32

(Pseudo)scalar correlators

We shall be concerned with the two-point correlators:

$$\Psi_5(q^2)^i_j \equiv i \int d^4x\, e^{iqx} \langle 0|\mathcal{T}\partial_\mu A^\mu(x)^j_i \left(\partial_\nu A^\nu(0)^j_i\right)^\dagger |0\rangle ,$$
$$\Psi(q^2)^i_j \equiv i \int d^4x\, e^{iqx} \langle 0|\mathcal{T}\partial_\mu V^\mu(x)^j_i \left(\partial_\nu V^\nu(0)^j_i\right)^\dagger |0\rangle , \qquad (32.1)$$

associated to the pseudoscalar and scalar currents:

$$\partial_\mu A^\mu_{ij} = (m_i + m_j)\bar\psi_i\,(i\gamma_5)\,\psi_j$$
$$\partial_\mu V^\mu_{ij} = (m_i - m_j)\bar\psi_i(i)\psi_j \qquad (32.2)$$

Here the indices $i,\ j$ correspond to the light quark flavours $u,\ d,\ s$; m_i is the mass of the quark i. It will be convenient to introduce the notation:

$$m_\pm = m_i \pm m_j\ . \qquad (32.3)$$

The result of the scalar current can be deduced from the one of the pseudoscalar by the change m_j into $-m_j$ or, equivalently, by the change m_+ into m_- and vice-versa.

32.1 Exact two-loop perturbative expression in the $\overline{MS}$ scheme

The *complete* two-loop result for the pseudoscalar correlator, using the $\overline{MS}$ renormalized mass is:

$$\Psi_5(q^2)^i_j = \frac{3}{8\pi^2}(m_i+m_j)^2\left[(q^2-(m_i-m_j)^2)\left[K+\left(\frac{\alpha_s}{\pi}\right)\frac{L}{3}\right]+M+\left(\frac{\alpha_s}{\pi}\right)\frac{N}{3}\right] , \qquad (32.4)$$

with:

$$K \equiv 1+\frac{1}{2}l_i+\frac{1}{2}(1+x_i)f_i \ +\ (i \longleftrightarrow j)\,,$$
$$L \equiv 3K^2+2K+6-2EI-10x_i f_i^2$$
$$\quad + m_i\big[(3K-2)\,\partial K/\partial m_i - 2\big(E+m_j^2\big)\partial I/\partial m_i\big] + (i \longleftrightarrow j)\,,$$
$$M \equiv -m_i^2(1+l_i)+\ \ (i \longleftrightarrow j)\,,$$

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$$N \equiv -\frac{1}{8}q^2(12K^2 - 4K + 5) + 3m_i^2(1 + f_i)(1 + x_j f_j)$$
$$\quad - 2m_i^2\left(5 + 5l_i + 3l_i^2\right) + (i \longleftrightarrow j)\ ,$$
$$I \equiv [F(x_i) + F(x_j) - F(x_i x_j) - F(1)]/q^2,$$
$$E \equiv \frac{1}{2}\left(m_i^2 + m_j^2 - q^2\right)\ ,$$
$$l_i \equiv \log\left(\nu^2/m_i^2\right), \quad f_i \equiv \frac{\log x_i}{(1 - x_i)}\ ,$$
$$x_{i,j} \equiv m_{i,j}^2/\{E + E[1 - (m_i m_j/E)^2]^{1/2}\}\ ,$$
$$F(x) \equiv \int_o^x dy \left(\frac{\log y}{1 - y}\right)^2 \log\left(\frac{x}{y}\right) = \sum_{n=1}^{\infty}[(2 - n\log x)^2 + 2]x^n/n^3\ . \tag{32.5}$$

This expression reproduces the massless result given previously in Section 11.14. The use of these results at $q = 0$ lead to the two-loop expression given in Eq. (27.14).

32.2 Three-loop expressions in the chiral limit

To order α_s^2, the correlator reads:[1]

$$(16\pi^2)\Psi_5(q^2)$$
$$= -q^2 m_+^2\Bigg[\left[-12 - 6\ln\frac{\nu^2}{-q^2}\right]$$
$$+ \left(\frac{\alpha_s}{\pi}\right)\left[-\frac{131}{2} + 24\,\zeta(3) - 34\ln\frac{\mu^2}{Q^2} - 6\ln^2\frac{\mu^2}{Q^2}\right]$$
$$+ \left(\frac{\alpha_s}{\pi}\right)^2\Bigg[-\frac{17645}{24} + 353\,\zeta(3) + \frac{3}{2}\,\zeta(4) - 50\,\zeta(5)$$
$$+ \frac{511}{18}n_f - 8\,\zeta(3)\,n_f - \frac{10801}{24}\ln\frac{\nu^2}{-q^2} + 117\,\zeta(3)\ln\frac{\nu^2}{-q^2}$$
$$+ \frac{65}{4}n_f\ln\frac{\nu^2}{-q^2} - 4\,\zeta(3)\,n_f\ln\frac{\nu^2}{-q^2} - 106\ln^2\frac{\nu^2}{-q^2} + \frac{11}{3}n_f\ln^2\frac{\nu^2}{-q^2}$$
$$- \frac{19}{2}\ln^3\frac{\nu^2}{-q^2} + \frac{1}{3}n_f\ln^3\frac{\nu^2}{-q^2}\Bigg]\Bigg]\ . \tag{32.6}$$

The same equation with $n_f = 3$ reads:

$$(16\pi^2)\Psi_5(q^2)$$
$$= -q^2 m_+^2\Bigg[\left[-12 - 6\ln\frac{\nu^2}{-q^2}\right]$$
$$+ \left(\frac{\alpha_s}{\pi}\right)\left[-\frac{131}{2} + 24\,\zeta(3) - 34\ln\frac{\nu^2}{-q^2} - 6\ln^2\frac{\nu^2}{-q^2}\right]$$

[1] From now, we shall omit the indices i and j on $\Psi_5(q^2)^i_j$.

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$$+\left(\frac{\alpha_s}{\pi}\right)^2\left[-\frac{15601}{24}+329\,\zeta(3)+\frac{3}{2}\,\zeta(4)-50\,\zeta(5)-\frac{9631}{24}\ln\frac{\nu^2}{-q^2}\right.$$
$$\left.+105\,\zeta(3)\ln\frac{\nu^2}{-q^2}-95\ln^2\frac{\nu^2}{-q^2}-\frac{17}{2}\ln^3\frac{\nu^2}{-q^2}\right]\Bigg]. \qquad (32.7)$$

32.3 Dimension-two

For a practical application, one should substract the mass singularities with the help of the Ward identity in Eqs. (2.17) and (27.14) and by working with the non-normal ordered condensate. To next-to-leading order in the quark mass terms, the IR stable result is:

$$\Psi_\pm(q^2)|^{(D=2)}=\frac{3}{8\pi^2}(m_i+m_j)^2\left[A\left(m_i^2\pm m_j^2\right)\mp Bm_im_j\right], \qquad (32.8)$$

where:

$$A\equiv 2l-2+C_F\left(\frac{\alpha_s}{\pi}\right)\left[-3l^2+8l-\frac{25}{2}+6\zeta(3)\right],$$
$$B\equiv 2l-4+C_F\left(\frac{\alpha_s}{\pi}\right)[-3l^2+14l-22+6\zeta(3)], \qquad (32.9)$$

with: $C_F=(N^2-1)/(2N)$ and $l=\log(-q^2/\nu^2)$; Ψ_+ and Ψ_- are the pseudoscalar and scalar correlators.

32.4 Dimension-four

In terms of the non-normal ordered quark condensate, where the $m^4\log m^2$ terms have been absorbed, one obtains:

$$\Psi_\pm(q^2)\Big|_{m^4}^{(D=4)}=\frac{3}{8\pi^2}(m_i\pm m_j)^2\frac{1}{-2q^2}\left[m_i^4+4m_i^2m_j^2+m_j^4\right.$$
$$\left.+2\left(m_i^4\mp 2m_i^3m_j\mp 2m_j^3m_i+m_j^4\right)l\right]. \qquad (32.10)$$

To lowest order in α_s and to *all orders* in the quark mass, the normal ordered quark condensate contribution reads:

$$\Psi_\pm(q^2)\Big|_{\psi}^{(D=4)}=-(m_i\pm m_j)^2\left[\frac{\langle:\bar\psi_i\psi_i:\rangle}{2m_i}\left[1-\frac{q_\pm^2}{q^2-m_i^2-m_j^2}f(z_i)\right]+(i\longleftrightarrow j)\right], \qquad (32.11)$$

where:

$$f(z_i)=\frac{1}{2z_i}[1-\sqrt{1-4z_i}],$$
$$z_i\equiv\frac{m_i^2q^2}{\left(q^2-m_j^2+m_i^2\right)^2},$$

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$$
\begin{aligned}
q_{\pm} &\equiv q^2 - (m_i \pm m_j)^2 \ , \\
u &\equiv \sqrt{1 - \frac{4m_i m_j}{q_-^2}} \ .
\end{aligned}
\tag{32.12}
$$

To order α_s and to leading order in the quark mass, one obtains:

$$
\begin{aligned}
\Psi_{\pm}(q^2 0\big|_{\psi}^{(D=4)} = \frac{(m_i \pm m_j)^2}{-q^2} &\left[\frac{1}{2} \left[1 + C_F \left(\frac{\alpha_s}{\pi} \right) \left(-\frac{3}{2} l + \frac{11}{4} \right) \right] \langle m_i \bar\psi_i \psi_i + m_j \bar\psi_j \psi_j \rangle \right. \\
&\left. \mp \left[1 + C_F \left(\frac{\alpha_s}{\pi} \right) \left(-\frac{3}{2} l + \frac{7}{2} \right) \right] \langle m_j \bar\psi_i \psi_i + m_i \bar\psi_j \psi_j \rangle \right] ,
\end{aligned}
\tag{32.13}
$$

To all orders in the quark mass, one obtains the contribution of the normal ordered gluon condensates:

$$
\begin{aligned}
\Psi_{\pm}(q^2)\big|_{G}^{(D=4)} &= -(m_i \pm m_j)^2 \frac{1}{48\pi} \langle \ :\alpha_s G G: \ \rangle \\
&\times \left[\frac{q^2}{q_{\pm}^4} \left[\frac{3(3+u^2)(1-u^2)}{2u^3} \log \frac{u+1}{u-1} - \frac{3u^4 + 4u^2 + 9}{u^2(1-u^2)} \right] \pm \frac{4}{m_i m_j} \right] ,
\end{aligned}
\tag{32.14}
$$

where the expression of the scalar correlator can be deduced from the former by the additional change of u into $1/u$. The previous expression still contains mass singularities. The introduction of *non-normal ordered* condensates as given in Eq. (27.17) leads to a cancellation of these terms. One obtains the IR stable result:

$$
\begin{aligned}
\Psi_{\pm}(q^2)\big|_{G}^{(D=4)} &= (m_i \pm m_j)^2 \frac{1}{-q^2} \frac{1}{8\pi} \langle \alpha_s G G \rangle \\
&\times \left[1 + \frac{2}{3} \frac{m_i^2 + m_j^2}{q^2} \pm \frac{2 m_i m_j}{q^2} (3 - 2l) \right] ,
\end{aligned}
\tag{32.15}
$$

where the mass-logs have cancelled.

32.5 Dimension-five

To all order in the quark mass, the contribution of the normal ordered mixed quark-gluon condensate reads:

$$
\begin{aligned}
\Psi_{\pm}(q^2)\big|_{\text{mix}}^{(D=5)} = -(m_i \pm m_j)^2 \langle \ :\bar\psi_i G \psi_i: \ \rangle \frac{1}{2 m_i^3 q_{\pm}^2} &\left[q^2 - m_j^2 \mp m_i m_j \right. \\
&\left. - \frac{\left[\left(q^2 - m_j^2\right)^2 \mp m_i m_j \left(q^2 - m_j^2 + m_i^2\right) - m_i^2 m_j^2 \right]}{q^2 - m_j^2 - m_i^2} f(z_i) \right] \\
&+ (i \longleftrightarrow j) \ .
\end{aligned}
\tag{32.16}
$$

The leading contribution of the non-normal ordered condensate is:

$$\Psi_\pm(q^2)\Big|_{\rm mix}^{(D=6)} = \mp\frac{(m_i \pm m_j)^2}{2q^2} g\langle m_i\bar\psi_j G\psi_j + m_j\bar\psi_i G\psi_i\rangle\ . \tag{32.17}$$

32.6 Dimension-six

The leading contributions are:

$$\begin{aligned}\Psi_\pm(q^2)\Big|_{4\psi}^{(D=6)} = \frac{(m_i \pm m_j)^2}{q^4}\pi\alpha_s\Bigg[&\mp 4\langle\bar\psi_i\sigma_{\mu\nu}T_A\psi_j\bar\psi_j\sigma_{\mu\nu}T_A\psi_i\rangle \\ &+\frac{4}{3}\Bigg\langle 9\bar\psi_i\gamma_\mu T_A\psi_i + \bar\psi_j\gamma_\mu T_A\psi_j)\sum_{u,d,s}\bar\psi_k\gamma^\mu T^A\psi_k\Bigg\rangle\Bigg]\ .\end{aligned} \tag{32.18}$$

Using the vacuum saturation-like parametrization, one can write:

$$\Psi_\pm(q^2)\Big|_{4\psi}^{(D=6)} = -\frac{(m_i \pm m_j)^2}{q^4}\left(\frac{4C_F}{3N}\right)\pi\alpha_s[\langle\bar\psi_i\psi_i\rangle^2 + \langle\bar\psi_j\psi_j\rangle^2 \mp 9\langle\bar\psi_i\psi_i\rangle\langle\bar\psi_j\psi_j\rangle]\ . \tag{32.19}$$

32.7 Exact two-loop expression of the spectral function

The complete two-loop pseudoscalar spectral function expressed in terms of the pole mass reads:

$$\begin{aligned}\frac{1}{\pi}{\rm Im}\Psi_5(t) = \frac{3}{8\pi^2}(m_i+m_j)^2\frac{\bar q^4}{t}v\Bigg[1+\frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\Bigg[&\frac{3}{8}(7-v^2) \\ &+\sum_i (v+v^{-1})\,[{\rm Li}_2(\alpha_i\alpha_j) - {\rm Li}_2(\alpha_i) - \log\alpha_i\log\beta_i] \\ &+A_i\log\alpha_i + B_i\log\beta_i\Bigg] + \mathcal{O}\big(\alpha_s^2\big)\Bigg]\ ,\end{aligned} \tag{32.20}$$

where:

$$\begin{aligned}{\rm Li}_2(x) &= -\int_0^x \frac{dx}{x}\log(1-x)\ ,\\ A_i &= \frac{3}{4}\left(\frac{3m_i+m_j}{m_i+m_j}\right) - \frac{19+2v^2+3v^4}{32v} - \frac{m_i(m_i-m_j)}{\bar q^2 v(1+v)}\left(1+v+\frac{2v}{1+\alpha_i}\right)\ ,\\ B_i &= 2+2\frac{\left(m_i^2-m_j^2\right)}{\bar q^2 v}\ ,\\ \alpha_i &= \frac{m_i}{m_j}\frac{1-v}{1+v}\ ,\\ \beta_i &= \frac{(1+v)^2}{4v}\sqrt{1+\alpha_i}\ ,\end{aligned} \tag{32.21}$$

with:

$$\bar{q}^2 \equiv t - (m_i - m_j)^2 ,$$
$$v \equiv \sqrt{1 - \frac{4m_i m_j}{\bar{q}^2}} , \tag{32.22}$$

while A_j, α_j, B_j and β_j can be obtained by interchanging the label i and j. The spectral function of the scalar current $\partial^\mu \left(\bar{\psi}_i \gamma_\mu \psi_j\right)$ can be obtained from the former by changing the sign of m_j. For $m_i = 0$, one has $A_i = 0$ and $\beta_i = 1$, which guarantee the absence of mass singularities. In this case, the expression simplifies and reads ($m \equiv m_i$):

$$\frac{1}{\pi}\mathrm{Im}\Psi_5(t) = \frac{3}{8\pi^2} x(1-x)^2 t^2 \left[1 + \frac{4}{3}\left(\frac{\alpha_s}{\pi}\right)\left[\frac{9}{4} + \mathrm{Li}_2(x) + \log x \log(1-x) - \frac{3}{2}\log\left(\frac{1}{x} - 1\right) - \log(1-x) + x\log\left(\frac{1}{x} - 1\right) - \frac{x}{1-x}\log x\right]\right]\theta(t - m^2) , \tag{32.23}$$

where $x \equiv m^2/t$.

The order α_s^2 corrections to the (pseudo)scalar spectral function has been also obtained recently for massive quarks [448] where the result is available in a Mathematica package Rvs.m from the url: *http://www-ttp.physik.uni-karlsruhe.de/Progdata/ttp00/ttp00-25*.

32.8 Heavy-light correlator

We have given in the previous section the expression of the spectral function when one of the quark mass goes to zero. In the following, we give useful lowest order expressions in α_s when $m \equiv m_i \ll M \equiv m_j$ for the (pseudo)scalar current. In taking the small mass m limit, different cares have been taken in order to have IR stable results. Expressing the correlator as:

$$\Psi_\pm = (m \pm M)^2 \left[\Psi_\pm^{pert} + \Psi_\pm^{\bar{\psi}\psi}\langle\bar{\psi}\psi\rangle + \Psi_\pm^{\bar{Q}Q}\langle\bar{Q}Q\rangle + \Psi_\pm^{G^2}\langle\alpha_s G^2\rangle + \Psi_\pm^{\bar{\psi}G\psi}\langle\bar{\psi}\frac{\lambda_a}{2}\sigma^{\mu\nu}G^a_{\mu\nu}\psi\rangle + \Psi_\pm^{\bar{Q}GQ}\langle\bar{Q}\frac{\lambda_a}{2}\sigma^{\mu\nu}G^a_{\mu\nu}Q\rangle\right] \tag{32.24}$$

One obtains[488] ($W \equiv M^2 - q^2$):

$$\Psi_\pm^{pert} = \frac{N}{8\pi^2}\left[2q^2 - 3M^2 + \frac{M^4}{q^2}\ln\frac{M^2}{W} - (2M^2 - q^2)\ln\frac{\mu^2}{W} \pm 2mM\left(2 - \frac{M^2}{q^2}\ln\frac{M^2}{W} + \ln\frac{\mu^2}{W}\right) - 2m^2\left(1 + \ln\frac{\mu^2}{W}\right) \mp 2\frac{m^3 M}{W}\left(1 - \frac{M^2}{q^2}\ln\frac{M^2}{W} - \ln\frac{m^2}{W}\right) + \frac{m^4}{W^2}\left[M^2 - \frac{3}{2}q^2 - \frac{M^4}{q^2}\ln\frac{M^2}{W} - (2M^2 - q^2)\ln\frac{m^2}{W}\right]\right]$$

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$$\Psi_{\pm}^{\bar\psi\psi} = \left[\mp\frac{Mq^2}{W} + \frac{mq^2(2M^2-q^2)}{2W^2} \mp \frac{m^2M^3q^2}{W^3}\right]$$

$$\Psi_{\pm}^{\bar QQ} = \left[-\frac{M}{2} \pm m \mp \frac{m^3}{W}\right] \qquad \text{for} \quad -q^2 > M^2$$

$$\Psi_{\pm}^{G^2} = \frac{1}{12\pi W}\left[\pm\frac{Mq^2}{m} - \frac{q^4}{2W} \pm mq^4MW^2\left(q^2+6M^2\ln\frac{mM}{W}\right)\right.$$
$$\left. -\frac{m^2q^4}{W^3}\left(q^2+7M^2+6M^2\ln\frac{mM}{W}\right)\right]$$

$$\Psi_{\pm}^{\bar\psi G\psi} = \left[\pm\frac{Mq^6}{2W^3} - \frac{mM^2q^6}{2W^4}\right]$$

$$\Psi_{\pm}^{\bar QGQ} = \pm\frac{mq^2}{2W} \qquad \text{for} \quad -q^2 > M^2 \tag{32.25}$$

One can notice that some of these terms are IR singular and behave like $\log m$ and $1/m$. In order to have an IR stable result, one should work with the renormalized condensates defined in Eq. (27.56). In this way, one obtains:

$$\bar\Psi_{\pm}^{pert} = \Psi_{\pm}^{pert} + \frac{3}{\pi^2}\left[\frac{M^3}{q^2}\left[\log\frac{M^2}{\mu^2}-1\right]\Psi_{\pm}^{\bar QQ} + \frac{m^3}{q^2}\left[\log\frac{m^2}{\mu^2}-1\right]\Psi_{\pm}^{\bar\psi\psi}\right],$$
$$\bar\Psi_{\pm}^{\bar QQ} = \Psi_{\pm}^{\bar QQ},$$
$$\bar\Psi_{\pm}^{\bar\psi\psi} = \Psi_{\pm}^{\bar\psi\psi},$$
$$\bar\Psi_{\pm}^{G^2} = \Psi_{\pm}^{G^2} + \frac{1}{12M}\Psi_{\pm}^{\bar QQ} + \frac{1}{12m}\Psi_{\pm}^{\bar\psi\psi} - \frac{M}{2q^2}\log\frac{M^2}{\mu^2}\Psi_{\pm}^{\bar QGQ} - \frac{m}{2q^2}\log\frac{m^2}{\mu^2}\Psi_{\pm}^{\bar\psi G\psi},$$
$$\bar\Psi_{\pm}^{\bar QGQ} = \Psi_{\pm}^{\bar QGQ},$$
$$\bar\Psi_{\pm}^{\bar\psi G\psi} = \Psi_{\pm}^{\bar\psi G\psi} = \mp\frac{M}{2}. \tag{32.26}$$

Therefore, one can deduce for small m:

$$\bar\Psi_{\pm}^{pert} = \frac{3}{16\pi^2}\left[4q_{\mp}^2 - (M^4+4M^2m^2+m^4)\frac{1}{q^2}\right.$$
$$\left. -2\left[q^2 \mp -M^2-m^2+(M^4\mp 2M^3m \mp 2Mm^3+m^4)\frac{1}{q^2}\right]\log\frac{-q^2}{\mu^2}\right].$$
$$\bar\Psi_{\pm}^{G^2} = -\frac{1}{8} - \frac{(M^2+m^2)}{12q^2} \mp \frac{Mm}{4q^2}\left[3-2\log\frac{-q^2}{\mu^2}\right], \tag{32.27}$$

where:

$$q_{\mp}^2 = q^2 - (M\mp m)^2, \tag{32.28}$$

and where all IR infinities have disappeared.

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