begin{cases} -2 - \sqrt{-m}, \text{ for } m < 0 \\ -2 + m, \text{ for } 0 \leqslant m < 1 \\ - - - - - - \\ [(-r^2 - r - 4)/2] + (r + 1) \cdot m, \text{ for } r \leqslant m < r + 1, \\ \text{where } r \text{ is a non-negative integer.} \end{cases}$$

LEMMA 1. *Any Moulton plane, $M_\phi(F)$, is isomorphic to a plane $M_{(\phi')}(F)$ with $\phi'(0) = 0$, $\phi'(1) = 1$.*

Proof. Initially, "lines" are given by $\{x = c\}$ for $c \in F$, and $\{y = b + mx\}$ if $x \geqslant 0$; $\{y = b + \phi(m) \cdot x\}$ if $x < 0$. Change co-ordinates, putting $x = x'$, $y = y'$, if $x \geqslant 0$; and $x = cx'$, $y = y' + ax'$, if $x > 0$, where $a = \phi(0)/[\phi(1) - \phi(0)]$ and $c = 1/[\phi(1) - \phi(0)]$ with $c > 0$ since ϕ is order-preserving. This transformation permutes "lines" $\{x = c\}$ among themselves, maps $\{y = b + mx\}$

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onto $\{y' = b + mx'\}$ for $x \geqq 0$, and $\{y = b + \phi(m) \cdot x\}$ onto $\{y' + ax' = b + c \cdot \phi(m) \cdot x'\}$ for $x < 0$; the latter reducing to $\{y' = b + [\phi'(m)] \cdot x'\}$, where $\phi'(m) = c \cdot \phi(m) - a$, $\phi'(0) = 0$, and $\phi'(1) = 1$.

THEOREM 2. *Every Moulton plane can be represented by co-ordinates from a Cartesian group, G (in the sense of Pickert* **(13)**, p. 90). *Addition for G coincides with that of F, but the multiplication,* $\circ$, *of G is defined as follows:* $u \circ v = uv$ or $\phi(u) \cdot v$, *according as* $v \geqslant 0$ *or* $v < 0$.

Proof. Apply Lemma 1 to represent the given plane as $M_\phi(F)$, where $\phi(0) = 0$, $\phi(1) = 1$. Since the elements of F already form a group under $+$, they will form a Cartesian group under the operations $\{+, \circ\}$ if and only if:

(i) The non-zero elements form a loop under $\circ$.

(ii) $x \in F \rightarrow 0 \circ x = x \circ 0 = 0$, $1 \circ x = x \circ 1 = x$.

(iii) For all $a, b, c, d \in F$, $a \circ c - a \circ d = b \circ c - b \circ d \rightarrow a = b$ or $c = d$ [Pickert **(13)**, p. 90 (9)].

(iv) Given $a, b, c \in F$, with $a \neq b$, $\exists\, x \in F$ such that $a \circ x - b \circ x = c$ **(13)**, p. 90 (10)].

(v) Given $a, b, c \in F$, with $a \neq b$, $\exists\, x \in F$ such that $x \circ a - x \circ b = c$. [**(13)**, p. 90 (11)].

In the presence of $\phi(0) = 0$ and $\phi(1) = 1$, properties (i) and (ii) are trivial. Property (iii) is immediate if $c/d > 0$, because $a \cdot (c - d) = b \cdot (c - d)$ or $\phi(a) \cdot (c-d) = \phi(b) \cdot (c-d)$ according as $c>0$ or $c<0$. To prove (iii) when $c/d < 0$, use the symmetry between c and d to suppose $c < 0$, $d > 0$. Then $\phi(a) \cdot c - ad = \phi(b) \cdot c - bd \rightarrow [\phi(a) - \phi(b)] \cdot c = (a - b) \cdot d$. Unless $a = b$, $[\phi(a) - \phi(b)]/(a - b) = d/c < 0$, contradicting the order-preservation.

To verify (iv), use $x = c/(a - b)$ if $c/(a - b) \geqslant 0$; otherwise, $x = c/[\phi(a) - \phi(b)]$.

Property (v) is obvious if $ab \geqslant 0$. Otherwise, after possible multiplication by $n < 0$, we can assume $a < 0$, $b > 0$. Property (b) of Theorem 1 asserts that the map $x \rightarrow \phi(x) - (b/a) \cdot x$ is "onto," thus supplying the desired value of x. The representation of lines follows at once from the Moulton construction, and the proof of the Theorem is complete.

Remark. The basic geometry of Moulton planes can be developed using direct, synthetic proofs. It is more efficient, however, to apply known results concerning Cartesian groups, as given by Pickert in *Projektive Ebenen* **(13)**. Identify Moulton points $(0, 0)$, $(1, 1)$, X_∞ (the ideal point on the x-axis), Y_∞ (the ideal point on the y-axis), and the infinite point on $\{y = x\}$, with the respective points O, E, U, V, W, of Pickert's co-ordinate system, and the (Moulton) ideal line, l_∞, with line $U \cup V$ (**(13)**, pp. 31–32). Put the Hall ternary (**(13)**, p. 35; Hall **(6)**), $T(u, x, v) = (u \circ x) + v$, so that Moulton lines have equations of the form $\{x = c\}$ and $\{y = T(m, x, b)\}$ (Pickert **(13)**, p. 35).

THEOREM 3. *Every Moulton plane ,M, is a Baer plane* (Baer **(1)**), *in the sense that it satisfies the Desarguesian* (Y_∞, l_∞)*-Theorem* (*Pickert* **(13)**, *pp.* 74–76). *Thus, M also satisfies the Reidemeister-condition for the* (X_∞, Y_∞, W)*-web* ("Gewebe")—(**(13)**, *p.* 52; *Reidemeister* **(15)**).

Proof. By Theorem 2, M can be co-ordinated over a Cartesian group. The present Theorem is then a restatement of Pickert's *Satz 36* (**(13)**, top of p. 100).

Note. The direct proof of Theorem 3 would present a neat geometric picture —the y-axis being used as an auxiliary line.

THEOREM 4. *In a Moulton plane* $M_\phi(F)$, *with* $\phi(0) = 0$, $\phi(1) = 1$, *the following assertions are equivalent:*

(a) *The Desarguesian* $(X_\infty, l_\infty; Y_\infty, \{y = 0\})$—*Theorem holds* [this involves triangles perspective from X_∞, with one pair of corresponding vertices, say P and P', on $\{y = 0\}$; a pair of corresponding sides, QR and $Q'R'$, through Y_∞; and PR parallel to $P'R'$ if and only if PQ is parallel to $P'Q'$].

(b) *The plane* $M_\phi(F)$ *is a translation plane with axis* l_∞ (*Pickert* **(13)**, p. 199*; Hall* **(7)**, p. 364–*a "Veblen–Wedderburn" plane in the latter's terminology*).

(c) *Desargues' Theorem is valid.*

(d) *The Cartesian group* $\{+, \circ\}$ *satisfies the right-distributive law* $u \circ (x + w) = (u \circ x) + (u \circ w)$, [**(13)**, p. 99, (18)].

(e) *The function* ϕ *is the Identity*! (Cf. footnote[2].)

Proof. Much of this Theorem is an immediate consequence of *Satz 37* (**(13)**, p. 100). By Theorem 3, $M_\phi(F)$ satisfies the Reidemeister-condition relative to $(U = X_\infty, V = Y_\infty, W)$—whence Pickert's condition (b) of *Satz 37* reduces to condition (a) above. By the associativity of addition, and by the "erste Zerlegbarkeitsbedingung," $T(u, x, v) = u \circ x + v$, condition (c) of *Satz 37* reduces to (d) of the present Theorem. (Cf. Satz, 35, p. 99.) Each of (a) and (d) becomes equivalent to the Desarguesian (Q, l_∞)-Theorem for two distinct choices—in this case X_∞ and Y_∞—of $Q \in l_\infty$, implying condition (b) of the present Theorem.

It remains only to show that (d) $\rightarrow$ (e), since the Theorem will then follow trivially. Suppose $\phi \neq \mathscr{J}$ (the identity), and let $\phi(u) \neq u$, for $u \in F$. If $x < 0$, we get $(u \circ x) + (u \circ 1) = \phi(u) \cdot x + u$, and $u \circ (x + 1) = \phi(u) \cdot (x + 1)$ or $u \cdot (x + 1)$, neihter of which equals $[\phi(u) \cdot x + u]$. Thus, $\phi \neq \mathscr{J}$ implies that the right-distributive law cannot hold, and (e) follows from (d).

Note. A direct verification of Theorem 4 could be based on the following neat proof that (a) $\rightarrow$ (e). Suppose ϕ non-trivial. Choose u and n such that $\phi(u) \neq u$, $n < 0$, and $n + 1 \leqq 0$. Consider the triangles with vertices, $(1, 0)$, $(0, 1)$, $(0, b)$; and $(n + 1, 0)$, $(n, 1)$, $(n, -u)$. As triangles in the classical

[2]The redundance of Theorem 4 may help to clarify the relation between this development and that of Pickert **(13)**. Henceforth, all references will be to the latter work unless otherwise specified.

plane over F, they are perspective from X_∞, axial from l_∞, have a pair of corresponding vertices on $\{y = 0\}$ and a pair of corresponding sides through Y_∞. In $M_\phi(F)$, all these properties still hold *except that* $(1, 0) \cup (0, -u)$ and $(n + 1, 0) \cup (n, -u)$ have respective Moulton "slopes" u and $\phi^{-1}(u)$—violating the $(X_\infty, l_\infty; Y_\infty, \{y = 0\})$-condition of (a).

THEOREM 5. *Let $M = M_\phi(F)$ denote a Moulton plane where $\phi(0) = 0$, $\phi(1) = 1$. Each of the following is necessary and sufficient for M to be (Y_∞, Y_∞)-transitive:*

(i) *The (Y_∞, n)-Desargues' Theorem holds for every line n through Y_∞.*
(ii) *The left-distributive law $(a + b) \circ c = a \circ c + b \circ c$ is valid.*
(iii) *The Cartesian group $\{+, \circ\}$ is a left quasi-field.*
(iv) *The function ϕ is additive.*

Proof. Condition (i) is a standard variation of (Y_∞, Y_∞)-transitivity. Conditions (ii) and (iii) involve *Satz 39* (page 101) and the definition of "Linksquasikörper." Let us check the equivalence of (ii) and (iv): the law $(a+b)\circ c = a\circ c + b \circ c$ is automatic if $c \geqslant 0$; but for $c < 0$, $(a + b) \circ c = a\circ c + b\circ c$ if and only if $[\phi(a + b)]\cdot c = \phi(a)\cdot c + \phi(b)\cdot c$; the latter holds for all $c < 0$, $a, b \in F$ if and only if ϕ is additive.

COROLLARY 3. *A finite Moulton plane M must be (Y_∞, Y_∞)-transitive.*

Proof. Use Lemma 1 to represent M as $M_\phi(F)$, where $\phi(0) = 0$, $\phi(1) = 1$. By the Theorem of Carlitz, ϕ is an automorphism—in particular, it is additive.

COROLLARY 4. *If $M_\phi(F)$ denotes a finite Moulton plane with $\phi(0) = 0$, $\phi(1) = 1$, Conditions* (i)–(iv) *of Theorem* 5 *all hold in $M_\phi(F)$.*

Proof. The additivity of ϕ implies the remaining conditions.

Remark. Examples already given show that the conditions of Theorem 5 are not valid in every Moulton plane.

THEOREM 6. *If $\phi(0) = 0$ and $\phi(1) = 1$, a Moulton plane $M_\phi(F)$ determines a Cartesian group with associative multiplication if and only if Desargues' $(X_\infty, \{x = 0\})$-Theorem holds. Associativity of multiplication and the right-distributive law $c \circ (a + b) = c \circ a + c \circ b$ together are equivalent to Desargues' $(Y_\infty, \{y = 0\})$-Theorem.*

Proof. Since $M_\phi(F)$ satisfies $T(m, x, b) = m \circ x + b$ ("erste Zerlegbarkeitsbedingung"), the first part of *Satz 45* reduces to an equivalence between the associativity of multiplication and the $(X_\infty, \{x = 0\})$-Theorem. The second part of *Satz 45* becomes the final statement of Theorem 6, since the right-distributive law is equivalent to $T(m, x, m \circ b) = m \circ (x + b)$ ("zweite Zerlegbarkeitsbedingung") when $T(m, x, b) = m \circ x + b$ (Satz 35).

THEOREM 7. *Let $+$ and $\circ$ determine the Cartesian group for a Moulton plane $M_\phi(F)$, where $\phi(0) = 0$, $\phi(1) = 1$. Then*

(i) $c \geqslant 0$ *implies* $(a \circ b) \circ c = a \circ (b \circ c)$, *for all* $a, b, c \in F$.

(ii) $c < 0$ *and* $b \geqslant 0$ *imply* $(a \circ b) \circ c = a \circ (b \circ c)$ *for all* $a \in F$ *if and only if* ϕ *is multiplicative on* F.

(iii) $c < 0$ *and* $b < 0$ *imply* $(a \circ b) \circ c = a \circ (b \circ c)$ *for all* $a \in F$ *if and only if* $\phi(a) \cdot b = \phi^{-1}(a \cdot \phi(b))$.

Proof. Note first that ϕ preserves sign, since $\phi(0) = 0$ and ϕ is order-preserving.

(i) $(a \circ b) \circ c = (a \circ b) \cdot c = \lambda(a) \cdot bc = a \circ (bc) = a \circ (b \circ c)$, where $\lambda = \phi$ or $\mathscr{I}$ (the identity) according as $b < 0$ or $b \geqslant 0$.

(ii) $(a \circ b) \circ c = \phi(ab) \cdot c$, and $a \circ (b \circ c) = \phi(a) \cdot \phi(b) \cdot c$ [the multiplicative property being exactly what we need].

(iii) $(a \circ b) \circ c = \phi\{\phi(a) \cdot b\} \cdot c$, and $a \circ (b \circ c) = a \cdot \phi(b) \cdot c$, whence the condition $\phi(a) \cdot b = \phi^{-1}(a \cdot \phi(b))$.

COROLLARY 7. *Under the hypotheses of Theorem* 7, *the operation* $\circ$ *is associative if and only if* ϕ *is multiplicative and* $\phi = \phi^{-1}$.

COROLLARY 8. *Under the same hypotheses, the Desarguesian* $(X_\infty, \{x = 0\})$-*Theorem is valid in* $M_\phi(F)$ *if and only if* ϕ *denotes a multiplicative function of order* 2.

Proof. This follows from Theorems 6 and 7, and Corollary 7.

4. Collineations and isomorphisms on $M_\phi(F)$. A sequel to this paper will prove that some Moulton planes support a rather large group of collineations. It will treat isomorphisms between Moulton planes, and will show that a large class of "new" planes is obtained from the construction.

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