

The geometry of measured geodesic laminations and measured train tracks

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Abstract. Thurston and Kerckhoff have shown that the space of measured geodesic laminations on a hyperbolic Riemann surface serves as a non-linear model of the tangent space to Teichmüller space at the surface. In this paper we show that the natural map between these manifolds has stronger than Hölder continuous regularity.

0. Introduction

Thurston discovered that train tracks can be used to obtain a PL manifold structure on his space of measured geodesic laminations (MGLs) on a hyperbolic Riemann surface. Required are two fundamental constructions: the first is a method of constructing train track approximations of a MGL and the second is the construction of a MGL from a measured train track (MTT).

In this paper we study the geometry of the latter construction. We examine how the leaves of MGLs move about on a surface as one changes the weights on a train track, and in particular, we show that if two weights on a train track are ε close, then the angle of leaves in the corresponding MGLs are $C\varepsilon \log^k(1/\varepsilon)$ close on the surface. Along the way, we study the growth rate of the number of train paths in a MTT with n edges that correspond to (non empty) leaf packets in the MGL and show this function has polynomial growth. We also find a Train Path Uncertainty Principle which states if two weights on a MTT are ε close, then the measures of any train path with n edges can be no more than $n\varepsilon$ apart.

Kerckhoff has shown that the natural map between the space of MGLs on a Riemann surface and the tangent space of Teichmüller space at the surface is a homeomorphism. An immediate consequence of our main theorem is that this map, between the manifolds, has stronger than Hölder continuous regularity.

Since very little about MGLs and MTTs has appeared in print, we begin with several short sections containing background material, along with a sketch of Thurston's two fundamental constructions.

1. Preliminaries on hyperbolic geometry

Definition. A hyperbolic Riemann surface is a smooth oriented surface of genus $g \geq 2$ equipped with a metric of constant curvature -1 .

By the uniformization theorem, every hyperbolic Riemann surface R is isometric to a surface of the form \mathbb{H}^2/Γ where \mathbb{H}^2 is the hyperbolic plane and Γ is a Fuchsian

group (a discrete subgroup of $\text{PSL}(2, \mathbb{R})$) acting by isometries, and isomorphic to $\pi_1(R)$. Let Σ_g denote a fixed smooth, closed, oriented surface of genus g .

Definition. The *Teichmüller space of genus g* , \mathcal{T}_g , is the space of discrete and faithful representations from $\pi_1(\Sigma_g) \rightarrow \text{PSL}(2, \mathbb{R})$ modulo conjugacy. Equivalently, \mathcal{T}_g may be defined as the space of hyperbolic metrics on Σ_g with two metrics identified if there is an isometry between them which is isotopic to the identity.

Every Möbius transformation A acting on \mathbb{H}^2 has a continuous extension to its boundary, denoted S^1_∞ , and called the ‘circle at ∞ ’. Pairs of distinct points on S^1_∞ are in 1-1 correspondence with (oriented) geodesics in \mathbb{H}^2 . Every hyperbolic Möbius transformation A ($|\text{tr } A| > 2$) has exactly two fixed points both on S^1_∞ . The geodesic corresponding to the fixed points of $A \in \Gamma$ projects to the unique closed geodesic in R contained in the free homotopy class of $i(A) \in \pi_1(\Sigma)$ under the isomorphism $i: \Gamma \rightarrow \pi_1(\Sigma)$.

Let f be a homeomorphism between two Riemann surfaces R_1 and R_2 , and let f_* be the induced isomorphism between their uniformizing groups Γ_1 and Γ_2 . This isomorphism gives a canonical 1-1 correspondence between elements of Γ_1 and Γ_2 , and hence between closed geodesics on R_1 and R_2 . Consequently, this isomorphism also induces a mapping between the sets of fixed points of elements of Γ_1 and Γ_2 , which are dense in S^1_∞ .

The following theorem of Nielsen states that this mapping has a unique extension to a homeomorphism between the circles at infinity. This homeomorphism enables us to identify geodesics on the two surfaces.

THEOREM. (Nielsen, [T].) *Let $f: R_1 \rightarrow R_2$ be a homeomorphism between two closed hyperbolic Riemann surfaces. Then any lift of f to a homeomorphism $\tilde{f}: \mathbb{H}^2 \rightarrow \mathbb{H}^2$ has a unique continuous extension \tilde{f} to a homeomorphism of $\mathbb{H}^2 \cup S^1_\infty$, and the extension is invariant under isotopy, depending only on the lift and on R_1 and R_2 . The extension is also equivariant so that the fixed points of any element $A_1 \in \Gamma_1$ are taken to the fixed points of $A_2 \in \Gamma_2$ where $\tilde{f}_*(A_1) = A_2$ ($\tilde{f}_*: \Gamma_1 \rightarrow \Gamma_2$ induced by \tilde{f}).*

2. Measured geodesic laminations

Let R be a closed hyperbolic Riemann surface of genus g .

Definition. A *geodesic lamination* \mathcal{G} on R is a foliation of a closed subset of R by complete, simple geodesics.

Let \mathcal{A} denote the set of simple, smooth open arcs contained in R which are transverse to \mathcal{G} , and have endpoint in $R - \mathcal{G}$.

Definition. A *Measured geodesic lamination* (MGL) is a pair (\mathcal{G}, μ) , where \mathcal{G} is a geodesic lamination and μ is a transverse measure satisfying:

- (1) $\text{support } (\mu) = \mathcal{G}$
- (2) If t_1 is homotopic to t_2 in \mathcal{A} , then $\mu(t_1) = \mu(t_2)$.

‘Trivial’ *Example of a MGL.* Let (\mathcal{G}, μ) be a finite disjoint union of simple closed geodesics ϕ_1, \dots, ϕ_n on R with counting measure $\mu = \sum_{k=1}^n a_k \delta_{\phi_k}$, $a_k > 0$.

We will follow custom and denote the MGL (\mathcal{G}, μ) by μ . The following three theorems about MGL's are due to Thurston [T]:

THEOREM. *If μ is a MGL on R and $t \in \mathcal{A}$, then $t \cap \mu$ is either a discrete set or the union of a Cantor set and a discrete set. Moreover, the isolated points of $t \cap \mu$ are exactly the intersections of t with simple closed leaves of μ .*

THEOREM. *If μ is a MGL on R , then $\text{Hyperbolic Area}(\mu) = 0$.*

THEOREM. *If μ is a MGL on R , then $R - \mu$ is a finite union of ideal polygons and Riemann surfaces with geodesic boundary.*

The transverse measure on an MGL μ induces a Lebesgue-Stieltjes measure on arcs contained in \mathcal{A} . This allows us to integrate functions along transverse arcs. For a transverse arc $t \in \mathcal{A}$, let θ be the angle that the leaves of μ make with t measured counterclockwise from t to μ . It is easy to see that θ is a Lipschitz function along t , and therefore may be integrated with respect to μ . We define the *total angle* and *total cosine* of an arc $t \in \mathcal{A}$ by:

$$\theta(t, \mu) \equiv \int_t \theta \, d\mu, \quad \cos(t, \mu) \equiv \int_t \cos \theta \, d\mu.$$

Similarly, we define the total angle and total cosine of a simple closed geodesic ϕ by:

$$\theta(t, \phi) \equiv \int_\phi \theta \, d\mu, \quad \cos(\phi, \mu) \equiv \int_\phi \cos \theta \, d\mu.$$

Let $\text{ML}(= \text{ML}(R))$ denote the set of MGL's on R . Thurston defined the following topology on ML : given a finite set of arcs $\{t_k\}_{k=1}^n \subset \mathcal{A}$, $\varepsilon > 0$, and $\mu \in \text{ML}$, a neighborhood basis of μ is given by:

$$\mathfrak{B}(\mu, \{t_k\}_{k=1}^n, \varepsilon) = \{ \nu \in \text{ML} : |(\mu(t_k), \theta(t_k, \mu)) - (\nu(t_k), \theta(t_k, \nu))| < \varepsilon, \\ k = 1, 2, \dots, n \}.$$

The following theorem of Thurston may be thought of as a strong existence statement for MGL's:

THEOREM. (Thurston, [T].) *ML is homeomorphic to \mathbb{R}^{6g-6} .*

Let \mathcal{S} denote the set of simple closed geodesics on R . We can embed $\mathcal{S} \times \mathbb{R}^+$ in ML by sending (ϕ, r) to the MGL consisting of ϕ with $\mu = r \cdot \delta_\phi$.

THEOREM. (Thurston, [T].) *$\mathcal{S} \times \mathbb{R}^+$ is dense in ML.*

This theorem allows one to extend notions defined for simple closed geodesics such as length and Dehn twist to MGL's.

Given two Riemann surfaces R_1 and R_2 of genus g , along with an isomorphism between their uniformizing groups, we have seen that the mapping between their circles at infinity may be used to identify MGL's on the two Riemann surfaces.

THEOREM. (Thurston, [T].) *Suppose R_1 and $R_2 \in \mathcal{T}_g$, thought of as hyperbolic structures on a fixed surface Σ_g . Let μ_{R_1} be a MGL on R_1 and let μ_{R_2} be the 'corresponding' MGL on R_2 . Then the leaves of μ_{R_1} are isotopic to the leaves of μ_{R_2} on Σ_g .*

3. Train tracks

Train tracks serve as ‘finite approximations’ to MGL’s and also allow us to define a PL (Piecewise linear) manifold structure on ML. The reader is referred to [HP] for an exhaustive treatment of train tracks.

A train track on Σ_g is a branched 1-dimensional submanifold with several added technical conditions to assure compatibility with MGLs.

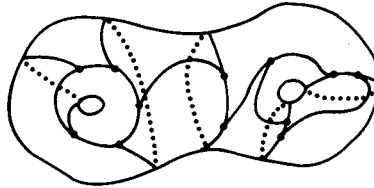


FIGURE 1. Train track.

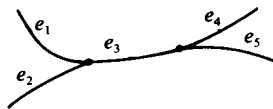
Definition. A curve c is *carried by* τ if there exists a C^1 map $\phi: \Sigma_g \rightarrow \Sigma_g$ such that $\phi(c) \subset \tau$, ϕ is homotopic to the identity, and $d\phi|_{\text{tangent space to } c} \neq 0$.

Definition. An (infinite) *train path* is a map $\rho: \mathbb{R} \rightarrow \tau$ which is a C^1 immersion and $\rho(k, k+1) = \text{edge}$ and $\rho(k) = \text{switch}$ for all $k \in \mathbb{Z}$.

Definition. A *measure* λ for τ is an assignment of a non-negative number $\mu(e)$ to each edge $e \subset \tau$ satisfying at each switch V the ‘switch condition’:

$$\mu(e_1) + \mu(e_2) + \dots + \mu(e_s) = \mu(e_{s+1}) + \dots + \mu(e_t),$$

where e_1, \dots, e_s are the incoming edges at v and e_{s+1}, \dots, e_t are the outgoing ones.



$$\mu(e_1) + \mu(e_2) = \mu(e_3) = \mu(e_4) + \mu(e_5)$$

FIGURE 2. Switch condition.

The set of measures on τ is a convex Euclidean cell and is denoted $V(\tau)$.

Definition. τ is *recurrent* if for each edge $e \subset \tau$, there exists non-trivial closed curves C_e carried by τ with supporting map ϕ and $\phi(C_e) \supset e$, or equivalently, τ supports a measure with all edge weights positive.

Definition. A *simple closed curve* c hits τ *efficiently* if there does not exist a (not necessarily embedded) bigon $B \subset \Sigma = \tau - c$ with ∂B a union of two C^1 segments $S_1 \subset C$ and $S_2 \subset \tau$.

Definition. τ is *transversely recurrent* if for each edge $e \subset \tau$, there exists a non-trivial simple closed curve C_e hitting τ efficiently and $C_e \cap e \neq \emptyset$.

Definition. τ is *birecurrent* if it is both recurrent and transversely recurrent.

We will assume that all edges are open and all switches are trivalent.

4. Fundamental construction #1

We will now outline two fundamental constructions of Thurston: The first will be a method of constructing train track approximations of an MGL and the latter will be the construction of an MGL from a transversely recurrent train track.

Construction #1. ϵ -train track approximation of an MGL μ . Before beginning the construction, we need the following preliminary result for a Riemann surface R with finite area.

Definition. $R_{(0,\epsilon]} = \{p \in R: \text{there exists non-trivial closed loop through } p \text{ with length } \leq \epsilon\}$.

PROPOSITION. (Margulis [BE].) *There exists a universal constant C such that $R_{(0,\epsilon]}$, $\epsilon \leq C$ is the disjoint union of cusps and tubular neighborhoods of simple closed geodesics with length $\leq \epsilon$.*

Step 1. Since $R - \mu = R_1 \cup R_2 \cup \dots \cup R_N$, $N < \infty$, where each R_k is either an ideal polygon (generically a triangle) or a Riemann surface with geodesic boundary, the double of R_k , dR_k , is a Riemann surface with finite area. Consider $(dR_k)_{(0,2\epsilon]}$. $(dR_k)_{(0,2\epsilon]}$ has a canonical foliation by curves of constant curvature, i.e., equidistant curves about a short geodesic and horocycles about cusps.

Step 2. Transfer the partial foliations on dR_k to R_k and then back to R .

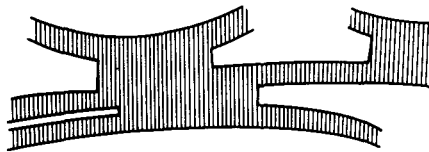


FIGURE 3. Partial foliation on R .

LEMMA. (Thurston, [W₁].) *The partial foliations on the R_k transferred to R extend to a unique Lipschitz foliation \mathcal{F} on the closure of $\bigcup_{k=1}^N (R_k)_{(0,\epsilon]}$.*

Step 3. Now collapse each leaf of \mathcal{F} down to a point. A well-known theorem of Moore [MOO] insures that the quotient ‘object’ \tilde{R} is a (topological) surface. The image of \mathcal{F} under the quotient map is the ϵ -train track approximation to μ denoted τ_ϵ .

Remark. It is clear that the switches of τ_ϵ come from collapsing leaves of \mathcal{F} near configurations such as:

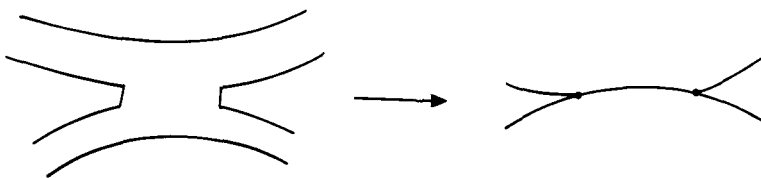


FIGURE 4

Since the number of switches cannot be greater than the total number of cusps in the R_k , it follows from simple Euler characteristic considerations that there are at most $4g - 4$ complimentary regions R_k and $\#$ switches of $\tau_\epsilon \leq 3(4g - 4) = 12g - 12$.

Step 4. (optional) (Thurston, [T, MOR].) Given $\delta > 0$, there exists $\epsilon_0 > 0$ such that if $\epsilon < \epsilon_0$, τ_ϵ (living on the quotient surface \tilde{R}) can be pulled back to a train track on R with geodesic curvature $< \delta$.

Our train track approximation τ_ϵ comes equipped with a natural measure. Each branch is assigned the weight corresponding to the total mass of the leaves in μ which collapse onto it. It is clear this defines a measure on τ_ϵ .

It is important to keep in mind that train tracks are topological objects having finite combinatorics. Thurston advocates not studying MGL's on a microscopic level, i.e., thinking about individual leaves, but to study them on a macroscopic level using ϵ -train track approximation.

5. Fundamental construction #2

Construction #2. Construction of an MGL from a measured transversely recurrent train track.

Let us start with a transversely recurrent train track τ on R with measure λ .

Step 1. [Construction of a Euclidean model for (λ, λ) .] For each edge $e \subset \tau$ take a Euclidean strip of width $k \cdot \lambda(e)$, where k is a scaling factor, $k > 0$, and splice these strips together obeying the switch condition at each vertex. The resulting object has a natural horizontal (singular) foliation obtained from the natural foliation on the Euclidean strips.

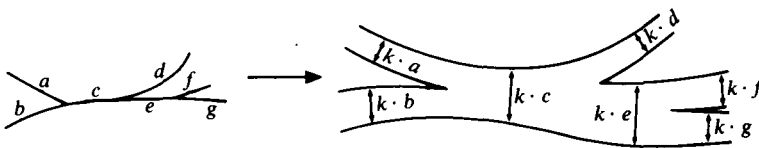


FIGURE 5. Euclidean model.

Step 2. Choose k small enough such that the foliated model $\subset \mathbb{R}^3$ embeds into R with τ as a spine. The model is not, and will not be isometrically embedded. All singular leaves of the foliation should be regarded as a pair of leaves, an ‘upper’ and a ‘lower’, making a consistent choice at each switch.

Step 3. Lift \mathcal{L} to a partial foliation $\tilde{\mathcal{L}}$ in \mathbb{H}^2 . The transverse recurrence of τ , insures that each $\tilde{l} \in \tilde{\mathcal{L}}$ has well defined endpoints on S_∞^1 [HP]. Now replace each leaf $\tilde{l} \in \tilde{\mathcal{L}}$ with the geodesic having the same endpoints on S_∞^1 . Denote the new collection of geodesic leaves $\tilde{\mathcal{G}}$.

Remarks.

- (1) Many leaves of $\tilde{\mathcal{L}}$ may amalgamate to create one leaf in $\tilde{\mathcal{G}}$
- (2) The assumption of transverse recurrence on τ is only needed to insure that the leaves of $\tilde{\mathcal{F}}$ have well defined endpoints on S_∞^1 . If it is known that τ has geodesic

curvature strictly less than one, then it follows from elementary hyperbolic geometry that the lifts have well defined endpoints on S^1_∞ .

$\tilde{\mathcal{G}}$ is a geodesic lamination on \mathbb{H}^2 , i.e., $\tilde{\mathcal{G}}$ is a closed disjoint union of simple geodesics. Now project $\tilde{\mathcal{G}}$ down to a geodesic lamination \mathcal{G} on \mathbb{R} .

Step 4. [Construction of the transverse measure on \mathcal{G} .] We now construct a measure on arcs in \mathcal{A} (open smooth arcs contained in R which are transverse to \mathcal{G}). Let $\mathcal{A}^* \subset \mathcal{A}$ consists of those arcs $t \in \mathcal{A}$ such that t intersects precisely those leaves of \mathcal{G} which collapse onto some edge $e_i \subset \tau$.

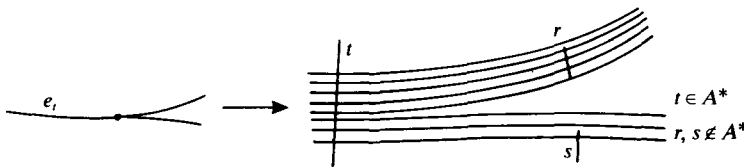


FIGURE 6

For $t \in \mathcal{A}^*$ define $\mu(t) = \lambda(e_i)$.

Claim. The transverse measure μ on \mathcal{A} is completely determined by μ on \mathcal{A}^* .

$$\mu(t) = \mu(t_1^1) + \mu(t_1^2) = \mu(t_2^1) + \mu(t_2^2) + \mu(t_2^3) = \lambda(e_{i_1}) + \lambda(e_{i_2}) + \lambda(e_{i_3}).$$

Since the leaves of \mathcal{G} are geodesics, they diverge from each other ‘exponentially fast’, hence the endpoints of arcs $t \in \mathcal{A}$ are only ‘finitely buried’ in \mathcal{G} . The mass of an arc $t \in \mathcal{A}$ is computed by ‘dragging’ the arc along \mathcal{G} until it splits into a union of arcs all of which are contained in \mathcal{A}^* .

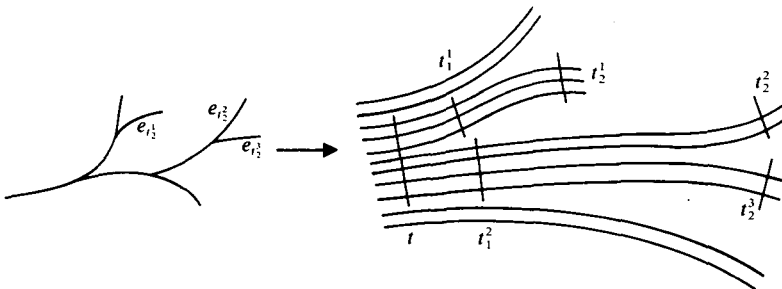


FIGURE 7. ‘Dragging’ a transverse arc.

THEOREM. (Thurston, [HP].) *If τ is a birecurrent train track and $\lambda_1, \lambda_2 \in V(\tau)$, then the MGL’s μ_1 and μ_2 constructed from (τ, λ_1) and (τ, λ_2) coincide iff $\lambda_1 = \lambda_2$.*

6. Measured train tracks and train track coordinates on ML

There is a natural equivalence relation on measured train tracks such that if (τ_1, λ_1) and (τ_2, λ_2) are two equivalent transversely recurrent train tracks, then the MGL’s constructed from them coincide.

Definition. MT is the space of equivalence classes of all measured train tracks equipped with the quotient topology

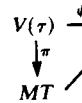
$$\coprod_{\tau \in \mathcal{R}} V(\tau) \xrightarrow{\pi} MT.$$

The following three theorems of Thurston are proved in [HP].

THEOREM A. MT is homeomorphic to \mathbb{R}^{6g-6} .

THEOREM B. Let ψ denote the mapping which constructs the MGL from a train track.

- (a) ψ is a continuous injection and π is a continuous mapping.
- (b) η is a bijection.
- (c) Since MT, ML are homeomorphic to \mathbb{R}^{6g-6} , it follows from ‘invariance of domain’ that η is a homeomorphism.



THEOREM C. ML is a PL (Piecewise Linear) manifold with PL coordinates $\psi_\tau: V(\tau) \rightarrow ML$, where τ is a transversely recurrent maximal (there does not exist a proper subtrack) train track.

Let ML denote the PL manifold structure on ML given by train track coordinates. The PL manifold structure on ML may alternately be described as follows: Let ψ_τ be a local parametrization for ML , and let e_1, \dots, e_N be the edges of N arcs $\alpha_1, \dots, \alpha_N$ transverse to all leaves in $\text{Im}(\psi_\tau)$ such that α_k intersects those leaves in $\text{Im}(\psi_\tau)$ which collapse onto $e_k, k = 1, 2, \dots, N$. Coordinates are given by $(\text{Im}(\psi_\tau), f_\tau)$, where:

$$\begin{aligned} f_\tau: \text{Im}(\psi_\tau) \subset ML &\rightarrow \mathbb{R}^N \\ \mu &\rightarrow (i(\mu, \alpha_1), \dots, i(\mu, \alpha_N)). \end{aligned}$$

We see that train track coordinates are equivalent to intersection numbers of ‘strategically placed’ transverse arcs.

7. Packet uncertainty principle

Let τ be a train track on R .

Definition. A finite train path in τ is a mapping $\rho: [0, N] \rightarrow \tau, N \in \mathbb{Z}^+, C^1$ -immersion, $\rho(k, k + 1) = \text{edge}$ and $\rho(k) = \text{switch}$ for $0 \leq k \leq N$.

We will usually denote a finite train path by listing its edges.

A measure λ on τ induces a measure on finite train paths in τ . The measure is inductive, and is illustrated below:

Let (τ, λ) be a MTT and let $\{e_1, e_2, \dots, e_n\}$ be a finite train path with n edges. We first define a subinterval of $[0, \lambda(e_n)]$, $I\{e_1, \dots, e_n\}$, by induction. We define $\lambda\{e_1, \dots, e_n\} = \text{length}(I\{e_1, \dots, e_n\})$. If $n = 1, I\{e_1\} = [0, \lambda(e_1)]$. If $I\{e_1, \dots, e_{n-1}\}$ has been defined. There are four possible configurations for $\{e_1, \dots, e_n\}$:

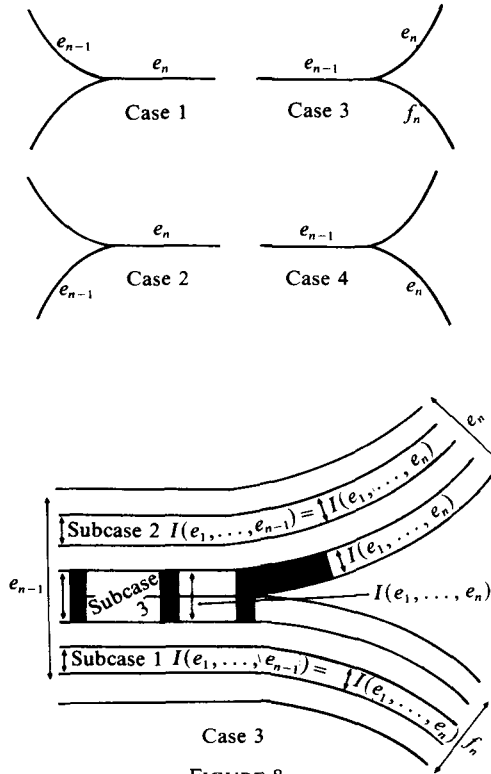


FIGURE 8

Cases 1 and 2 are trivial. $I\{e_1, \dots, e_n\} = I\{e_1, \dots, e_{n-1}\}$, hence $\lambda\{e_1, \dots, e_n\} = \lambda\{e_1, \dots, e_{n-1}\}$.

Case 3 has 3 subcases:

subcase 1: $\max I\{e_1, \dots, e_{n-1}\} \leq f_n$

$$I\{e_1, \dots, e_n\} = I\{e_1, \dots, e_{n-1}\},$$

hence $\lambda\{e_1, \dots, e_n\} = \lambda\{e_1, \dots, e_{n-1}\}$

subcase 2: $\min I\{e_1, \dots, e_{n-1}\} \geq f_n$

$$I\{e_1, \dots, e_n\} = [\min I\{e_1, \dots, e_{n-1}\} - f_n, \max I\{e_1, \dots, e_{n-1}\} - f_n],$$

hence $\lambda\{e_1, \dots, e_n\} = \lambda\{e_1, \dots, e_{n-1}\}$

subcase 3: $\min I\{e_1, \dots, e_{n-1}\} < f_n$ and $\max I\{e_1, \dots, e_{n-1}\} > f_n$

$$I\{e_1, \dots, e_n\} = [0, \max I\{e_1, \dots, e_{n-1}\} - f_n],$$

hence $\lambda\{e_1, \dots, e_n\} = \max I\{e_1, \dots, e_{n-1}\} - f_n$

Case 4 is identical to Case 3 and is omitted.

Definition. Given two measures λ_1 and λ_2 on τ ,

$$|\lambda_1 - \lambda_2|_\infty = \max_{e \subset \tau} |\lambda_1(e) - \lambda_2(e)|,$$

where the maximum is over all edges $e \subset \tau$.

Example.

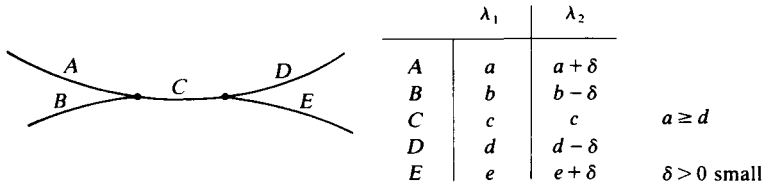


FIGURE 9

It is clear that $|\lambda_1 - \lambda_2|_\infty = \delta$. Consider the train path $P = \{ACE\}$.

$$\lambda_1(P) = a - d$$

$$\lambda_2(P) = (a + \delta) - (d - \delta) = a - d + 2\delta.$$

Hence, $|\lambda_1(P) - \lambda_2(P)| = 2\delta = 2|\lambda_1 - \lambda_2|_\infty$. This trivial example illustrates the phenomenon of ‘train path uncertainty’.

LEMMA 1.1. [*Train Path Uncertainty.*] Let τ be a train track on R , $\lambda_1, \lambda_2 \in V(\tau)$, and let P be a train path with n edges. Then

$$|\lambda_1(P) - \lambda_2(P)| \leq n|\lambda_1 - \lambda_2|_\infty.$$

Proof. Let $P = \{e_1, \dots, e_n\}$ be a train path with n edges. There are at most n edges of τ incident upon P . Let f_1, \dots, f_n denote those edges of τ that are incident to P , where e_k and f_j are incident upon the same switch. It follows from the previous discussion on weights on finite train paths, that

$$\lambda_1(P) = \lambda_1(e_1) \pm \lambda_1(f_2) \pm \lambda_1(f_3) \pm \dots \pm \lambda_1(f_n).$$

Then,

$$\lambda_2(P) = [\lambda_1(e_1) \pm \delta_1] \pm [\lambda_1(f_2) \pm \delta_2] \pm \dots \pm [\lambda_1(f_n) \pm \delta_n],$$

where the pattern of + or - between the bracketed terms in $\lambda_2(P)$ is the same as in $\lambda_1(P)$, and $0 \leq \delta_1, \delta_2, \dots, \delta_n \leq |\lambda_1 - \lambda_2|_\infty$. There is no bound on the sign combinatorics, and the worst case of the sign pattern could occur, hence $|\lambda_1(P) - \lambda_2(P)| \leq n|\lambda_1 - \lambda_2|_\infty$. \square

We will now translate the uncertainty principle into MGL language. Given an MGL μ on R , let (τ, λ) be a train track approximation to μ .

Definition. Let $P = \{e_1, \dots, e_n\}$ be a train path in τ . The leaf packet in μ corresponding to P , \mathcal{P} , consists of all those segments of leaves in μ which collapse exactly onto P .

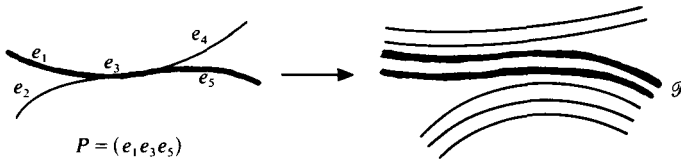


FIGURE 10

Note the leaf packet corresponding to $P = \{e_2, e_3, e_4\}$ is empty.

COROLLARY 1.2 [*Leaf Packet Uncertainty Principle.*] *Let τ be a birecurrent train track on R , $\lambda_1, \lambda_2 \in V(\tau)$, and P a train path in τ with n -edges. Let μ_1, μ_2 be the MGL's constructed from $(\tau, \lambda_1), (\tau, \lambda_2)$ respectively, and let $\mathcal{P}_1, \mathcal{P}_2$ be the packets contained in μ_1 and μ_2 respectively, corresponding to P . Then:*

$$|\mu_1(\mathcal{P}_1) - \mu_2(\mathcal{P}_2)| \leq n|\lambda_1 - \lambda_2|_\infty.$$

Proof.

$$|\mu_1(\mathcal{P}_1) - \mu_2(\mathcal{P}_2)| = |\lambda_1(P) - \lambda_2(P)| \leq n|\lambda_1 - \lambda_2|_\infty.$$

8. Polynomial growth estimate

Let (τ, λ) be a transversely recurrent train track on R and μ its associated MGL. Let $R(n)$ denote the number of train paths in τ with n edges that have geometric realizations as (non-empty) packets of leaves in μ . A priori, $R(n)$ grows exponentially in n , since you can turn left or right at each switch.

Birman and Series [BS] have devised a clever coding scheme for simple geodesic arcs on a Riemann surface. Although their coding scheme was devised expressly to encode simple geodesic arcs on a Riemann surface, we use it to encode train paths (leaf packets) and prove that $R(n)$ grows as a polynomial in n . This distinction between subexponential/polynomial is essential to our later analysis.

THEOREM 1.3. *Let (τ, λ) be a transversely recurrent train track on R and μ its associated MGL. Let $R(n)$ denote the number of train paths in τ with n edges that have geometric realizations as (non-empty) packets of leaves in μ . Then:*

$$R(n) \leq Cn^{\binom{12g-6}{2}+2}, \text{ and } C = C(g, \tau, \text{fixed fundamental polygon for } R).$$

Our strategy is to show a bounded correspondence between train track coding and Birman-Series coding of train paths. The proposition then readily follows since the Birman-Series coding has the polynomial growth built in. Most of the following is taken verbatim from [BS].

Choose and fix a convex fundamental polygon $\mathcal{D} \subset \mathbb{H}^2$ for $\pi_1(R)$.

In [BS], the authors show that the set of simple geodesics on R is a nowhere dense subset of R (and has measure 0). Using their result, we can choose \mathcal{D} such that no simple geodesic intersects any vertex of \mathcal{D} and $\partial\mathcal{D}$ does not contain any simple geodesic arcs. This simplification is not essential, but it gives cleaner statements of results.

Let A denote the set of sides of \mathcal{D} , ordered in any fixed manner.

Let J_μ denote the set of oriented simple geodesic arcs $\gamma \subset \mu$ such that $\partial\gamma \subseteq \pi(\partial\mathcal{D})$ and no subarc of γ coincides with a side of $\pi(\partial\mathcal{D})$, where $\pi \in \mathbb{H}^2 \rightarrow R$ is the universal covering map.

We refer to the components of $\gamma \cap \pi(\mathcal{D})$ as the segments of γ and the points of $\gamma \cap \pi(\partial\mathcal{D})$ as the partition points of γ . We label the partition points t_0, t_1, \dots, t_n

in the order in which they occur along γ and we set $\|\gamma\| = n$. It is clear that any lift of γ enters n translates of \mathcal{D} .

Let $B = A \times A - \{\text{diagonal}\}$ and define $h_1: J_\mu \rightarrow Z^B, h_1(\gamma) = \{n(a, a')\}_{(a,a') \in B}$, where $n(a, a')$ is the number of segments joining a to a' in (the lift of γ to) \mathcal{D} . Clearly,

$$\|\gamma\| = \sum_{(a,a') \in B} n(a, a').$$

When no confusion will occur, we will henceforth not distinguish between an arc γ on R and its lift to a broken arc on \mathcal{D} .

We shall need to record information about the position of the initial and final points t_0, t_n of γ . For $i = 0, n$ let $a(t_i)$ be the element of A containing t_i and let $j(t_i) \in N$ be the position of t_i among the partition points of γ which lie along $a(t_i)$ counting in the anticlockwise direction around $\partial\mathcal{D}$. Now define:

$$h_2: J_\mu \rightarrow (A \times Z)^2$$

$$h_2(\gamma) = (a(t_0), j(t_0), a(t_n), j(t_n))$$

and let

$$h = (h_1, h_2): J_\mu \rightarrow Z^B \times (A \times Z)^2 = \Omega.$$

For

$$w \in \Omega, \quad \text{let} \quad |w| = \sum_{(a,a') \in B} |w_{a,a'}|,$$

where $w_{a,a'}$ is the projection of w on the (a, a') factor of Z^B . For $\gamma \in J_\mu$, define $\|\gamma\|_{BS} = |h(\gamma)|$. For $m \in N$, let $\Omega_m = \{w \in \Omega : |w| \leq m\}$.

LEMMA BS1. [BS, Lemma 2.1.] *Suppose that $\gamma, \gamma' \in J_\mu$ and that $h(\gamma) = h(\gamma')$. Let t_0, \dots, t_n and t'_0, \dots, t'_n be the partition points of γ, γ' . Then t_i and t'_i lie in the same element of A for each $i = 0, 1, \dots, n$, hence γ and γ' are homotopic rel open edges in $\pi(\partial\mathcal{D})$.*

LEMMA BS2. [BS, Lemma 2.2.]

$$\#(\Omega_n \cap \text{Image}(h)) \leq Cn^{\binom{\#A}{2} + 2},$$

where C depends on \mathcal{D} .

THEOREM. [BE]. *Let G be a finitely generated Fuchsian group of the first kind and let P be any convex fundamental polygon for G . Suppose that P has N sides. Then $4g \leq N \leq 12g - 6$.*

Applying this theorem to Lemma BS2 yields:

$$\#(\Omega_n \cap \text{Image}(h)) \leq Cn^{\binom{12g-6}{2} + 2}.$$

Remark 1. The mapping $h: \{\gamma \in J_\mu : \|\gamma\|_{BS} \leq n\} \rightarrow (\Omega_n \cap \text{Image}(h))$ is a many-to-one mapping, however, by Lemma BS1, the mapping:

$$h: \left\{ \begin{array}{l} \text{Homotopy classes} \\ \text{of } \gamma \in J_\mu \end{array} \begin{array}{l} \text{with } \|\gamma\|_{BS} \leq n \text{ and the homotopy} \\ \text{is rel open edges in } \pi(\partial\mathcal{D}) \end{array} \right\} \rightarrow (\Omega_n \cap \text{Image}(h))$$

is a one-to-one mapping.

Remark 2. Using the coding scheme, we may formally code any simple arc γ as long as $\partial\gamma \subset \pi(\partial\mathcal{D} \setminus \text{vertices of } \mathcal{D})$, $\pi(\partial\mathcal{D})$ does not contain any subarc of γ , and γ has no inessential intersections with \mathcal{D} . A moment's thought will show that Lemma BS1 is valid if γ and γ' are just simple arcs on R and not necessarily geodesic arcs.

Remark 3. Let P be a train path in τ . Since ∂P need not lie on $\pi(\partial\mathcal{D})$, P may not have a unique coding, but could have as many as $(\neq A)^2$ different codings corresponding to all possible extensions of the first and the last edge to $\pi(\partial\mathcal{D})$.

For the next proposition, we need a slightly more general notion of train path: we will drop the requirement that a train path must begin and end at a switch.

The following proposition shows that the correspondence between train paths in τ with geometric realizations and Birman-Series codes is at most N to 1.

LEMMA 1.4. *Let (τ, λ) be a transversely recurrent train track such that τ does not intersect any vertex of $\pi(\mathcal{D})$, $\pi(\partial\mathcal{D})$ does not contain any subarc of τ , and no train path in τ has an inessential intersection with \mathcal{D} , and let μ be its associated MGL. There exists $N = N(\mathcal{D}, \tau)$ such that if P_1, \dots, P_r are distinct train paths in τ which correspond to leaf packets in μ , $\partial P_k \subset \pi(\partial\mathcal{D})$, and $h(P_1) = \dots = h(P_r)$, then $r \leq N$.*

Proof. Suppose P_1, \dots, P_r are distinct train paths in τ which correspond to leaf packets in μ , $\partial P_k \subset \pi(\partial\mathcal{D})$, and $h(P_1) = \dots = h(P_r)$. Studying fundamental constructions 1 and 2 carefully, one can show that μ can be collapsed onto τ in the following strong way: there exists a C^1 homotopy $F: I \times R \rightarrow R$ and $\varepsilon > 0$ such that:

- (a) $F(0, p) = p$ for all $p \in R$.
- (b) $F(1, \mu) \subset \tau$.
- (c) $F(*, \mu) \subset N_\varepsilon(\mu)$, where $N_\varepsilon(\mu)$ is the ε -neighborhood of μ .
- (d) $F(*, \text{leaf})$ is transverse to the foliation \mathcal{F} obtained in Fundamental Construction $\neq 1$.

Choose geodesic arcs $\alpha_1, \dots, \alpha_r \subset J_\mu$ which collapse onto P_1, \dots, P_r via the homotopy described above.

Unfortunately $h(\alpha_k)$ might not be equal to $h(P_k)$, since when we lift these curves to \mathcal{D} , 'glancing' may occur near a vertex. However, using property (d) of the homotopy described above, we can find simple arcs β_1, \dots, β_r , $\partial\beta_k \subset \pi(\partial\mathcal{D})$, such that β_k collapses onto P_k and $h(\beta_k) = h(P_k)$. By hypothesis $h(\beta_1) = \dots = h(\beta_r)$. Lemma BS1 implies that β_1, \dots, β_r are all homotopic rel open edges of $\pi(\partial\mathcal{D})$, hence P_1, \dots, P_r are all homotopic rel open edges of $\pi(\partial\mathcal{D})$. If P_1, \dots, P_r were infinite train paths we could conclude that $P_1 = \dots = P_r$ [HP]. Since P_1, \dots, P_r are



FIGURE 11. 'Last fundamental domain'.

finite train paths, we can only conclude that (the lifts of) $P_1 = \dots = P_r$, outside of the first l and last m fundamental domains, where l and m are determined by \mathcal{D} and τ and independent of the number of edges of the train paths. We choose N to be the total number of train paths in the lift of τ contained in \mathcal{D} with first and last edges of $\partial\mathcal{D}^{l+m}$. □

Proof of Theorem 1.3. We will assume that τ does not intersect any vertex of $\pi(\mathcal{D})$, $\pi(\partial\mathcal{D})$ does not contain any subarc of τ , and no train path in τ has an inessential intersection with $\pi(\partial\mathcal{D})$. This is no restriction, since we can always find a fundamental polygon for R which satisfies the above conditions with respect to the fixed train track τ [N].

$$R(n) \leq \# \left\{ \begin{array}{l} P: P \text{ is a train path in } \tau, P \text{ corresponds to a leaf packet} \\ \text{in } \mu, \text{ the first and last edges (or switches) intersect} \\ \pi(\partial\mathcal{D}), \text{ and if } \bar{P} \text{ is the retraction of } P \text{ such that} \\ \partial(\bar{P}) \subset \pi(\mathcal{D}), \text{ then } \|\bar{P}\|_{BS} \leq n \end{array} \right\}$$

By Lemma 1.4,

$$R(n) \leq N \# (\Omega_n \cap \text{Image}(h)),$$

and by Lemma BS2,

$$R(n) \leq NCn^{(12g-6)/2+2}.$$

□

9. The geometry of fundamental construction #2

In the proof of the next theorem, we will need the following easy lemmas:

LEMMA 1.6. For large b , $\cos \theta \approx 2 \tanh(a/2)e^{-b/2}$.

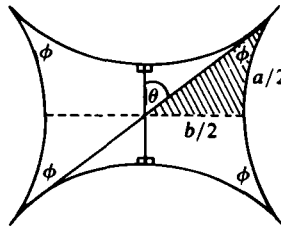


FIGURE 12

Proof. Since the shaded triangle is a right triangle, [BE]

$$\begin{aligned} \tan\left(\frac{\pi}{2} - \theta\right) &= \frac{\tanh(a/2)}{\sinh(b/2)} = \cot \theta \\ \cos \theta &= \frac{\cot \theta}{\sqrt{1 + \cot^2 \theta}} = \frac{1}{\sqrt{\frac{\sinh^2(b/2)}{\tanh^2(a/2)} + 1}} \stackrel{b \ll 1}{\approx} \frac{\tanh(a/2)}{\sinh(b/2)} \approx 2 \tanh(a/2)e^{-b/2} \quad \square \end{aligned}$$

LEMMA 1.7. Let (τ, λ) be a transversely recurrent train track and let μ be its associated

MGL. The hyperbolic length of a subleaf in μ is commensurable with the number of edges in τ it collapses onto, counted with multiplicity.

Proof. For each edge $e \subset \tau$, let

$$\mathcal{C}_e = \{ \alpha \text{ subleaf of } \mu : \alpha \text{ collapses onto } e \text{ under the homotopy described in Lemma 1.4} \}.$$

Let $l^* \subset \mu$, and suppose l^* collapses onto n edges of τ (counted with multiplicity), then

$$n[\min_{e \subset \tau} \inf_{\alpha \in \mathcal{C}_e} l_{\text{HYP}}(\alpha)] \leq l_{\text{HYP}}(l^*) \leq n[\max_{e \subset \tau} \sup_{\alpha \in \mathcal{C}_e} l_{\text{HYP}}(\alpha)]. \quad \square$$

THEOREM 1.8. Let τ be a transversely recurrent train track on R , $\lambda_1, \lambda_2 \in V(\tau)$, μ_1 and μ_2 the associated MGL's, and t be any simple geodesic arc transverse to μ_1 and μ_2 which collapses onto one edge of τ . Then:

$$\left| \int_t \cos \theta d\mu_1 - \int_t \cos \theta d\mu_2 \right| \leq C |\lambda_1 - \lambda_2|_\infty \log^r \left(\frac{1}{|\lambda_1 - \lambda_2|_\infty} \right),$$

where $r = (12\frac{g-6}{2}) + 3$, and C depends on $\tau, t, \mu_1(t)$ and $\mu_2(t)$.

Proof. Let c_1 and c_2 be the collapsing maps for μ_1 and μ_2 onto τ . We may assume that c_1 and c_2 collapse t onto the same edge $e \subset \tau$. Let P^n be a train path in τ , consisting of $2n + 1$ edges, having e as its middle edge, and let \mathcal{P}_k^n denote the packets of leaves in μ_k which collapse onto P^n . Lift μ_1, μ_2 and τ to \mathbb{H}^2 along with a fixed lift of t which we denote \tilde{t}^* . Then:

$$\begin{aligned} \left| \int_t \cos \theta d\mu_1 - \int_t \cos \theta d\mu_2 \right| &= \left| \sum_{P^n} \left(\int_{\tilde{t}^*} \cos \theta d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{t}^*} \cos \theta d(\mu_2 \cap \mathcal{P}_2^n) \right) \right| \\ &= \left| \sum_{\substack{P^n: \\ \mathcal{P}_1^n \neq \emptyset \text{ and } \mathcal{P}_2^n \neq \emptyset}} \left(\int_{\tilde{t}^*} \cos \theta d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{t}^*} \cos \theta d(\mu_2 \cap \mathcal{P}_2^n) \right) \right| \\ &\quad + \left| \sum_{\substack{P^n: \\ \mathcal{P}_1^n = \emptyset \text{ and } \mathcal{P}_2^n = \emptyset \text{ or} \\ \mathcal{P}_1^n = \emptyset \text{ and } \mathcal{P}_2^n \neq \emptyset}} \left(\int_{\tilde{t}^*} \cos \theta d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{t}^*} \cos \theta d(\mu_2 \cap \mathcal{P}_2^n) \right) \right|. \end{aligned}$$

We will refer to those leaf packets which occur in the first sum as type 1 packets and those appearing in the second sum as type 2 packets. We begin by estimating the first sum. By the triangle inequality, it follows that:

$$\begin{aligned} (1) &\leq \sum_{\substack{P^n \\ \text{type 1}}} \int_{\tilde{t}^*} \left| \cos \theta - \left(\frac{\min \cos}{\mathcal{P}_1^n \cap \tilde{t}^*} \right) \right| d(\mu_1 \cap \mathcal{P}_1^n) \\ &\quad + \sum_{\substack{P^n \\ \text{type 1}}} \int_{\tilde{t}^*} \left| \cos \theta - \left(\frac{\min \cos}{\mathcal{P}_2^n \cap \tilde{t}^*} \right) \right| d(\mu_2 \cap \mathcal{P}_2^n) \\ &\quad + \sum_{\substack{P^n \\ \text{type 1}}} \left| \left(\frac{\min \cos}{\mathcal{P}_1^n \cap \tilde{t}^*} \right) \right| \left| \int_{\tilde{t}^*} d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{t}^*} d(\mu_2 \cap \mathcal{P}_2^n) \right|. \end{aligned}$$

Suppose $P^n = (e_0, \dots, e_{n-1}, e, e_{n+1}, \dots, e_{2n+1})$ is a type 1 train path. Let v_0 and v_{2n+1} be its initial and terminal switches. \mathcal{P}_k^n connects $c_k^{-1}(v_0)$ to \tilde{i}^* to $c_k^{-1}(v_{2n+1})$. Since the collection $\{c_k^{-1}(v)\}_{k=1,2}$, where v is a switch in τ , is finite, Lemmas 1.6 and 1.7 imply there exists constants c_1 and c_2 such that:

$$\left| \begin{pmatrix} \max \cos \\ \mathcal{P}_1^n \cap \tilde{i}^* \\ \mathcal{P}_2^n \cap \tilde{i}^* \end{pmatrix} - \begin{pmatrix} \min \cos \\ \mathcal{P}_1^n \cap \tilde{i}^* \\ \mathcal{P}_2^n \cap \tilde{i}^* \end{pmatrix} \right| \leq c_1 e^{-c_2 n}.$$

Hence:

$$\begin{aligned} (1) &\leq \sum_{\substack{P^n \\ \text{type 1}}} c_1 e^{-c_2 n} \int_{\tilde{i}^*} d(\mu_1 \cap \mathcal{P}_1^n) + \sum_{\substack{P^n \\ \text{type 1}}} c_1 e^{-c_2 n} \int_{\tilde{i}^*} d(\mu_2 \cap \mathcal{P}_2^n) \\ &+ \sum_{\substack{P^n \\ \text{type 1}}} \left| \int_{\tilde{i}^*} d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{i}^*} d(\mu_2 \cap \mathcal{P}_2^n) \right| \\ &\leq c_1 e^{-c_2 n} \mu_1(\tilde{i}^*) + c_1 e^{-c_2 n} \mu_2(\tilde{i}^*) + \sum_{\substack{P^n \\ \text{type 1}}} \left| \int_{\tilde{i}^*} d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{i}^*} d(\mu_2 \cap \mathcal{P}_2^n) \right|. \end{aligned}$$

The packet uncertainty principle (Corollary 1.2) implies that:

$$\left| \int_{\tilde{i}^*} d(\mu_1 \cap \mathcal{P}_1^n) - \int_{\tilde{i}^*} d(\mu_2 \cap \mathcal{P}_2^n) \right| \leq (2n+1) |\lambda_1 - \lambda_2|_\infty.$$

The number of type 1 packets is bounded by the total number of packets, $R(2n+1)$, which by Theorem 1.3 is bounded by $c_3 n^r$, where $r = (\binom{12g-6}{2}) + 2$. Thus, we may estimate the first sum by:

$$(1) \leq c_1 e^{-c_2 n} (\mu_1(\tilde{i}^*) + \mu_2(\tilde{i}^*)) + (2n+1) |\lambda_1 - \lambda_2|_\infty c_3 n^r.$$

We will assume that $\mu_1(\tilde{i}^*)$ and $\mu_2(\tilde{i}^*)$ are bounded by some constant, hence,

$$(1) \leq c_4 e^{-c_2 n} + (2n+1) |\lambda_1 - \lambda_2|_\infty c_3 n^r.$$

The second sum may be crudely estimated by:

$$(2) \leq \sum_{\substack{P^n \\ \text{type 2}}} \int_{\tilde{i}^*} d(\mu_l \cap \mathcal{P}_l^n),$$

where the choice of l depends on the particular packet, $l = 1$ or 2 .

It is an immediate consequence of the packet uncertainty principle (Corollary 1.2) that for type 2 packets,

$$\int_{\tilde{i}^*} d(\mu_l \cap \mathcal{P}_l^n) \leq (2n+1) |\lambda_1 - \lambda_2|_\infty.$$

Unfortunately, we have no control over the number of type 2 packets. Again, all we know is the number of type 2 packets is bounded by the total number of packets $R(2n+1)$, which by Theorem 1.3 is bounded by $c_3 n^r$, where $r = (\binom{12g-6}{2}) + 2$. Thus, we may estimate the second sum by:

$$(2) \leq (2n+1) |\lambda_1 - \lambda_2|_\infty c_3 n^r.$$

It follows that the entire sum may be estimated by:

$$(1) + (2) \leq c_4 e^{-c_2 n} + 2(2n+1) |\lambda_1 - \lambda_2|_\infty c_3 n^r.$$

Equating the two terms, we choose

$$n = \frac{1}{c_2} \log \left(\frac{c_4}{4c_3} \frac{1}{|\lambda_1 - \lambda_2|_\infty} \right),$$

and after absorbing constants, we obtain:

$$(1) + (2) \leq c_5 |\lambda_1 - \lambda_2|_\infty + c_6 |\lambda_1 - \lambda_2|_\infty \log^{r+1} \left(c_7 \frac{1}{|\lambda_1 - \lambda_2|_\infty} \right) + c_8 |\lambda_1 - \lambda_2|_\infty \log^r \left(c_7 \frac{1}{|\lambda_1 - \lambda_2|_\infty} \right).$$

For $|\lambda_1 - \lambda_2|_\infty \ll 1$,

$$(1) + (2) \leq c_9 |\lambda_1 - \lambda_2|_\infty \log^{r+1} \left(\frac{1}{|\lambda_1 - \lambda_2|_\infty} \right). \quad \square$$

Remark. This theorem may be interpreted as saying that if two MGL’s have intersection numbers which are ε close with respect to suitably chosen transverse arcs, then their total cosines with respect to closed geodesics are $C\varepsilon \log^{r+1}(1/\varepsilon)$ close.

10. Earthquakes

Thurston has generalized the Dehn twist or Fenchel-Nielsen deformation about a simple closed geodesic to a ‘twisting’ or ‘shearing’ about a MGL, called an earthquake deformation.

Recall that for R , the set $\mathcal{S} \times \mathbb{R}^+$ consisting of weighted simple closed geodesics in R is dense in ML . The Earthquake deformation about a MGL μ is defined by the limit in \mathcal{T}_g of any sequence of Dehn twists along weighted simple closed geodesics which converge to μ in ML . Formally:

Definition. (Thurston.) The left *Earthquake deformation* of R at time t determined by $\mu \in ML$ is the limit in \mathcal{T}_g of the time t (Dehn) twist deformations of R for any sequence $(\alpha_i, \gamma_i) \in \mathcal{S} \times \mathbb{R}^+$ converging to μ in ML . It will be denoted $E(R, \mu, t)$.

In [K₁], Kerckhoff shows that this definition is valid, i.e., that the limits exist and are independent of the approximating sequence, and hence unique.

We will think of earthquake deformations as defining paths in \mathcal{T}_g as t varies.

THEOREM. (Kerckhoff, [K₃].) *Every tangent vector in \mathcal{T}_g is tangent to a unique earthquake path in \mathcal{T}_g . In fact:*

$$\psi : ML \mapsto (\mathcal{T}_g)_R$$

$$\mu \mapsto \left. \frac{d}{dt} \right|_{t=0^+} E(R, \mu, t)$$

is a homeomorphism.

Choose $6g - 6$ simple closed geodesics $\alpha_1, \dots, \alpha_{6g-6}$ whose lengths $l_{\alpha_1}, \dots, l_{\alpha_{6g-6}}$ serve as local coordinates in $R \in U \subset \mathcal{T}_g$. In these coordinates,

$$\left. \frac{d}{dt} \right|_{t=0^+} E(R, \mu, t) = \sum_{k=1}^{6g-6} \left[\left. \frac{d}{dt} \right|_{t=0^+} l_{\alpha_k}(E(R, \mu, t)) \right] \frac{\partial}{\partial l_{\alpha_k}}.$$

In $[K_2]$, Kerckhoff shows:

$$\frac{d}{dt} \Big|_{t=0^+} l_{\alpha_k}(E(R, \mu, t)) = \cos(\alpha_k, \mu) = \int_{\alpha_k} \cos \theta \, d\mu$$

hence,

$$\frac{d}{dt} \Big|_{t=0^+} E(R, \mu, t) = \sum_{k=1}^{6g-6} \left(\int_{\alpha_k} \cos \theta \, d\mu \right) \frac{\partial}{\partial l_{\alpha_k}}.$$

Definition. $f: \mathbb{R}^m \rightarrow \mathbb{R}^n$ is ' k Log Hölder' iff there exists C, K such that

$$|f(x) - f(y)| \leq C|x - y| \log^K \left(\frac{1}{|x - y|} \right).$$

Remark. It is easy to show that k Log Hölder implies α -Hölder continuous for all $\alpha < 1$, but it does not imply Lipschitz.

Theorem 1.8 applied to $\alpha_1, \alpha_2, \dots, \alpha_{6g-6}$ readily implies:

COROLLARY 1.9. $\psi: ML(R) \rightarrow (\mathcal{T}_g)_R$ is k Log Hölder.

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