BULL. AUSTRAL. MATH. SOC. 20E45, 20D10
VOL. 67 (2003) [163–169]

FINITE GROUPS WITH TWO p-REGULAR CONJUGACY CLASS LENGTHS

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Let G be a finite p-solvable group for a fixed prime p. We determine the structure of G when the set of p-regular conjugacy class sizes of G is $\{1, m\}$ for an arbitrary integer $m > 1$.

1. INTRODUCTION

The question of how certain arithmetical conditions on the sizes of the conjugacy classes of finite groups G influence the structure of G has been widely studied during the last years. For a fixed prime p, our purpose in this note is to obtain information on the structure of finite p-solvable groups from the set of conjugacy class sizes of p'-elements, that is, from its p-regular class sizes. A classical result of Ito ([4]) asserts that if 1 and $m > 1$ are the only lengths of conjugacy classes of a finite group G, then there exists a prime q such that $G = Q \times A$, where Q is a Sylow q-subgroup of G and A is Abelian. Thus, in particular m is a power of q. Our main result extends Ito's Theorem when this problem is transferred to p-regular classes in a p-solvable group.

THEOREM A. *Suppose that G is a finite p-solvable group and that $\{1, m\}$ are the p-regular conjugacy class sizes of G. Then $m = p^a q^b$, with q a prime distinct from p and $a, b \geqslant 0$. If $b = 0$ then G has Abelian p-complement. If $b \neq 0$, then $G = PQ \times A$, with $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $A \leqslant Z(G)$. Furthermore, if $a = 0$ then $G = P \times Q \times A$.*

2. PROOF OF THEOREM A

In order to obtain Theorem A, we first need some preliminary lemmas. The first one is a generalisation for p-regular elements of a lemma of Ito (in fact, the lemma of Ito is stated in [4] for arbitrary elements and it is assumed that the group is centreless).

We shall denote by $G_{p'}$ the set of p-regular elements of G.

LEMMA 1. *Let G be a finite group, $x \in G_{p'}$ and $C_G(x) < G$. Assume the following:*

1. *If $C_G(a) \leqslant C_G(x)$ for $a \in G_{p'}$, then $C_G(a) = C_G(x)$.*

Received 29th July, 2002

2. *If $C_G(x) \leqslant C_G(b)$ for $b \in G_{p'}$, then $C_G(x) = C_G(b)$ or $b \in Z(G)$. Then $C_G(x) = P \times L$, with P a Sylow p-subgroup of $C_G(x)$ and $L \leqslant Z(C_G(x))$, or $C_G(x) = PQ \times A$, with P a p-Sylow of $C_G(x)$, Q a q-Sylow of $C_G(x)$, for some prime $q \neq p$, and $A \leqslant Z(G)$.*

PROOF: Write $x = x_1 x_2 \dots x_s$ where the order of each x_i is a power of a prime distinct from p and the x_i commute pairwise. As $x \notin Z(G)$, there exists an i such that $x_i \notin Z(G)$. By applying hypothesis 2 we have $C_G(x) = C_G(x_i)$, whence there is no loss if we assume that x is a q-element for some prime $q \neq p$.

Suppose that there exists a prime divisor r of $|C_G(x)|$ such that $p \neq r \neq q$ (in other case the lemma is proved) and take R a Sylow r-subgroup of $C_G(x)$. If $y \in R$, since x and y have coprime orders then $C_G(yx) = C_G(x) \cap C_G(y)$. By hypothesis 1 it follows that $C_G(x) = C_G(yx) \subseteq C_G(y)$. Thus $R \leqslant Z(C_G(x))$, so we can write $C_G(x) = PQ \times A$, for some $P \in \mathrm{Syl}_p(C_G(x))$, $Q \in \mathrm{Syl}_q(C_G(x))$ and $A \leqslant Z(C_G(x))$. If $A \subseteq Z(G)$ we are finished. Suppose then that there exists a non-central $u \in A$. Since u is a $\{p, q\}'$-element which commutes with x, by applying hypotheses 1 and 2, we obtain $C_G(ux) = C_G(x) = C_G(u)$. Now, take $z \in Q$. Then $C_G(uz) = C_G(u) \cap C_G(z) \subseteq C_G(u) = C_G(x)$. By hypothesis 1, we get $C_G(uz) = C_G(x)$, so $C_G(x) \subseteq C_G(z)$. Therefore $z \in Z(C_G(x))$. If we put $L = Q \times A$ then $C_G(x) = P \times L$, with $L \leqslant Z(C_G(x))$ as wanted. ∎

In the following lemma, we characterise those groups whose non-central p-regular conjugacy class sizes are powers of the prime p.

LEMMA 2. *Let G be a finite group. Then all conjugacy class sizes in $G_{p'}$ are p-numbers if and only if G has Abelian p-complements.*

PROOF: We show first that G is solvable in both directions of the Lemma. Suppose first that all p-regular conjugacy class lengths of G are p-numbers. Our assumption is inherited by normal subgroups and factor groups, and hence, arguing by induction on $|G|$ we may assume that G is simple nonabelian. However, Burnside's Theorem ([**3**, Theorem 15.2] for instance) asserts that a simple group cannot possess a conjugacy class of prime power length and thus, the claim follows. Conversely, suppose that G has an Abelian p-complement. Then G can be written as the product of two nilpotent subgroups, that is, an Abelian p-complement and a Sylow p-subgroup of G. By Kegel-Wielandt's Theorem ([**2**, Theorem VI.4.3]), it follows that G is solvable too.

Suppose now that every p-regular conjugacy class of G has p-power size and work by induction on $|G|$ to show that G has Abelian p-complement. We assume first that $O_p(G) \neq 1$. By induction, $G/O_p(G)$ has Abelian p-complement and trivially so does G. Thus, we can assume that $O_p(G) = 1$ and consequently $O_{p'}(G) \neq 1$. Let x be a non-central element in $G_{p'}$. As G is solvable and $|G : C_G(x)|$ is a p-power there exists a p-complement of G, say H, such that $x \in H \subseteq C_G(x)$. Observe that $1 \neq F(G)$

$\subseteq O_{p'}(G) \subseteq H$, where $F(G)$ is the Fitting subgroup. This implies that

$$x \in C_G(H) \subseteq C_G(F(G)) \subseteq F(G) \subseteq O_{p'}(G).$$

Therefore, any non-central p-regular element of G belongs to $O_{p'}(G)$ (and any central p-regular element too), whence G has normal p-complement. Moreover, notice that this complement is Abelian.

The converse direction is trivial, just noticing that since G is solvable then any two p-complements of G are conjugated, whence all of them are Abelian. □

The following result determines the structure of p-solvable groups whose p-regular class sizes are p'-numbers.

LEMMA 3. *Suppose that G is a finite p-solvable group and that p is not a divisor of the lengths of p-regular conjugacy classes. Then $G = P \times H$ where P is a Sylow p-subgroup and H is a p-complement of G.*

PROOF: See Proposition 2 of [**1**]. □

PROOF OF THEOREM A: By Lemma 2, if $m = p^a$ then G has an Abelian p-complement, so this case is finished. We shall assume then that m is not a p-power, that is, $b \neq 0$, and procceed by induction on $|G|$ in several steps to prove that G has the given structure. The last assertion in the statement will be followed then as a trivial consequence of Lemma 3.

STEP 1. We can assume that $C_G(x) = P_x \times L_x$, with P_x a Sylow p-subgroup of $C_G(x)$ and $L_x \leqslant Z(C_G(x))$, for any non-central $x \in G_{p'}$.

By Lemma 1, we know that for any non-central p-regular element x of G then $C_G(x) = P_x \times L_x$, with P_x a Sylow p-subgroup of $C_G(x)$ and $L_x \leqslant Z(C_G(x))$, or $C_G(x) = P_x Q_x \times A$, with P_x and Q_x a Sylow p-subgroup and a Sylow q-subgroup of $C_G(x)$, respectively, for some prime $q \neq p$, and $A \leqslant Z(G)$.

Suppose that the second possibility holds for some non-central $x \in G_{p'}$. We can also assume that there exists some non-central r-element z in G, for some prime r distinct from p and q. Then

$$A < \langle A, z \rangle \leqslant C_G(z),$$

contradicting the fact that $|C_G(z)|_{\{p,q\}'} = |C_G(x)|_{\{p,q\}'} = |A|$. Therefore, we can write $G = PQ \times A$, for some $P \in \mathrm{Syl}_p(G)$ and $Q \in \mathrm{Syl}_q(G)$, so the theorem is proved.

STEP 2. Let x and y be two non-central p-regular elements. Suppose that $C_G(x) \neq C_G(y)$, then $(C_G(x) \cap C_G(y))_{p'} = Z(G)_{p'}$.

Suppose that there exists a non-central element $a \in (C_G(x) \cap C_G(y))_{p'}$. By applying Step 1, we have $C_G(x) \subseteq C_G(a)$ and $C_G(y) \subseteq C_G(a)$. As $C_G(a) \neq G$ and all proper centralisers of p-regular elements have the same order, we conclude $C_G(x) = C_G(a) = C_G(y)$, a contradiction.

For the rest of the proof we are going to distinguish two cases, depending on whether all centralisers of non-central elements in $G_{p'}$ are conjugated in G or not.

CASE 1. Suppose that $C_G(x)$ and $C_G(y)$ are conjugated in G for any non-central $x, y \in G_{p'}$.

STEP 3. We can assume that $O^p(G) = G$.

Suppose that $O^p(G) < G$. Let x be a p-regular element of $O^p(G)$ such that $x \notin Z(O^p(G))$, so $x \notin Z(G)$. By Step 1, we write $C_G(x) = P_x \times L_x$, with P_x a Sylow p-subgroup of $C_G(x)$ and $L_x \leqslant Z(C_G(x))$. As $L_x \subseteq O^p(G)$, then

$$O^p(G) \cap C_G(x) = L_x(P_x \cap O^p(G)).$$

Furthermore,

$$\frac{|G|}{|O^p(G)|}\frac{|O^p(G)|}{|O^p(G) \cap C_G(x)|} = \frac{|G|}{|C_G(x)|}\frac{|P_x|}{|P_x \cap O^p(G)|}$$

and this yields to

$$\frac{|O^p(G)|}{|O^p(G) \cap C_G(x)|} = m\frac{|P_xO^p(G)|}{|G|} = \frac{m}{p^l}$$

where l is a positive integer or zero. Since we are assuming that all centralisers of non-central $x \in G_{p'}$ are G-conjugated, the corresponding subgroups P_x must be conjugated too. Consequently, p^l is a fixed power not depending on x. This implies that $\{1, m/p^l\}$ are the only lengths of p-regular conjugacy classes in $O^p(G)$. Also, in any case m/p^l is not a power of p, so we can apply the inductive hypothesis to $O^p(G)$ to obtain $O^p(G) = P_0Q \times A_0$, with P_0 a Sylow p-subgroup of $O^p(G)$, Q a Sylow q-subgroup of G for some prime $q \neq p$, and $A_0 \leqslant Z(O^p(G))$.

If all centralisers of non-central p-regular elements are equal, say to $C_G(x)$ with $x \in G_{p'}$, then every p-regular element belongs to $C_G(x)_{p'}$, which is Abelian by Step 1. Then G has an Abelian p-complement and $m = p^a$ by Lemma 2, a contradiction. Thus, we can take x, y non-central elements of $G_{p'}$ such that $C_G(x) \neq C_G(y)$. Also, notice that $x, y \in O^p(G)$, whence $A_0 \subseteq (C_G(x) \cap C_G(y))_{p'} = Z(G)_{p'}$ by Step 2. Consequently, we can write $G = PQ \times A_0$, with $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $A_0 \leqslant Z(G)$, so the theorem is proved.

STEP 4. $C_G(x) < N_G(C_G(x))$ for every non-central $x \in G_{p'}$.

Suppose that $N_G(C_G(x)) = C_G(x)$ for some non-central $x \in G_{p'}$. By Step 1 we know that $C_G(x) = P \times L$, with $P \in \mathrm{Syl}_p(C_G(x))$ and L Abelian. We claim that if $q \neq p$ is a prime dividing m, then q also divides $|L/Z(G)_{p'}|$. If q does not divide $|L/Z(G)_{p'}|$, since all centralisers of non-central p-regular elements have the same order, then any q-element is central, so q does not divide m and the claim is proved. Therefore, as we are assuming that m is not a p-power, there exists a Sylow subgroup of $C_G(x)$, say $L_i \subseteq L$, which is non-central in G and such that L_i is properly contained in some Sylow subgroup P_i of G. In particular, $L_i < N_{P_i}(L_i) \subseteq N_G(L_i)$, so we can take $y \in N_G(L_i) - L_i$. If $L^y \neq L$, then

$L^y \cap L = Z(G)_{p'}$ by Step 2. As $L_i^y = L_i$, then $L_i \leqslant L^y \cap L = Z(G)_{p'}$, a contradiction. Thus, $L^y = L$. Moreover,

$$P \subseteq C_G(L) = C_G(L^y) \subseteq C_G(x^y) = P^y \times L^y.$$

Hence $P = P^y$ and as a consequence $y \in N_G\big(C_G(x)\big) = C_G(x)$, which implies the contradiction $y \in L_i$.

STEP 5. Conclusion in Case 1.

By Step 3, we can consider $O^{p'}(G) < G$. Fix a non-central element $x \in G_{p'}$. Let $g \in G$ and write $g = g_p g_{p'}$, where g_p and $g_{p'}$ are the p-part and p'-part of g, respectively. If $g_{p'} \in Z(G)$, it certainly follows that $g \in O^{p'}(G)Z(G)_{p'}$. If $g_{p'} \notin Z(G)$, since we are assuming that $C_G(g_{p'}) = C_G(x)^n$ for some $n \in G$, then $g \in C_G(x)^n$. Therefore

$$G = \bigcup_{n \in G} C_G(x)^n \ \bigcup \ O^{p'}(G)Z(G)_{p'}.$$

By counting elements, this implies

$$|G| \leqslant \Big|G : N_G\big(C_G(x)\big)\Big|\Big(\big|C_G(x)\big| - 1\Big) + \big|O^{p'}(G)Z(G)_{p'}\big|,$$

hence

$$1 \leqslant \frac{|C_G(x)| - 1}{|N_G(C_G(x))|} + \frac{|O^{p'}(G)Z(G)_{p'}|}{|G|}.$$

We put $\Big|N_G\big(C_G(x)\big)\Big| = n_1$. Suppose first that $O^{p'}(G)Z(G)_{p'} < G$. Since $n_1 \geqslant 2\big|C_G(x)\big|$ by Step 4, we obtain the following contradiction

$$1 \leqslant \frac{1}{2} - \frac{1}{n_1} + \frac{1}{2}.$$

Accordingly, we can assume that $O^{p'}(G)Z(G)_{p'} = G$.

Now, $O^{p'}(G)$ cannot be a p-group, otherwise G has a central p-complement, which yields to a contradiction. Therefore, there exist non-central p-regular elements in $O^{p'}(G)$ and for any such element y we have

$$\frac{|O^{p'}(G)|}{|C_G(y) \cap O^{p'}(G)|} = \frac{|G|}{|C_G(y)|} = m.$$

Thus, we can apply the inductive hypothesis to $O^{p'}(G)$ to conclude that $O^{p'}(G) = PQ_0 \times S_0$, with $P \in \mathrm{Syl}_p(G)$, $Q_0 \in \mathrm{Syl}_q\big(O^{p'}(G)\big)$ for some prime $q \neq p$, and $S_0 \leqslant Z\big(O^{p'}(G)\big) \leqslant Z(G)$. Since $O^{p'}(G)Z(G)_{p'} = G$, we get $G = PQ \times A$, with $A \leqslant Z(G)$ and $Q \in \mathrm{Syl}_q(G)$, so Case 1 is finished.

CASE 2. We assume now that the centralisers of non-central elements in $G_{p'}$ are not all G-conjugated. We put $\overline{G} = G/Z(G)_{p'}$ and use bars to work in the factor group.

STEP 6. Let $\bar{x}, \bar{y} \neq 1$ be two p-regular elements in $\overline{G}$ such that $\bar{x}\bar{y} = \bar{y}\bar{x}$ and $C_G(x) \neq C_G(y)$. Then $o(\overline{x}) = o(\overline{y})$ is a prime.

Notice that x and y are p-regular elements. Moreover, since $\overline{x}$ and $\overline{y}$ commute, then $\bar{x}\bar{y} = \bar{x}\bar{y}$ is p-regular and consequently, so is xy. Suppose first that $o(\overline{x}) < o(\overline{y})$, then $(\bar{x}\bar{y})^{o(\bar{x})} = \bar{y}^{o(\bar{x})} \neq 1$. Furthermore,

$$1 \neq (\bar{x}\bar{y})^{o(\overline{x})} = \overline{xy}^{o(\overline{x})} \in \overline{C_G(xy)} \cap \overline{C_G(y)}.$$

By applying Step 2, we deduce that $C_G(y) = C_G(xy)$, so in particular $x \in C_G(y)$. Again by Step 2, we obtain $C_G(x) = C_G(y)$, contradicting the hypothesis of this step. Therefore, $o(\overline{x}) = o(\overline{y})$.

On the other hand, if s is a prime divisor of $o(\overline{x})$ and $\overline{x}^s \neq 1$, then $C_G(x) \subseteq C_G(x^s) < G$, whence we obtain $C_G(x) = C_G(x^s)$. Moreover, $\overline{x}^s\overline{y} = \bar{y}\bar{x}^s$. By the above paragraph it follows that $o(\overline{x}^s) = o(\overline{y}) = o(\overline{x})$, a contradiction.

STEP 7. Let g be a non-central element in $G_{p'}$ and consider the conjugacy class of $\overline{g}$ in $\overline{G}$, $\overline{g}^{\overline{G}}$. Then there exists some non-central $x \in G_{p'}$ such that $\overline{g}^{\overline{G}} \cap \overline{C_G(x)} = \emptyset$.

Suppose that this is false. Then for every non-central $x \in G_{p'}$ we have that $\overline{C_G(x)}$ must contain some conjugate of $\overline{g}$, say $\overline{g}^{\overline{n}}$ for some $\overline{n} \in \overline{G}$. Thus, $\overline{g^n} = \overline{g}^{\overline{n}} \in \overline{C_G(x)}$, and consequently $g^n \in C_G(x)_{p'}$. By applying Step 1 we deduce that $C_G(x) \subseteq C_G(g^n)$, and hence the equality holds. It follows that any two centralisers are conjugated in G, a contradiction.

STEP 8. The order of every non-trivial p-regular element in $\overline{G}$ is a prime.

Suppose that $o(\overline{g})$ is composite for some p-regular element $\overline{g}$. Notice that g is p-regular too. By Step 7, there exists a non-central element $x \in G_{p'}$ such that $\overline{g}^{\overline{G}} \cap \overline{C_G(x)} = \emptyset$. Write $\overline{C_{p'}}$ for $\overline{C_G(x)_{p'}}$ and observe that $\overline{C_{p'}}$ operates on $\overline{g}^{\overline{G}}$ by conjugation. Furthermore, by Step 6 no element in $\overline{C_{p'}}$ distinct from 1 centralises any element in $\overline{g}^{\overline{G}}$, and hence all orbits of $\overline{C_{p'}}$ on $\overline{g}^{\overline{G}}$ have the same length, $|\overline{C_{p'}}|$, which implies that $|\overline{C_{p'}}|$ divides $|\overline{g}^{\overline{G}}|$.

On the other hand, again by applying Step 6, we deduce that $\overline{C_G(g)_{p'}}$ operates without fixed points on $\overline{g}^{\overline{G}} - \overline{g}^{\overline{G}} \cap \overline{C_G(g)}$. Therefore, $|\overline{C_G(g)_{p'}}|$ divides $|\overline{g}^{\overline{G}}| - |\overline{g}^{\overline{G}} \cap \overline{C_G(g)}|$. As $|\overline{C_G(g)_{p'}}| = |\overline{C_{p'}}|$, we conclude that $|\overline{C_G(g)_{p'}}|$ also divides $|\overline{g}^{\overline{G}} \cap \overline{C_G(g)}|$, which is a contradiction because

$$0 < |\overline{g}^{\overline{G}} \cap \overline{C_G(g)}| < |\overline{C_G(g)_{p'}}|.$$

STEP 9. Conclusion in Case 2.

By Step 1, all $C_G(x)_{p'}$ are Abelian and have the same order for any non-central $x \in G_{p'}$. Therefore, by applying Step 8 we obtain that $|\overline{C_G(x)_{p'}}|$ is a power of some prime $q \neq p$. Hence $\overline{G} = G/Z(G)_{p'}$ is a $\{p, q\}$-group and consequently we can write $G = PQ \times A$, where $P \in \mathrm{Syl}_p(G)$, $Q \in \mathrm{Syl}_q(G)$ and $A \leqslant Z(G)$. ∎

REMARK. The proof of Theorem A has been divided into two cases depending on the fact that all centralisers of non-central p-regular elements may be conjugated or not. It is easy to find examples where both cases actually occur when G has exactly two p-regular conjugacy class lengths.

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