By writing y for $\pi/\sqrt{x}$ in the above relations, we obtain a corresponding set of expressions for $\sum_{n=1}^{\infty} \Phi_{2r+1}(ny)$ for $y \geqq 10$. For example from (8) we have

$$\sum_{n=1}^{\infty} \Phi_{2r+1}(ny) \doteq (-1)^{r+1}(2r)!/\pi^{\frac{1}{2}}r!$$

which is independent of y ($\geqq 10$), and for $r = 1$ gives

$$\sum_{n=1}^{\infty} e^{-\frac{1}{4}n^2z^2} D_2(nz) \doteq \tfrac{1}{2}, \text{ if } z > 14.$$

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A Generalisation of Dirichlet's Multiple Integral

By I. J. GOOD.

The purpose of this note is to generalise the Dirichlet-Liouville formula which expresses a certain type of multiple integral in terms of a single integral.[1] In our formula the multiple integral will involve several arbitrary functions instead of only one, and it will be expressed as a product of single integrals.

Let n be a positive integer. Let $f_1(t), f_2(t), \ldots, f_n(t)$ be Lebesgue measurable functions when $0 \leqq t \leqq 1$. A finite sequence of n real numbers $m_1, m_2, \ldots, m_n$ is given. We write $m_{n+1} = 0$ and

$$M_r = m_1 + m_2 + \ldots + m_r$$

$$X_r = x_1 + x_2 + \ldots + x_r$$

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[1] See, for example, G. F. Meyer, *Vorlesungen über die Theorie der bestimmten Integrale* (Leipzig, 1871), 566 *et seq.*; or E. T. Whittaker and G. N. Watson, *Modern Analysis* (4th edn., Cambridge, 1935), section 12.5; or H. Jeffreys and B. S. Jeffreys, *Methods of Mathematical Physics* (Cambridge, 1946), section 15.08; or L. J. Mordell, "Dirichlet's integrals," *Edin. Math. Notes*, No. 34 (1944), 15-17.

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and so on, and we assume that all variables of integration, such as $x_1, x_2, \ldots, x_n$, are non-negative. Then

$$\int_{X_n \leqslant 1} \cdots \int \prod_{r=1}^{n-1} \left\{ x_r^{m_r} f_r\left(\frac{X_r}{X_{r+1}}\right)\right\} x_n^{m_n} f_n(X_n)\, dx_1\, dx_2 \ldots dx_n$$
$$= \prod_{r=1}^{n} \int_0^1 f_r(x)\,(1-x)^{m_{r+1}}\, x^{M_r + r - 1}\, dx \qquad (1)$$

provided that the n single integrals on the right all exist.

Proof. We proceed in a formal spirit. The proof can easily be made rigorous by working backwards from the final result and making use of Fubini's theorem.

Denote the left-hand side of (1) by $I_n(f_1, f_2, \ldots, f_n)$. In this n-fold integral we first change the variables of integration from $x_1, x_2, \ldots, x_n$ to $x_1, x_2, \ldots, x_{n-1}, X_n$ and second put $x_r = X_n y_r$ $(r = 1, 2, \ldots, n-1)$. We obtain $I_n(f_1, f_2, \ldots, f_n)$

$$= \int_0^1 f_n(X_n)\, dX_n \int_{X_{n-1} \leqslant X_n} \cdots \int (X_n - X_{n-1})^{m_n} \prod_{r=1}^{n-1} \left\{ x_r^{m_r} f_r\left(\frac{X_r}{X_{r+1}}\right) dx_r \right\}$$

$$= \int_0^1 f_n(X_n)\, X_n^{M_n + n - 1}\, dX_n \int_{Y_{n-1} \leqslant 1} \cdots \int \prod_{r=1}^{n-2} \left\{ y_r^{m_r} f_r\left(\frac{Y_r}{Y_{r+1}}\right)\right\}$$
$$y_{n-1}^{m_{n-1}} f_{n-1}(Y_{n-1})\,(1 - Y_{n-1})^{m_n}\, dy_1 \ldots dy_{n-1}$$

$$= I_{n-1}(f_1, f_2, \ldots, f_{n-2}, g_{n-1}) \int_0^1 f_n(x)\, x^{M_n + n - 1}\, dx,$$

where $g_{n-1}(t) = f_{n-1}(t)\,(1-t)^{m_n}$. The theorem now follows by induction.

As a special case let $f_r(t) = t^{\Lambda_r}$ where $\Lambda_r = \lambda_1 + \lambda_2 + \ldots + \lambda_r$ and assume $m_r > -1$, $\lambda_r + m_r > -1$. Then

$$\int_{X_n \leqslant 1} \cdots \int \prod_{r=1}^{n} \{ x_r^{m_r} X_r^{\lambda_r}\, dx_r \} = \prod_{r=1}^{n} \frac{\Gamma(\Lambda_r + M_r + r)\,\Gamma(m_{r+1} + 1)}{\Gamma(\Lambda_r + M_{r+1} + r + 1)}$$

where $m_{n+1} = 0$.

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