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CONSTRUCTION OF STEINER TRIPLE SYSTEMS HAVING EXACTLY ONE TRIPLE IN COMMON

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1. Introduction. A *Steiner triple system* is a pair (Q, t) where Q is a set and t a collection of three element subsets of Q such that each pair of elements of Q belong to exactly one triple of t. The number $|Q|$ is called the order of the Steiner triple system (Q, t). It is well-known that there is a Steiner triple system of order n if and only if $n \equiv 1$ or $3 \pmod 6$. Therefore in saying that a certain property concerning Steiner triple systems is true for all n it is understood that $n \equiv 1$ or $3 \pmod 6$. Two Steiner triple systems (Q, t_1) and (Q, t_2) are said to be *disjoint* provided that $t_1 \cap t_2 = \emptyset$. Recently, Jean Doyen has shown the existence of a pair of disjoint Steiner triple systems of order n for every $n \geqq 7$ [**1**]. In this same paper Doyen raises the question as to whether it is possible to construct a pair of Steiner triple systems of order n having *exactly one* triple in common for every $n \geqq 3$. The purpose of this paper is to show that such a pair exists for every $n \geqq 3$.

2. Preliminaries. A *Steiner quasigroup* is a quasigroup satisfying the identities $x^2 = x$, $x(xy) = y$, and $(yx)x = y$. It is well-known that a Steiner triple system is algebraically a Steiner quasigroup. We will say that the Steiner triple system (Q, t) and Steiner quasigroup $(Q, \circ)$ are *associated* with each other provided that $\{x, y, z\} \in t$ if and only if $x \circ y = z$. In much of what follows we will consider Steiner triple systems algebraically. Steiner quasigroups $(Q, \circ_1)$ and $(Q, \circ_2)$ are disjoint or intersect in exactly one triple provided that their associated triple systems (Q, t_1) and (Q, t_2) have this property. Therefore the Steiner quasigroups $(Q, \circ_1)$ and $(Q, \circ_2)$ are disjoint provided that $x \circ_1 y \neq x \circ_2 y$ for all $x \neq y \in Q$ and intersect in exactly one triple provided that for some three element subset T of Q that $(T, \circ_1)$ is a subquasigroup of $(Q, \circ_1)$, $(T, \circ_2)$ is a subquasigroup of $(Q, \circ_2)$, $x \circ_1 y = x \circ_2 y$ for all $x, y \in T$, and $x \circ_1 y \neq x \circ_2 y$, $x \neq y$, if at least one of $x, y \in Q \backslash T$. In this case we will say that $(Q, \circ_1)$ and $(Q, \circ_2)$ *agree exactly* on T. We will call a pair of Steiner quasigroups $(Q, \circ_1)$ and $(Q, \circ_2)$ of order q a $(q, 0)$ pair if they are disjoint and a $(q, 1)$ pair if they interesct in exactly one triple.

Most of the constructions in this paper are based on the following generalized singular direct product of quasigroups. (See [**3**; **4**; **5**; **6**; **7**].) We begin by defining the *conjugate* of a quasigroup $(Q, \otimes)$. Let $(Q, \otimes)$ be any quasigroup and on the set Q define six binary operations $\otimes(1, 2, 3)$, $\otimes(1, 3, 2)$, $\otimes(2, 1, 3)$, $\otimes(2, 3, 1)$, $\otimes(3, 1, 2)$, and $\otimes(3, 2, 1)$ as follows:

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$a \otimes b = c$ if and only if

$$\begin{aligned} a &\otimes(1, 2, 3)b = c,\\ a &\otimes(1, 3, 2)c = b,\\ b &\otimes(2, 1, 3)a = c,\\ b &\otimes(2, 3, 1)c = a,\\ c &\otimes(3, 1, 2)a = b,\\ c &\otimes(3, 2, 1)b = a. \end{aligned}$$

The six (not necessarily distinct) quasigroups $(Q, \otimes(i, j, k))$ are called the conjugates of $(Q, \otimes)$ [**8**]. We will denote by $(T, *)$ the Steiner quasigroup of order 3, where $T = \{1, 2, 3\}$. Let $(V, \odot)$ be any Steiner quasigroup and (V, t) the associated Steiner triple system. Let $t_1, t_2, \ldots, t_r$ be the triples in t. Then each $(t_i, \odot)$ is a subquasigroup of $(V, \odot)$ and is isomorphic to $(T, *)$. Let α_i be a fixed isomorphism of $(t_i, \odot)$ onto $(T, *)$. Let Q be a set and for each v in V let $\circ(v)$ be a binary operation on Q so that $(Q, \circ(v))$ is a Steiner quasigroup. Further suppose that $P \subseteq Q$ is such that all of the operations agree on P and that $(P, \circ(v))$ is a subquasigroup of $(Q, \circ(v))$. Let $(\bar{P} = Q\backslash P, \otimes)$ be any quasigroup. If $p, q \in \bar{P}$ and $v \neq w \in V$, by $p \otimes(v, w, v\odot w)\, q$ is meant the element $p \otimes(v\alpha_i, w\alpha_i, v\alpha_i * w\alpha_i)\, q$ of $(Q, \otimes(v\alpha_i, w\alpha_i, v\alpha_i * w\alpha_i))$, where $\{v, w, v \odot w\} = t_i$. On the set $P \cup (\bar{P} \times V)$ define the binary operation $\otimes$ as follows:

(1) $p \oplus q = p \circ(v)\, q = p \circ(w)\, q$, if $p, q \in P$;
(2) $p \oplus (q, v) = (p \circ(v)\, q, v)$, if $p \in P, q \in \bar{P}, v \in V$;
(3) $(q, v) \oplus p = (q, \circ(v)\, p, v)$, if $p \in P, q \in \bar{P}, v \in V$;
(4) $(p, v) \oplus (q, v) = p \circ(v)\, q$, if $p \circ(v)\, q \in P$, and
$\qquad = (p \circ(v)\, q, v)$ if $p \circ(v)\, q \in \bar{P}$;
(5) $(p, v) \oplus (q, w) = (p \otimes (v, w, v \odot w)q, v \odot w)$, $v \neq w$.

The quasigroup so constructed is denoted by $V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$. We remark here that the operations $\circ(v)$ are not necessarily related other than agreeing on P and that although $(V, \odot)$ and $(Q, \circ(v))$, all $v \in V$, are Steiner quasigroups, the quasigroup $(\bar{P}, \otimes)$ is not necessarily Steiner. Finally, although we have used the same quasigroup $(\bar{P}, \otimes)$ for every triple in t this is not necessary. Different quasigroups can be associated with each triple in t. In [**6**] the following theorem is proved.

THEOREM 1. *The singular direct product* $V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$ *defined above is a Steiner quasigroup of order* $v(q - p) + p$, *where* $|V| = v$, $|Q| = q$, *and* $|P| = p$.

3. Basic constructions. In this section we give the basic constructions to be used in constructing a pair of $(q, 1)$ Steiner quasigroups for every $q \geqq 3$.

THEOREM 2. *Let* $q > 3$. *If there is a pair of* $(q, 1)$ *Steiner quasigroups, there is a pair of* $(v(q - 1) + 1, 1)$ *Steiner quasigroups for all* $v \equiv 1$ *or* $3 \pmod 6$.

Proof. Let $(V, \odot)$ be any Steiner quasigroup based on $1, 2, \ldots, v$ and $(Q, \circ_1)$ and $(Q, \circ_2)$ a pair of $(q, 1)$ Steiner quasigroups based on $1, 2, \ldots, q$ agreeing exactly on the subset $T = \{1, 2, 3\}$ of Q. Let $(Q, \bar{\circ}_1)$ and $(Q, \bar{\circ}_2)$ be a pair of $(q, 0)$ Steiner quasigroups. Take $P = \{1\}$ so that P is a subquasigroup of all of the above quasigroups. Finally, set $\bar{P} = Q \backslash P$ and let $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ be a pair of totally disjoint quasigroups; i.e., $x \otimes_1 y \neq x \otimes_2 y$ for all $x, y \in \bar{P}$. Now form the singular direct products $(S, \oplus_1)$ and $(S, \oplus_2)$ defined as follows:

(1) $(S, \oplus_1) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(1) = \circ_1$, $\circ(i) = \bar{\circ}_1$, $i > 1$, $P = \{1\}$, and $(\bar{P}, \otimes_1)$ is used to define $(\bar{P}, \otimes (u, v, w))$.

(2) $(S, \oplus_2) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(1) = \circ_2$, $\circ(i) = \bar{\circ}_2$, $i > 1$, $P = \{1\}$, and $(\bar{P}, \otimes_2)$ is used to define $(\bar{P}, \otimes (u, v, w))$.

Set $\bar{T} = \{1, (2, 1), (3, 1)\}$. Then both $\oplus_1$ and $\oplus_2$ agree on T. We show that $\oplus_1$ and $\oplus_2$ agree exactly on $\bar{T}$. We consider three cases.

(1) $x \neq y \in P \cup (\bar{P} \times \{1\})$. Since $(P \cup (\bar{P} \times \{1\}), \oplus_1)$ is a copy of $(Q, \circ_1)$ and $(P \cup (\bar{P} \times \{1\}), \oplus_2)$ is a copy of $(Q, \circ_2)$ it follows that $x \oplus_1 y = x \oplus_2 y$ if and only if both x and y are in T.

(2) $x \neq y \in P \cup (\bar{P} \times \{v\})$, $v \neq 1$. Since $(P \cup (\bar{P} \times \{v\}), \oplus_1)$ is a copy of $(Q, \bar{\circ}_1)$ and $(P \cup (\bar{P} \times \{v\}), \oplus_2)$ is a copy of $(Q, \circ_2)$ we must have $x \oplus_1 y \neq x \oplus_2 y$.

(3) $x = (p, v)$, $y = (q, w)$, $v \neq w$. In this case we have

$$(p, v) \oplus_1 (q, w) = (p \otimes_1 (v, w, v \odot w) q, v \odot w)$$

and

$$(p, v) \oplus_2 (q, w) = (p \otimes_2 (v, w, v \odot w) q, v \odot w).$$

Since $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ have the property that $x \otimes_1 y \neq x \otimes_2 y$ for all $x, y \in \bar{P}$ it follows that their corresponding conjugates also have this property. Hence $p \otimes_1 (v, w, v \odot w) q \neq p \otimes_2 (v, w, v \otimes w) q$.

Now combining cases (1), (2), and (3) shows that the singular direct products $(S, \oplus_1)$ and $(S, \oplus_2)$ agree exactly on $\bar{T}$ completing the proof.

THEOREM 3. *Let $q > 3$. If there is a pair of $(q, 1)$ Steiner quasi-groups, there is a pair of $(v(q - 3) + 3, 1)$ Steiner quasigroups for all $v \equiv 1$ or 3 (mod 6).*

Proof. Again let $(V, \odot)$ be any Steiner quasigroup and $(Q, \circ_1)$ and $(Q, \circ_2)$ a pair of $(q, 1)$ Steiner quasigroups agreeing exactly on the three element subset T of Q. Set $\bar{P} = Q \backslash T$ and let $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ be a pair of completely disjoint quasigroups; i.e., $x \otimes_1 y \neq x \otimes_2 y$ for all $x, y \in \bar{P}$. Form the singular direct products $(S, \oplus_1)$ and $(S, \oplus_2)$ defined below.

(1) $(S, \oplus_1) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(v) = \circ_1$, all $v \in V$, $P = T$, and $(\bar{P}, \otimes_1)$ is used to define $(\bar{P}, \otimes (u, v, w))$.

(2) $(S, \oplus_2) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(v) = \circ_2$, all $v \in V$, $P = T$, and $(\bar{P}, \otimes_2)$ is used to define $(\bar{P}, \otimes (u, v, w))$.

Clearly both of $\oplus_1$ and $\oplus_2$ agree on T. There are two cases to consider.

(1) $x \neq y \in P \cup (\bar{P} \times \{v\})$, $v \in V$. Since $(P \cup (\bar{P} \times \{v\}), \oplus_1)$ is a copy of

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$(Q, \circ_1)$ and $(P \cup (\bar{P} \times \{v\}), \oplus_2)$ is a copy of $(Q, \circ_2)$ it follows that $x \oplus_1 y = x \oplus_2 y$ if and only if $x, y \in T$.

(2) $x = (p, v), y = (q, w), v \neq w$. This is identical to case (3) in the proof of Theorem 2.

Combining cases (1) and (2) shows that $(S, \oplus_1)$ and $(S, \oplus_2)$ are a pair of $(v(q-3)+3, 1)$ Steiner quasigroups.

THEOREM 4. *If* $v \equiv 1$ *or* 3 (mod 6) *there is a pair of* $(2v+1, 1)$ *Steiner quasigroups.*

Proof. Let $(V, \odot)$ be a Steiner quasigroup based on $1, 2, \ldots, v$ and let $(Q, \circ)$ be the Steiner quasigroup of order 3 with $Q = \{1, 2, 3\}$. Take $P = \{1\}$ as a subquasigroup of order 1. Set $\bar{P} = \{2, 3\}$ and define quasigroups $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ by the following tables.

$\otimes_1$	2	3
2	2	3
3	3	2

$(\bar{P}, \otimes_1)$

$\otimes_2$	2	3
2	3	2
3	2	3

$(\bar{P}, \otimes_2)$

Note that $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ are both totally symmetric and therefore invariant under conjugation. Also $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ are totally disjoint; i.e., $x \otimes_1 y \neq x \otimes_2 y$ for all $x, y \in \bar{P}$. Now form the following singular direct products.

(1) $(S, \oplus_1) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(v) = \circ$, all $v \in V$, $P = \{1\}$, and $(\bar{P}, \otimes_1)$ is used to define $(\bar{P}, \otimes(u, v, w))$.

(2) $(S, \oplus_2) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(v) = \circ$, all $v \in V$, $P = \{1\}$, and $(\bar{P}, \otimes_2)$ is used to define $(\bar{P}, \otimes(u, v, w))$.

Let (S, t_1) and (S, t_2) be the associated Steiner triple systems. It is a routine matter to show that (S, t_1) and (S, t_2) have exactly the triples $\{1, (2, v), (3, v)\}$, all $v \in V$, in common. Let α be the permutation on V defined by $\alpha = (2\,3)(4\,5) \ldots (v - 1v)$ and denote by $(S, t_1\alpha)$ the Steiner triple system obtained from (S, t_1) by replacing all ordered pairs $(2, v)$ by $(2, v\alpha)$. Claim: $(S, t_1\alpha)$ and (S, t_2) intersect exactly in the triple $\{1, (2, 1), (3, 1)\}$. Since α fixes 1, the Steiner triple systems $(S, t_1\alpha)$ and (S, t_2) have the triple $\{1, (2, 1), (3, 1)\}$ in common. Now let t be any triple in $t_1\alpha$ other than $\{1, (2,1), (3, 1)\}$. There are two cases to consider.

(i) $1 \in t$. Since each triple in t_1 containing 1 is of the form $\{1, (2, v), (3, v)\}$, each triple in $t_1\alpha$ containing 1 is of the form $\{1, (2, v\alpha), (3, v)\}$. Since $v \neq 1$, $v\alpha \neq v$ so that t cannot belong to t_2.

(ii) $1 \notin t$. In t_1 each triple not containing 1 is of the form $\{(p, v), (q, w), (p \otimes_1 q, v \odot w)\}$. Hence in $t_1\alpha$ each triple not containing 1 is of the form

$$\{(p, v\alpha), (q, w\alpha), (p \otimes_1 q, v\alpha \odot w\alpha)\}.$$

This triple cannot belong to t_2 because $p \otimes_1 q \neq p \otimes_2 q$ for all $p, q \in \bar{P}$. Com-

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bining cases (i) and (ii) shows that $(S, t_1\alpha)$ and (S, t_2) intersect in exactly one triple completing the proof.

THEOREM 5. *If* $v \equiv 1$ *or* 3 (mod 6) *there is a pair of* $(40v + 9, 1)$ *Steiner quasigroups.*

Proof. We begin by constructing a certain pair of (49, 1) Steiner quasigroups. Let $(V, \odot)$ be the Steiner quasigroup of order 7 based on 1, 2, 3, 4, 5, 6, 7. We can assume that $\{1, 2, 3\}$ belongs to the associated triple system. Let $(Q, \circ_1)$ and $(Q, \circ_2)$ be a pair of (7, 1) Steiner quasigroups based on 1, 2, 3, 4, 5, 6, 7 and agreeing exactly on $T = \{1, 2, 3\}$. Let $(Q, \bar{\circ}_1)$ and $(Q, \bar{\circ}_2)$ be a pair of (7, 0) Steiner quasigroups and $(Q, \otimes_1)$ and $(Q, \otimes_2)$ a pair of totally disjoint quasigroups. Finally, let $(Q, \overline{\otimes}_1)$ and $(Q, \overline{\otimes}_2)$ be the quasigroups defined below.

$\otimes_1$	1	2	3	4	5	6	7
1	1	3	2	6	7	4	5
2	3	2	1	5	4	7	6
3	2	1	3	7	6	5	4
4	6	5	7	4	2	1	3
5	7	4	6	2	5	3	1
6	4	7	5	1	3	6	2
7	5	6	4	3	1	3	7

$(Q, \overline{\otimes}_1)$

$\otimes_2$	1	2	3	4	5	6	7
1	1	3	2	7	4	5	6
2	3	2	1	6	5	4	7
3	2	1	3	4	7	6	5
4	7	6	4	5	3	2	1
5	4	5	7	3	6	1	2
6	5	4	6	2	1	7	3
7	6	7	5	1	2	3	4

$(Q, \overline{\otimes}_2)$

Note that both $\overline{\otimes}_1$ and $\overline{\otimes}_2$ agree on $\{1, 2, 3\}$ while if at least one of $x, y \notin \{1, 2, 3\}$ then $x \overline{\otimes}_1 y \neq x \overline{\otimes}_2 y$ and this is true even if $x = y$. It also remains true for corresponding conjugates. Note also that $(\{1, 2, 3\}, \overline{\otimes}_1) = (\{1, 2, 3\}, \overline{\otimes}_2)$ is the Steiner quasigroup of order 3 so that while the quasigroups $(Q, \overline{\otimes}_1)$ and $(Q, \overline{\otimes}_2)$ are not necessarily invariant under conjugation the subquasigroups $(\{1, 2, 3\}, \overline{\otimes}_1)$ and $(\{1, 2, 3\}, \overline{\otimes}_2)$ are. Now form the following singular direct products.

(1) $(S, \oplus_1) = V(\odot) \times Q(\circ(v), \emptyset, \bar{P} \otimes (u, v, w))$, where $\circ(1) = \circ(2) = \circ(3) = \circ_1$, $\circ(4) = \circ(5) = \circ(6) = \circ(7) = \bar{\circ}_1$, $P = \emptyset$, $(Q, \overline{\otimes}_1)$ is used to define $(\bar{P}, \otimes (u, v, w))$ for the triple $\{1, 2, 3\}$, and $(Q, \otimes_1)$ is used to define $(\bar{P}, \otimes (u, v, w))$ for the remaining triples.

(2) $(S, \oplus_2) = V(\odot) \times Q(\circ(v), \emptyset, \bar{P} \otimes (u, v, w))$, where $\circ(1) = \circ(2) = \circ(3) = \circ_2$, $\circ(4) = \circ(5) = \circ(6) = \circ(7) = \bar{\circ}_2$, $P = \emptyset$, $(Q, \overline{\otimes}_2)$ is used to define $(\bar{P}, \otimes (u, v, w))$ for the triple $\{1, 2, 3\}$, and $(Q, \otimes_2)$ is used to define $(\bar{P}, \otimes (u, v, w))$ for the remaining triples.

Clearly $|S| = 49$. We show that $(S, \oplus_1)$ and $(S, \oplus_2)$ agree exactly on the 9 element subquasigroup $T = \{(i, j) | i, j \in \{1, 2, 3\}\}$. Let $x, y \in S$. There are two main cases to consider.

(1) $x = (p, v)$, $y = (q, v)$, $p \neq q$. If $v = 1, 2$, or 3, since $(Q \times \{v\}, \oplus_1)$ is a copy of $(Q, \circ_1)$ and $(Q \times \{v\}, \oplus_2)$ is a copy of $(Q, \circ_2)$ it follows that $x \oplus_1 y = x \oplus_2 y$ if and only if x and y both belong to T. If $v = 4, 5, 6$ or 7, since $(Q \times \{v\},$

$\oplus_1$) is a copy of $(Q, \bar{\circ}_1)$ and $(Q \times \{v\}, \oplus_2)$ is a copy of $(Q, \bar{\circ}_2)$ it follows that $x \oplus_1 y \neq x \oplus_2 y$.

(2) $x = (p, v)$, $y = (q, w)$, $v \neq w$. If $v, w \in \{1, 2, 3\}$, since the quasigroups associated with this triple in $(S, \oplus_1)$ and $(S, \oplus_2)$ are $(Q, \overline{\otimes}_1)$ and $(Q, \overline{\otimes}_2)$ respectively, it follows that $x \oplus_1 y = x \oplus_2 y$ if and only if $p, q \in \{1, 2, 3\}$; i.e., if and only if $x, y \in T$. If at least one of v, w does not belong to $\{1, 2, 3\}$, then the quasigroups associated with the triple $\{v, w, v \odot w\}$ in $(S, \oplus_1)$ and $(S, \oplus_2)$ are $(Q, \otimes_1)$ and $(Q, \otimes_2)$ respectively and so $x \oplus_1 y \neq x \oplus_2 y$. Now combining cases (1) and (2) shows that $(S, \oplus_1)$ and $(S, \oplus_2)$ agree exactly on T. Now let $(T, *_1)$ and $(T, *_2)$ be a pair of (9, 1) Steiner quasigroups. Unplug the subquasigroups $(T, \oplus_1)$ and $(T, \oplus_2)$ from $(S, \oplus_1)$ and $(S, \oplus_2)$ and replace them with the quasigroups $(T, *_1)$ and $(T, *_2)$. The result is a pair of (49, 1) Steiner quasigroups containing a pair of (9, 1) Steiner quasigroups.

We are now in a position to prove the statement of the theorem. Let $(V, \odot)$ be any Steiner quasigroup and $(Q, \circ_1)$ and $(Q, \circ_2)$ a pair of (49, 1) Steiner quasigroups containing a pair of (9, 1) Steiner quasigroups $(P, \circ_1)$ and $(P, \circ_2)$. Set $\bar{P} = Q \backslash P$ and let $(\bar{P}, \otimes_1)$ and $(\bar{P}, \otimes_2)$ be a pair of totally disjoint quasigroups. Now form the following singular direct products.

(1) $(S, \oplus_1) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(v) = \circ_1$, all $v \in V$ and $(\bar{P}, \otimes_1)$ is used to define $(\bar{P}, \otimes (u, v, w))$.

(2) $(S, \oplus_2) = V(\odot) \times Q(\circ(v), P, \bar{P} \otimes (u, v, w))$, where $\circ(v) = \circ_2$, all $v \in V$, and $(\bar{P}, \otimes_2)$ is used to define $(\bar{P}, \otimes (u, v, w))$.

Clearly S has order $v(49 - 9) + 9$. The proof that $(S, \oplus_1)$ and $(S, \oplus_2)$ are a pair of $(40v + 9, 1)$ Steiner quasigroups is analogous to the proof of Theorem 3.

4. Construction of a pair of (q, 1) Steiner quasigroups of every order. We begin by exhibiting pairs of Steiner triple systems intersecting in exactly one triple of orders 7, 9, and 13.

(1) $n = 7$. $S = \{1, 2, 3, 4, 5, 6, 7\}$,

$$t_1 = \{\{1, 2, 3\}, \{2, 6, 7\}, \{6, 4, 1\}, \{4, 5, 2\}, \{5, 3, 6\}, \{3, 7, 4\}, \{7, 1, 5\}\},$$

and

$$t_2 = \{\{1, 2, 3\}, \{6, 2, 5\}, \{4, 3, 6\}, \{5, 3, 7\}, \{4, 5, 1\}, \{7, 4, 2\}, \{1, 7, 6\}\}.$$

(2) $n = 9$. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$,

$$t_1 = \{\{1, 2, 3\}, \{1, 4, 7\}, \{1, 5, 9\}, \{1, 6, 8\}, \{4, 5, 6\}, \{2, 5, 8\}, \{2, 6, 7,\} \{2, 4, 9\}, \{7, 8, 9\}, \{3, 6, 9\}, \{3, 4, 8\}, \{3, 5, 7\}\},$$

and

$$t_2 = \{\{1, 2, 3\}, \{1, 5, 8\}, \{1, 6, 4\}, \{1, 7, 9\}, \{5, 6, 7\}, \{2, 6, 9\}, \{2, 7, 8\}, \{2, 5, 4\}, \{8, 9, 4\}, \{3, 7, 4\}, \{3, 5, 9\}, \{3, 6, 8\}\}.$$

(3) $n = 13$. $S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$,

$t_1 = \{\{1, 2, 3\}, \{2, 11, 12\}, \{11, 4, 13\}, \{4, 5, 1\}, \{5, 6, 2\}, \{6, 7, 11\},$
$\{7, 8, 4\}, \{8, 9, 5\}, \{9, 10, 6\}, \{10, 3, 7\}, \{3, 12, 8\}, \{12, 13, 9\},$
$\{13, 1, 10\}, \{1, 11, 8\}, \{2, 4, 9\}, \{11, 5, 10\}, \{4, 6, 3\}, \{5, 7, 12\},$
$\{6, 8, 13\}, \{7, 9, 1\}, \{8, 10, 2\}, \{9, 11, 3\}, \{10, 12, 4\}, \{3, 13, 5\},$
$\{12, 1, 6\}, \{13, 2, 7\}\},$

and

$t_2 = \{\{1, 2, 3\}, \{11, 2, 6\}, \{4, 11, 7\}, \{3, 4, 8\}, \{6, 3, 9\}, \{7, 6, 10\}, \{8, 7, 5\},$
$\{9, 8, 12\}, \{10, 9, 13\}, \{5, 10, 1\}, \{12, 5, 2\}, \{13, 12, 11\}, \{1, 13, 4\},$
$\{11, 1, 9\}, \{4, 2, 10\}, \{5, 3, 11\}, \{6, 4, 12\}, \{7, 3, 13\}, \{8, 6, 1\},$
$\{9, 7, 2\}, \{10, 8, 11\}, \{5, 9, 4\}, \{12, 10, 3\}, \{13, 5, 6\}, \{1, 12, 7\},$
$\{2, 13, 8\}\}.$

The following equalities are due to A. J. W. Hilton [**2**].

$$36t + 1 = 3((12t + 1) - 1) + 1,$$
$$36t + 3 = 2(18t + 1) + 1,$$
$$36t + 7 = (6t + 1)(7 - 1) + 1,$$
$$36t + 9 = (6t + 1)(9 - 3) + 3,$$
$$36t + 13, \text{ see below},$$
$$36t + 15 = 2(18t + 7) + 1$$
$$36t + 19 = (6t + 3)(7 - 1) + 1$$
$$36t + 21 = (6t + 3)(9 - 3) + 3$$
$$36t + 25 = 3((12t + 9) - 1) + 1,$$
$$36t + 27 = 2(18t + 13) + 1,$$
$$36t + 31 = 2(18t + 15) + 1,$$
$$36t + 33 = 3(12t + 13) - 3) + 3.$$

In view of these equalities, Theorems 2, 3, 4 and 5, and the examples at the beginning of this section, a pair of $(q, 1)$ Steiner quasigroups of every order can be constructed provided we can fill in the details when $n = 36t + 13$. In [**2**], Hilton has shown that $36t + 13$ can always be expressed in one of the following forms.

$$(6n + 1)(k - 1) + 1, k \equiv 1 \text{ or } 3 \pmod 6,$$
$$(18n + 15)(s - 1) + 1, s \equiv 1 \text{ or } 3 \pmod 6,$$
$$u(13 - 3) + 3, u \equiv 1 \text{ or } 3 \pmod 6, \text{ or}$$
$$w(49 - 9) + 9, w \equiv 1 \text{ or } 3 \pmod 6.$$

Theorem 5 is necessary for the use of the very last equality. Combining all of the results in this section gives the following theorem.

THEOREM 6. *There is a pair of Steiner triple systems of order n having exactly one triple in common for every $n \geqq 3$.*

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