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# ON MENNICKE GROUPS OF DEFICIENCY ZERO II

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ABSTRACT. Let $M$ be the group defined by the presentation $\langle x, y, z \mid x^y = x^m, y^z = y^n, z^x = z^r \rangle, m, n, r \in Z$. $M$ is one of the few 3-generator finite groups of deficiency zero. These groups have been considered by Mennicke [3], Macdonald, Wamsley [10], Johnson and Robertson [7], and Albar. Properties like the order of $M$, the nilpotency and solvability were studied. In this paper we give a better upper bound for $M$ than the one given by Johnson and Robertson [7]. We also describe the structure of some cases of $M$.

**Introduction.** The Mennicke groups are defined by the presentations:

$$M(m,n,r) = \langle x, y, z \mid x^y = x^m, y^z = y^n, z^x = z^r \rangle$$

where $m, n, r \geq 2$.

LEMMA 1. *The defining relations of $M$ imply that $y^{-u}x^v y^u = x^{vm^u}$ for any integers $u, v$ with $u \geq 0$, together with two cyclic permutants.*

REMARK 1. The following identity holds in any group $G$: $(xy)^n = y^n \prod_{k=n}^{1} x^{y^k}$ for any $x, y \in G$ where

$$y^n \prod_{k=n}^{1} x^{y^k} = y^n[y^{-n}xy^n][y^{-(n-1)}xy^{(n-1)}]\cdots[y^{-2}xy^2][y^{-1}xy].$$

We write the relations of $M$ in the forms:

$$(a) \qquad y^{-1}xy = x^m, \quad x^{-1}yx = yx^{-(m-1)}$$

$$(b) \qquad z^{-1}yz = y^n, \quad y^{-1}zy = zy^{-(n-1)}$$

$$(c) \qquad x^{-1}zx = z^r, \quad z^{-1}xz = xz^{-(r-1)}.$$

We begin this paper by using Lemma 1, Remark 1 and relations (a), (b) and (c) to find different bounds for the orders of $x, y, z$. We then use these bounds to find a bound for the order of $M$.

Conjugating (a) by $z$ we get $(y^{-1}xy)^z = (x^m)^z$.

Using (b) and (c) we obtain $y^{-n}xz^{-(r-1)}y^n = z^{-1}x^m z$.

Using the lemma we have

$$x^{m^n}y^{-n}z^{-(r-1)}y^n = z^{-1}x^m z$$

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which implies

$$x^{m^n-m}y^{-n}z^{-(r-1)}y^n = x^{-m}z^{-1}x^m z.$$

Using (c) and the lemma we get

$$x^{m^n-m}y^{-n}z^{-(r-1)}y^n = z^{-r^m}z = z^{-r^m+1}$$
$$\Rightarrow x^{m^n-m}y^{-n} = z^{-r^m+1}y^{-n}z^{(r-1)}$$
$$\Rightarrow x^{m^n-m}y^{-n} = z^{-r^m+r}z^{-(r-1)}y^{-n}z^{(r-1)}.$$

Using (b) and the lemma we get

$$x^{m^n-m}y^{-n} = z^{-r^m+r}y^{-n^r}$$
$$\Rightarrow x^{m^n-m}y^{n^r-n}z^{r^m-r} = e.$$

We let $d = r^m - r, f = m^n - m, g = n^r - n$. So we have

(*) $$x^f y^g z^d = e.$$

Using equations (*), (b) and the lemma we get

$$z^{-1}(z^d x^f)z = (z^d x^f)^n$$
$$\Rightarrow z^{-1}z^d x^f z = (z^d x^f)(z^d x^f)^{n-1}$$
$$\Rightarrow x^{-f}z^{-1}x^f z = (z^d x^f)^{n-1}.$$

Now using (c) with the lemma we get:

$$z^{-r^f+1} = (z^d x^f)^{n-1}.$$

We use the identity in Remark 1 to obtain:

$$z^{-r^f+1} = (x^f)^{n-1}\prod_{k=n-1}^{1}(z^d)^{(x^f)^k} = x^{f(n-1)}\prod_{k=n-1}^{1}(z^d)^{x^{fk}} = x^{f(n-1)}\prod_{k=n-1}^{1}z^{dr^{fk}}.$$

This implies that $x^{f(n-1)} = z^l$ for some integer $l$.

Conjugating by $x$ and using (c) we get $x^{f(n-1)} = z^{lr} = x^{f(n-1)r} \Rightarrow x^{f(n-1)(r-1)} = e$. Therefore, we obtain $x^{(n-1)(r-1)(m^n-m)} = e$. Now using (a) we get

(1) $$x^{(n-1)(r-1)(m^{n-1}-1)} = e.$$

Similar arguments give us:

(2) $$y^{(r-1)(m-1)(n^{r-1}-1)} = e.$$

(3) $$z^{(n-1)(m-1)(r^{m-1}-1)} = e.$$

Johnson and Robertson [7] used relation (*) to show that

(1′) $$x^{(m-1)^2(m^{n-1}-1)} = e$$

(2′) $$y^{(n-1)^2(n^{r-1}-1)} = e$$

(3′) $$z^{(r-1)^2(r^{m-1}-1)} = e.$$

Using relations (a) and (2) we get $x^{m^{(r-1)(m-1)(n^{r-1}-1)}-1} = e$ which we simplify as follows:

$$\begin{aligned} m^{(m-1)(r-1)(n^{r-1}-1)} - 1 &= m^{(m-1)(r-1)(n-1)(n^{r-2}+\cdots+n+1)} - 1 \\ &= [m^{n-1}]^{(m-1)(r-1)(n^{r-2}+\cdots+n+1)} - 1 \\ &= (m^{n-1} - 1)[(m^{n-1})^{(m-1)(r-1)(n^{r-2}+\cdots+n+1)-1} + \cdots + (m^{n-1}) + 1]. \end{aligned}$$

Hence we get

(1″) $$x^{(m^{n-1}-1)[(m^{n-1})^{(m-1)(r-1)(n^{r-2}+\cdots+n+1)-1}+\cdots+m^{n-1}+1]} = e.$$

Using similar arguments we get

(1″) $$y^{(n^{r-1}-1)[(n^{r-1})^{(m-1)(n-1)(r^{m-2}+\cdots+r+1)-1}\cdots+n^{r-1}+1]} = e$$

(3″) $$z^{(r^{m-1}-1)(r^{m-1})^{(n-1)(r-1)(m^{n-2}+\cdots+m+1)-1}+\cdots+r^{m-1}+1|} = e.$$

Using relations (2′) and (a) we get $x^{m^{(n-1)^2(n^{r-1}-1)}-1} = e$ which we simplify as follows:

$$\begin{aligned} m^{(n-1)^2(n^{r-1}-1)} - 1 &= (m^{n-1})^{(n-1)(n^{r-1}-1)} - 1 \\ &= (m^{n-1} - 1)[(m^{n-1})^{(n-1)(n^{r-1}-1)-1} + \cdots + m^{n-1} + 1]. \end{aligned}$$

Therefore we get

(1‴) $$x^{(m^{n-1}-1)[(m^{n-1})^{(n-1)(n^{r-1}-1)-1}+\cdots+m^{n-1}+1]} = e.$$

Similar arguments give

(2‴) $$y^{(n^{r-1}-1)[(n^{r-1})^{(r-1)(r^{m-1}-1)-1}+\cdots+n^{r-1}+1|} = e$$

and

(3‴) $$z^{(r^{m-1}-1)[(r^{m-1})^{(m-1)(m^{n-1}-1)-1}+\cdots+r^{m-1}+1]} = e.$$

We summarize equations (1) to (3), (1′) to (3′), (1″) to (3″) and (1‴) to (3‴) in the theorem in the following section.

**A bound for the order of the group.**

THEOREM 1. *(i)* $x^{(m^{n-1}-1)K_{1i}} = e$ *where* $1 \leq i \leq 4$, $K_{11} = (r-1)(n-1)$, $K_{12} = (m-1)^2$, $K_{13} = [m^{n-1}]^{(r-1)(m-1)(n^{r-2}+\cdots+n+1)-1} + \cdots + m^{n-1} + 1$, *and* $K_{14} = [m^{n-1}]^{(n-1)(n^{r-1}-1)-1} + \cdots + m^{n-1} + 1$.

*(ii)* $y^{(n^{r-1}-1)K_{2i}} = e$ *where* $1 \leq i \leq 4$, $K_{21} = (r-1)(m-1)$, $K_{22} = (n-1)^2$, $K_{23} = [n^{r-1}]^{(m-1)(n-1)(r^{m-2}+\cdots+r+1)-1} + \cdots + n^{r-1} + 1$ *and* $K_{24} = [n^{r-1}]^{(r-1)(r^{m-1}-1)-1} + \cdots + n^{r-1} + 1$.

*(iii)* $z^{(r^{m-1}-1)K_{3i}} = e$ *where* $1 \leq i \leq 4$, $K_{31} = (m-1)(n-1)$, $K_{32} = (r-1)^2$, $K_{33} = [r^{m-1}]^{(n-1)(r-1)(m^{n-2}+\cdots+m+1)-1} + \cdots + r^{m-1} + 1$ *and* $K_{34} = [r^{m-1}]^{(m-1)(m^{n-1}-1)-1} + \cdots + r^{m-1} + 1$.

COROLLARY 1. *(i)* $x^{(m^{n-1}-1)K_1} = e$ *where* $K_1 = gcd\{K_{1i} | 1 \leq i \leq 4\}$.
*(ii)* $y^{(n^{r-1}-1)K_2} = e$ *where* $K_2 = gcd\{K_{2i} | 1 \leq i \leq 4\}$.
$z^{(r^{m-1}-1)K_3} = e$ *where* $K_3 = gcd\{K_{3i} | 1 \leq i \leq 4\}$.

PROOF. Follows easily from the fact that $g^s = g^t = e$ in a group implies $g^{(s,t)} = e$.

COROLLARY 2. $|M(m,n,r)| \leq K_1K_2K_3(m^{n-1}-1)(n^{r-1}-1)(r^{m-1}-1)$.

PROOF. The relations of the group imply that any element has the form $x^iy^jz^k$ for some integers $i,j$ and $k$. Since the integers $(m^{n-1}-1)K_1$, $(n^{r-1}-1)K_2$ and $(r^{m-1}-1)K_3$ divide the orders of $x, y$ and $z$ respectively the result follows easily.

REMARK 2. When we apply Corollary 2 to special cases of $M(m,n,r)$ we notice that the integers $K_{13}, K_{14}, K_{23}, K_{24}, K_{33}, K_{34}$ are usually very large. A weaker form of the corollary could be used where $K_1$ is the *gcd* of $K_{11}$ and $K_{12}$ and similarly for $K_2$ and $K_3$.

REMARK 3. If $K_1$, $K_2$ and $K_3$ are all one, we have $|M(m,n,r)| = (m^{n-1}-1)(n^{r-1}-1)(r^{m-1}-1)$.

REMARK 4. We notice that the bound of the order obtained by Johnson and Robertson is

$$|M(m,n,r)| \leq K_{12}K_{22}K_{32}(m^{n-1}-1)(n^{r-1}-1)(r^{m-1}-1)$$

and so the bound given in Corollary 2 is an improvement to the bound given by them.

We now use Theorem 1 to investigate some cases of $M$. Before that we begin by some preliminaries.

**Some special cases.**

DEFINITION. A group $G$ is an $n$-generator group if it can be generated by $n$ elements. The rank of $G$ is the least $n$ for which the group is $n$-generator.

We observe that $M$ is a 3-generator group but does not have the rank 3 in general.

We let $G$ be the finite split metacyclic group $\langle x,y \mid x^m = y^n = e, x^y = x^r\rangle$ where $r^n \equiv 1 \pmod{m}$.

THEOREM 2 [6]. *$G$ is the split extension $Z_m$ by $Z_n$.*

THEOREM 3 [6]. *The derived subgroup of $G$ is cyclic of order $\frac{m}{(m,r-1)}$.*

REMARK 5. It follows from Theorem 2 that $|G| = mn$.

We now consider general cases of $M(m,n,r)$.

a) $M(m,n,0)$ $m > 2$, $n > 2$: Using Tietze transformations together with Lemma 1, we get the following presentation for $M = \langle x,y \mid x^{m^{n-1}-1} = y^{n-1} = e, x^y = x^m\rangle$. Hence $M$ is a finite metacyclic group. Therefore $M = Z_d \rtimes Z_{n-1}$, $|M| = (n-1)d$, $M'$ is cyclic of order $\frac{d}{(d,m-1)}$ and $M$ is metabelian of rank 2 where $d = m^{n-1}-1$.
b) $M(m,2,r)$ $m, r \geq 3$ and $(m-1, r-1) = 1$: Using The Reidemeister-Schreier process we find that $M' = \langle a,y \mid y^{2^{r-1}-1} = a^d = e, ay = ya\rangle$ where $a = z^{r-1}$ and $d = \frac{r^{m-1}-1}{r-1}$. Therefore $M$ is a metabelian group of order $(m-1)(r^{m-1}-1)(2^{r-1}-1)$ and rank $\leq 3$.

   To explore the structure of $M$ we use Theorem 1 to write the following presentation for $M = \langle x,y,z \mid x^y = x^m, y^z = y^2, z^x = z^r, x^{m-1} = y^{2^{r-1}-1} = z^{r^{m-1}-1} = e\rangle$. Thus we get $x^y = x$, $x^{y^{-1}} = x$, $y^z = y^2$, $y^{z^{-1}} = y^{z^{r^{m-1}-2}} = y^{2^{r^{m-1}-2}}$. Hence

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the subgroup $H = \langle y \mid y^{2^{r-1}-1} = e\rangle$ of $M$ is normal. Using the presentations of group extensions [2] we easily see that $M$ is the split extension of $H$ by $K = \langle x, z \mid x^{m-1} = z^{r^{m-1}-1} = e,\ z^x = z^r\rangle$. Since $K$ is metacyclic of order $(m-1)(r^{m-1}-1)$, this also shows that the order of $M$ is $(m-1)(r^{m-1}-1)(2^{r-1}-1)$. Note. The results of case (b) hold for $M(m,2,2)$.

c) $M(n,n,n)$: We notice that $M' = \langle x^{n-1}, y^{n-1}, z^{n-1}\rangle$. Using the Witt identity [10] we find that $M'$ is abelian if $n^{n(n^{n-1}-1)} \equiv 1[\bmod(n-1)^2(n^{n-1}-1)]$. This congruence relation holds only if $n = 1, 2$ or 3. It is easy to see that $M(1,1,1) \cong Z \times Z \times Z$ and $M(2,2,2) = E$ [3]. If $n \geq 3$ the group $M$ is 3-generated because $\frac{M}{M'} \cong Z_{n-1} \times Z_{n-1} \times Z_{n-1}$. $M(3,3,3)$ is metabelian of order $2^{11}$ [3]. We observe that in Mennicke's paper [8] the order of $M(3,3,3)$ is incorrectly found to be $2^{10}$. It follows easily from Mennicke's work that $M(n,n,n)$ is metabelian exactly when $n-1$ is prime to 3.

REMARK 6. Using Tietze transformations it is possible to show that $M(-m,2,3) \cong M(m+2,2,3)$ and $M(-m,2,2) \cong M(m+2,2,2)$ for $m > 2$.

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