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ON RATIONAL APPROXIMATION ON THE POSITIVE REAL AXIS

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1. Introduction and statement of results. In their study of the uniform approximation of the reciprocal of e^z by reciprocals of polynomials on the positive real axis, Cody, Meinardus, and Varga [**3**] showed that if $\mathscr{P}_n$ denotes the class of all polynomials of degree at most n and

$$(1)\qquad \lambda_{m,n}(e^{-z}) = \inf_{\substack{p(z)\in\mathscr{P}_m\\ q(z)\in\mathscr{P}_n}} \left\{ \sup_{0\leqq x<\infty} \left| e^{-x} - \frac{p(x)}{q(x)} \right| \right\}$$

then

$$(2)\qquad \frac{1}{6} \leqq \lim_{n\to\infty} (\lambda_{0,n}(e^{-z}))^{1/n} \leqq 0.43501\ldots.$$

Subsequently, Schönhage [**8**] proved that

$$(3)\qquad \lim_{n\to\infty} (\lambda_{0,n}(e^{-z}))^{1/n} = \frac{1}{3}.$$

One of the most important problems in the theory of approximation which was settled by P. L. Chebyshev is the uniform approximation on the unit interval of a polynomial of degree $n+1$ by polynomials of lower degree. Chebyshev also discovered the following analogous result for rational approximation [**1**, pp. 278–280]:

Let a_ν, $\nu = 0, 1, \ldots, n$ be prescribed real numbers with $a_0 \neq 0$, and set

$$f(x) = \sum_{\nu=0}^{n} a_\nu 2^{-\nu} x^{N-\nu}$$

where $N > n$. Then

$$(4)\qquad \inf_{\substack{p(z)\in\mathscr{P}_{N-1}\\ q(z)\in\mathscr{P}_n}} \left\{ \sup_{-1\leqq x\leqq 1} \left| f(x) - \frac{p(x)}{q(x)} \right| \right\} = \frac{|\lambda|}{2^{N-1}}$$

where λ is the smallest eigenvalue of the Hankel matrix

$$\begin{bmatrix} c_n & c_{n-1} & \cdots & c_1 & c_0 \\ c_{n-1} & c_{n-2} & \cdots & c_0 & 0 \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ c_1 & c_0 & \cdots & 0 & 0 \\ c_0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

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with

$$c_r = \sum_{\nu=0}^{[r/2]} \binom{N - r + 2\nu}{\nu} a_{r-2\nu} \quad (r = 0, 1, \ldots, n).$$

If we take in particular

$$f(x) = ((1 + x)/2d)^{n+1} - (1/2d)^{n+1}, d > 0,$$

and replace x by $(1 - (xc/d))/(1 + (xc/d))$, $c > 0$, we obtain

$$(5) \qquad \gamma_n = \inf_{\substack{p(z) \in \mathcal{P}_n \\ q(z) \in \mathcal{P}_n}} \left\{ \sup_{0 \leq x < \infty} \left| \frac{1}{(cx + d)^{n+1}} - \frac{p(x)}{q(x)} \right| \right\}$$

which gives the best uniform approximation of the reciprocal of $(cz + d)^{n+1}$ by rational functions of degree at most n on $[0, \infty)$. Due to the fact that Chebyshev gave γ_n in terms of the smallest eigenvalue of a certain matrix the dependence of γ_n on n is not easily seen. We will, however, show by an elementary method that

$$(6) \qquad \overline{\lim_{n \to \infty}} \gamma_n^{1/n} \geqq \frac{1}{27d}$$

which means that the quantity γ_n cannot go to zero faster than geometrically.

In analogy with the above result of Cody, Meinardus and Varga we consider the uniform approximation of the reciprocal of the polynomial $(cz + d)^N$, $c > 0$, $d > 0$, by reciprocals of polynomials of degree at most $n < N$. We prove:

THEOREM. *If the ratio* $r = N/(n + 1) \geqq 1$ *is fixed then*

$$(7) \qquad \lim_{n \to \infty}' \{\lambda_{0,n}((cz + d)^{-N})\}^{1/n} = \frac{r^r (3r - 1)^{3r-1}}{(3r)^{3r} (r - 1)^{r-1} d^r}$$

where lim′ *indicates that the integer n assumes only those values for which* $(n + 1)r$ *is an integer.*

In the special case $c = d = r = 1$ our result gives

COROLLARY 1.

$$\lim_{n \to \infty} \{\lambda_{0,n}((z + 1)^{-n-1})\}^{1/n} = \frac{4}{27}.$$

Earlier it was shown by Erdös and Reddy [**4**] that

$$1/8 \leqq \{\lambda_{0,n}((z + 1)^{-n-1})\}^{1/n} \leqq 1/2.$$

Besides, putting $c = 1/N$, $d = 1$ the function considered becomes $(1 + z/N)^{-N}$ which tends uniformly to e^{-z} on the interval $[0, \infty)$ as $N \to \infty$. Furthermore, $(r^r(3r - 1)^{3r-1})/((3r)^{3r}(r - 1)^{r-1})$ increases monotonically to $1/3$ as r tends to infinity. From this point of view our result touches the scope of Schönhage's result (3) and even leads to a part of it. In fact, by a limiting process in our proof we can conclude that $\lim_{n\to\infty}(\lambda_{0,n}(e^{-z}))^{1/n} \geqq 1/3$ must hold.

We note that (7) also implies the following fact which is rather curious:

COROLLARY 2. *Given a sufficiently large positive integer N the function $(cz + d)^{-N}$ can be approximated by reciprocals of polynomials $p_n(z)$ of degree at most $n < N$ with*

$$\sup_{0\leqq x<\infty} \left| (cx + d)^{-N} - \frac{1}{p_n(x)} \right| < \rho^n, \quad \textit{where } \rho < 1$$

if and only if $d > 4/27$.

It follows from (5) and (6) that even the quantity

$$\inf_{\substack{p(z)\in\mathscr{P}_n \\ q(z)\in\mathscr{P}_n}} \left\{ \sup_{0\leqq x<\infty} \left| (cx + d)^{-N} - \frac{p(x)}{q(x)} \right| \right\}$$

does not go to zero faster than geometrically as $n \to \infty$, if the ratio $N/(n + 1)$ maintains a fixed value $\geqq 1$.

Our approach to our theorem is analogous to that of Schönhage in the sense that the best uniform approximation by reciprocals of polynomials in $\mathscr{P}_n$ turns out to be comparable to a certain weighted least square approximation by polynomials in $\mathscr{P}_n$.

2. Lemmas. For the proof of our theorem we need to introduce the finite sequence of orthonormal polynomials on $[1, \infty)$ with respect to the weight function $w(x) = x^{-R}$. As an important tool to obtain quantitative results we shall use the following well known identity.

LEMMA 1. ([**2**, p. 195; **7**, Chapter 7, Problem 3]). *For complex numbers a_ν, b_ν $(\nu = 1, 2, \ldots, k)$ such that $a_i + b_j \neq 0$ for all $1 \leqq i, j \leqq k$ we have*

$$\begin{vmatrix} \frac{1}{a_1+b_1} & \frac{1}{a_1+b_2} & \cdots & \frac{1}{a_1+b_k} \\ \frac{1}{a_2+b_1} & \frac{1}{a_2+b_2} & \cdots & \frac{1}{a_2+b_k} \\ \vdots & \vdots & & \vdots \\ \frac{1}{a_k+b_1} & \frac{1}{a_k+b_2} & \cdots & \frac{1}{a_k+b_k} \end{vmatrix} = \frac{\prod_{1\leqq i<j\leqq k} (a_i - a_j)\cdot(b_i - b_j)}{\prod_{i=1}^{k}\prod_{j=1}^{k} (a_i + b_j)}.$$

LEMMA 2. *Let $R \geqq 3$, $k = [(R - 3)/2]$ and $w(x) = x^{-R}$. Then there exists a sequence of orthonormal polynomials $\{\psi_\nu(R, x)\}_{\nu=0,1,\ldots,k}$ on $[1, \infty)$ with respect*

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to the weight function $w(x)$, *i.e.*

$$(8)\qquad \int_1^\infty w(x)\psi_\nu(R, x)\psi_\mu(R, x)dx = \delta_{\nu\mu} \quad (0 \leqq \nu, \mu \leqq k).$$

Moreover, with

$$(9)\qquad \begin{cases} \lambda_\nu = \dfrac{\sqrt{R-2\nu-1}}{\nu!} \displaystyle\prod_{i=1}^{\nu} (R-\nu-i), \\ \alpha_\nu = \dfrac{R-\nu+1}{R-2\nu+2} + \dfrac{\nu R}{(R-2\nu+2)(R-2\nu)}, \\ \beta_\nu = \dfrac{\nu}{\sqrt{(R-2\nu-1)(R-2\nu+1)}} \dfrac{R-\nu}{R-2\nu}, \end{cases}$$

for $\nu = 0, 1, 2, \ldots, k$ *the recurrence relation*

$$(10)\qquad \frac{\psi_{\nu+1}(R, x)}{\lambda_{\nu+1}} = (x-\alpha_{\nu+1}) \frac{\psi_\nu(R, x)}{\lambda_\nu} - \beta_\nu^2 \frac{\psi_{\nu-1}(R, x)}{\lambda_{\nu-1}} \qquad (\nu = 1, 2, \ldots, k-1)$$

where

$$(11)\qquad \psi_0(R, x) = \lambda_0 \quad \text{and} \quad \psi_1(R, x) = \lambda_1 \cdot (x - \alpha_1),$$

holds.

Proof. For the given k all the integrals

$$c_\nu = \int_1^\infty w(x)x^\nu dx = \frac{1}{R-\nu-1} \quad (\nu = 0, 1, \ldots, 2k)$$

exist. There is, therefore (cf. [**10**, §§ 2.1–2.2]), a unique sequence of polynomials $\psi_\nu(R, x)$ of degree $\nu (\nu = 0, 1, \ldots, k)$ satisfying (8). Furthermore, these polynomials are given by $\psi_0(R, x) = c_0^{-1/2}$ and

$$(12)\qquad \psi_\nu(R, x) = \frac{1}{\sqrt{D_{\nu-1}D_\nu}} \begin{vmatrix} c_0 & c_1 & \ldots & c_\nu \\ c_1 & c_2 & \ldots & c_{\nu+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{\nu-1} & c_\nu & \ldots & c_{2\nu-1} \\ 1 & x & \ldots & x^\nu \end{vmatrix} \qquad (1 \leqq \nu \leqq k),$$

where

$$(13)\qquad D_\nu = \begin{vmatrix} c_0 & c_1 & \ldots & c_\nu \\ c_1 & c_2 & \ldots & c_{\nu+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_\nu & c_{\nu+1} & \ldots & c_{2\nu} \end{vmatrix} \qquad (0 \leqq \nu \leqq k).$$

To prove the other assertions we write $\psi_\nu(R, x)$ as

$$(14) \quad \psi_\nu(R, x) = \lambda_\nu x^\nu + \lambda_\nu^* x^{\nu-1} + \varphi_{\nu-2}(x),$$

where $\varphi_{\nu-2}(x)$ is a polynomial of degree at most $\nu - 2$. It is known (see [**9**; **10**, §3.2]) that the polynomials $\psi_\nu(R, x)$ indeed satisfy the equations (10) and (11), if we set $\beta_0 = 0$,

$$(15) \quad \alpha_\nu = \frac{\lambda_{\nu-1}^*}{\lambda_{\nu-1}} - \frac{\lambda_\nu^*}{\lambda_\nu} \quad \text{and} \quad \beta_\nu = \frac{\lambda_{\nu-1}}{\lambda_\nu} \quad (\nu = 1, 2, \ldots, k).$$

It is only (9) that remains to be verified.

We readily see from (12) that

$$\lambda_\nu = \sqrt{\frac{D_{\nu-1}}{D_\nu}} \quad (\nu = 1, 2, \ldots, k), \quad \frac{\lambda_1^*}{\lambda_1} = -\frac{c_1}{c_0} = -\frac{R-1}{R-2},$$

and

$$(16) \quad \frac{\lambda_\nu^*}{\lambda_\nu} = \frac{-1}{D_{\nu-1}} \begin{vmatrix} c_0 & c_1 & \ldots & c_{\nu-2} & c_\nu \\ c_1 & c_2 & \ldots & c_{\nu-1} & c_{\nu+1} \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot \\ c_{\nu-1} & c_\nu & \ldots & c_{2\nu-3} & c_{2\nu-1} \end{vmatrix} \quad (\nu = 2, 3, \ldots, k).$$

To calculate D_ν we apply Lemma 1 with

$$a_i = -i, \quad b_i = R - i + 1 \quad (i = 1, 2, \ldots, \nu + 1)$$

and obtain

$$(17) \quad D_\nu = \frac{\prod_{1 \leq i < j \leq \nu+1} (i-j)^2}{\prod_{i=1}^{\nu+1} \prod_{j=1}^{\nu+1} (R - i - j + 1)} \quad (\nu = 0, 1, \ldots, k).$$

Hence,

$$\lambda_\nu = \sqrt{\frac{D_{\nu-1}}{D_\nu}} = \frac{\sqrt{R - 2\nu - 1}}{\nu!} \prod_{i=1}^{\nu} (R - i - \nu) \quad (\nu = 1, 2, \ldots, k)$$

and, therefore,

$$\beta_\nu = \frac{\lambda_{\nu-1}}{\lambda_\nu} = \nu . \sqrt{\frac{R - 2\nu + 1}{R - 2\nu - 1}} \frac{R - \nu}{(R - 2\nu + 1)(R - 2\nu)} \, (\nu = 1, 2, \ldots, k),$$

as stated in (9). To handle the determinant appearing in (16) we put

$$a_i = -i \quad (i = 1, 2, \ldots, \nu),$$

$$b_i = \begin{cases} R - i + 1 & \text{for } i = 1, 2, \ldots, \nu - 1, \\ R - \nu & \text{for } i = \nu \end{cases}$$

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and with the help of Lemma 1 obtain

$$(18)\quad \begin{vmatrix} c_0 & \cdots & c_{\nu-2} & c_\nu \\ c_1 & \cdots & c_{\nu-1} & c_{\nu+1} \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & \cdot & \cdot \\ c_{\nu-1} & & c_{2\nu-3} & c_{2\nu-1} \end{vmatrix} =$$

$$= \frac{\prod_{1\leqq i<j\leqq \nu-1} (i-j)^2 \prod_{i=1}^{\nu-1} (i-\nu)(i-1-\nu)}{\prod_{i=1}^{\nu-1}\prod_{j=1}^{\nu-1} (R-i-j+1) \prod_{i=1}^{\nu-1} (R-i+1-\nu) \prod_{j=1}^{\nu} (R-\nu-j)}.$$

Now set

$$M_1 = \frac{\prod_{1\leqq i<j\leqq \nu-1} (i-j)^2}{\prod_{1\leqq i<j\leqq \nu} (i-j)^2} = \frac{1}{\{(\nu-1)!\}^2},$$

$$M_2 = \prod_{i=1}^{\nu-1} (i-\nu)(i-1-\nu) = \nu\{(\nu-1)!\}^2,$$

and

$$M_3 = \frac{\prod_{i=1}^{\nu}\prod_{j=1}^{\nu} (R-i-j+1)}{\prod_{i=1}^{\nu-1}\prod_{j=1}^{\nu-1} (R-i-j+1) \prod_{i=1}^{\nu-1} (R-i+1-\nu) \prod_{j=1}^{\nu} (R-\nu-j)}$$

$$= \prod_{j=1}^{\nu} \frac{R-j-\nu+1}{R-j-\nu} = \frac{R-\nu}{R-2\nu}.$$

Then, from (16), (17), and (18) we get

$$\frac{\lambda_\nu^*}{\lambda_\nu} = -M_1M_2M_3 = -\nu\frac{R-\nu}{R-2\nu},$$

valid for $\nu = 0, 1, \ldots, k$. Thus,

$$\alpha_\nu = \frac{\lambda_{\nu-1}^*}{\lambda_{\nu-1}} - \frac{\lambda_\nu^*}{\lambda_\nu} = \frac{R-\nu+1}{R-2\nu+2} + \frac{\nu R}{(R-2\nu+2)(R-2\nu)} \qquad (\nu = 1, 2, \ldots, k),$$

which completes the proof of Lemma 2.

The next lemma gives some useful information about the location of the zeros of the orthogonal polynomials defined above.

LEMMA 3. *Let* $r \geqq 1$ *and put* $R = 4r(n+1)$, $n \in \mathbf{N}$. *Then* k *of Lemma* 2 *is greater than* n *and the first* n *polynomials* $\psi_\nu(R, x)$ $(\nu = 1, 2, \ldots, n)$ *have all their zeros in the interval* $(1, 4r^2/(2r-1)^2)$.

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Proof. Since the polynomials $\psi_\nu(R, x)$ satisfy the recurrence relation (10) it follows from a known result (see e.g. [**9**]) that the zeros of $\psi_\nu(R, x)$ are the eigenvalues of the matrix

$$\begin{bmatrix} \alpha_1 & \beta_1 & & & & & \\ \beta_1 & \alpha_2 & \beta_2 & & & \mathbf{0} & \\ & \cdot & \cdot & \cdot & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & \mathbf{0} & & & \beta_{\nu-2} & \alpha_{\nu-1} & \beta_{\nu-1} \\ & & & & & \beta_{\nu-1} & \alpha_\nu \end{bmatrix}$$

Hence, by Gershgorin's theorem all the zeros of $\psi_\nu(R, x)$ $(\nu = 1, 2, \ldots, n)$ lie in the disc

$$|z| \leqq \max_{1 \leqq \nu \leqq n} |\alpha_\nu| + 2 \max_{1 \leqq \nu \leqq n} |\beta_\nu| = \rho.$$

With the help of the values of α_ν and β_ν given in (9) we readily obtain

$$\max_{1 \leqq \nu \leqq n} |\alpha_\nu| = |\alpha_n| < 1 + 2\frac{4r - 1}{(4r - 2)^2}.$$

Similarily,

$$\max_{1 \leqq \nu \leqq n} |\beta_\nu| = |\beta_n| < \frac{4r - 1}{(4r - 2)^2}.$$

These estimates give $\rho < 4r^2/(2r - 1)^2$. Furthermore, the polynomials $\psi_\nu(R, x)$ being orthonormal on the interval $[1, \infty)$ must have all their zeros in $(1, \infty)$. Hence the result holds.

Unfortunately, the upper bound for the zeros of $\psi_\nu(R, x)$ obtained in Lemma 3 does not have a form appropriate for our later applications. We therefore prove:

LEMMA 4. *For $r > 1$ the inequality*

$$\frac{4r^2}{(2r - 1)^2} < \left\{\frac{(3r)^{3r}(r - 1)^{r-1}}{r^r(3r - 1)^{3r-1}}\right\}^{1/r} \tag{19}$$

holds.

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Proof. It is clearly enough to show that the function

$$\phi(r) = (3r - 1)\log(3r - 1) - 2r\log(2r - 1) - (r - 1)\log(r - 1) - r\log(27/4)$$

is negative for $r \in (1, \infty)$. This is readily verified for all large r and for all r sufficiently close to 1. So, $\phi(r)$ cannot become positive in $(1, \infty)$ unless $\phi'(r)$ vanishes somewhere in the interval. But $\phi'(r)$ is always positive since $\lim_{r\to\infty} \phi'(r) = 0$ and $\phi''(r) < 0$ in $(1, \infty)$.

The next lemma gives the development of x^N in terms of the orthonormal polynomials $\psi_\nu(4N, x)$, $\nu = 0, 1, \ldots, N$.

LEMMA 5. *Let N be a positive integer. Then x^N can be represented as*

$$x^N = \sum_{\nu=0}^{N} a_\nu^* \psi_\nu(4N, x), \tag{20}$$

where

$$a_\nu^* = \sqrt{4N - 2\nu - 1}\,\frac{N!}{(3N-1)!}\,\frac{(3N-\nu-2)!}{(N-\nu)!} > 0 \quad (\nu = 0, 1, \ldots, N). \tag{21}$$

Proof. Since $\{\psi_\nu(4N, x)\}$ is a sequence of orthonormal polynomials on $[1, \infty]$ with respect to the weight function x^{-4N}, the coefficients a_ν^* in (20) are given by

$$a_\nu^* = \int_1^\infty \frac{1}{x^{4N}} \cdot x^N \psi_\nu(4N, x)dx \quad (\nu = 0, 1, \ldots, N).$$

Using the representation (12) of $\psi_\nu(R, x)$ with $R = 4N$ we obtain by termwise integration in the last row of the determinant

$$a_\nu^* = \frac{1}{\sqrt{D_\nu D_{\nu-1}}} \begin{vmatrix} c_0 & c_1 & \cdots & c_\nu \\ c_1 & c_2 & \cdots & c_{\nu+1} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ c_{\nu-1} & c_\nu & \cdots & c_{2\nu-1} \\ \dfrac{1}{3N-1} & \dfrac{1}{3N-2} & \cdots & \dfrac{1}{3N-\nu-1} \end{vmatrix} \quad (\nu = 1, 2, \ldots, N).$$

Now, Lemma 1 with

$$a_i = \begin{cases} 4N - i + 1 & \text{for } i = 1, 2, \ldots, \nu \\ 3N & \text{for } i = \nu + 1 \end{cases}$$

and

$$b_i = -i \quad (i = 1, 2, \ldots, \nu + 1)$$

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can be used to handle this determinant. An elementary straightforward calculation completes the proof of Lemma 5.

3. Proof of the theorem. Let $r \geqq 1$ be a fixed rational number and let n be an arbitrary positive integer subject to the condition that $N = r(n + 1)$ is also an integer. Set

$$\lambda_n = \inf_{\pi(z) \in \mathscr{P}_n} \left\{ \sup_{1 \leqq x < \infty} \left| \frac{1}{x^N} - \frac{1}{\pi(x)} \right| \right\}.$$

Note that λ_n is simply an abbreviation for the quantity $\lambda_{0,n}((z + 1)^{-N})$. We shall compare the uniform approximation by reciprocals of polynomials with a certain weighted least square approximation, namely

$$(22) \quad \mu_n = \min_{\pi(z) \in \mathscr{P}_n} \left\{ \int_1^\infty \frac{1}{x^{4N}} (x^N - \pi(x))^2 dx \right\}^{1/2}.$$

If we denote by $g_n(x)$ the polynomial furnishing the minimum then as is well known (see e.g. [**10**, §3.1])

$$(23) \quad g_n(x) = \sum_{\nu=0}^{n} a_\nu^* \psi_\nu(4N, x)$$

and

$$\mu_n = \left(\sum_{\nu=n+1}^{N} |a_\nu^*|^2 \right)^{1/2},$$

where a_ν^* $(\nu = 0, 1, \ldots, N)$ is given in (21). Since the coefficients a_ν^* are decreasing we have

$$a_{n+1}^* \leqq \mu_n \leqq (N - n) a_{n+1}^*.$$

Therefore, if $N = r(n + 1)$ then

$$\operatorname{lim}'_{n \to \infty} (\mu_n)^{1/n} = \operatorname{lim}'_{n \to \infty} (a_{n+1}^*)^{1/n} = \operatorname{lim}'_{n \to \infty} \left\{ \frac{N!(3N - n - 3)!}{(3N - 1)!(N - n - 1)!} \right\}^{1/n}.$$

To calculate this limit we use Stirling's formula according to which

$$K! = K^K e^{-K} \sqrt{2\pi K}\, e^{\vartheta/12K} \quad (0 \leqq \vartheta \leqq 1)$$

and obtain

$$(24) \quad \operatorname{lim}'_{n \to \infty} (\mu_n)^{1/n} = \frac{r^r (3r - 1)^{3r-1}}{(3r)^{3r} (r - 1)^{r-1}}.$$

The right hand side is always less than 1 and hence the weighted least square approximation in question is geometrically convergent.

Upper estimate. Set

$$(25) \quad h_n(x) = (3N - 1) x^{4N-1} \int_x^\infty \frac{1}{t^{4N}} g_n(t) dt.$$

Subtracting the two sides of the identity

$$x^N = (3N-1)x^{4N-1}\int_x^\infty \frac{1}{t^{4N}}\,t^N dt$$

from the corresponding sides of (25) and then using Schwarz's inequality we obtain

$$(26)\quad |x^N - h_n(x)| \leqq (3N-1)x^{4N-1}\int_x^\infty \frac{1}{t^{4N}}\,|t^N - g_n(t)|dt$$
$$\leqq (3N-1)x^{4N-1}\left(\int_x^\infty \frac{1}{t^{4N}}\,dt\right)^{1/2}\left(\int_1^\infty \frac{1}{t^{4N}}\,(t^N - g_n(t))^2 dt\right)^{1/2}$$
$$\leqq \mu_n\sqrt{3N}\,x^{2N} \quad \text{for } x \in [1,\infty).$$

Next, putting $b_n := (2\mu_n\sqrt{3N})^{-1/N}$ we know from (24) that $b_n > 1$ for sufficiently large n. Thus (26) yields

$$(27)\quad h_n(x) \geqq x^N(1 - \mu_n\sqrt{3N}\,x^N) > \tfrac{1}{2}\,x^N \quad \text{for } x \in [1, b_n].$$

This inequality enables us to write (26) as

$$|x^N - h_n(x)| \leqq 2\mu_n\sqrt{3N}\,x^N h_n(x),$$

or equivalently

$$(28)\quad \left|\frac{1}{x^N} - \frac{1}{h_n(x)}\right| \leqq 2\mu_n\sqrt{3N} \quad \text{provided } x \in [1, b_n].$$

To settle the case $x \in [b_n, \infty)$, we first deduce from (25)

$$(29)\quad h_n'(x) = (3N-1)\left\{(4N-1)x^{4N-2}\int_x^\infty \frac{g_n(t)}{t^{4N}}\,dt - \frac{1}{x}\,g_n(x)\right\}.$$

Now, if ρ_n denotes the largest zero of $\psi_n(4N, x)$ then from (23) and Lemma 5 we see that $g_n(x)$ is strictly increasing for $x \geqq \rho_n$. Therefore, (29) shows that

$$(30)\quad h_n'(x) > 0 \quad \text{for } x \geqq \rho_n.$$

But by (24) and the Lemmas 3 and 4 we find that $\rho_n < b_n$ for sufficiently large n. Hence, according to (25) and (30), $h_n(x)$ is positive and strictly increasing for $x \geqq b_n$. Using (27), we obtain

$$(31)\quad \left|\frac{1}{x^N} - \frac{1}{h_n(x)}\right| < \max\left\{\frac{1}{b_n^N}, \frac{1}{h_n(b_n)}\right\} < \frac{2}{b_n^N} = 4\mu_n\sqrt{3N} \quad \text{for } x \in [b_n, \infty).$$

Together with (28) this inequality yields

$$(32)\quad \overline{\lim_{n\to\infty}}'\,(\lambda_n)^{1/n} \leqq \lim_{n\to\infty}{}'\,(\mu_n)^{1/n} < 1,$$

and so in particular the sequence (λ_n) is geometrically convergent.

Lower estimate. It is clear that there exists a polynomial $p_n(x) \in \mathscr{P}_n$ such that

$$\left| \frac{1}{x^N} - \frac{1}{p_n(x)} \right| \leqq \lambda_n,$$

or equivalently

$$(33) \quad |p_n(x) - x^N| \leqq \lambda_n x^N p_n(x) \quad \text{for } x \in [1, \infty].$$

Next, putting $c_n = (2\lambda_n)^{-1/N}$ we see from (32) that $c_n > 1$ for sufficiently large n. Thus (33) yields

$$p_n(x) \leqq \frac{x^N}{1 - \lambda_n x^N} \leqq 2x^N \quad \text{for } x \in [1, c_n].$$

This inequality enables us to write (33) as

$$(34) \quad |p_n(x) - x^N| \leqq 2\lambda_n x^{2N} \quad \text{for } x \in [1, c_n].$$

Hence, we know that

$$(35) \quad \inf_{\pi(z) \in \mathscr{P}_n} \sup_{1 \leqq x \leqq c_n} \left\{ \frac{1}{x^{2N}} |x^N - \pi(x)| \right\} \leqq 2\lambda_n.$$

Let $q_n(x) \in \mathscr{P}_n$ be the solution of the weighted uniform approximation problem arising at the left hand side of (35). We shall show that

$$(36) \quad 0 \leqq q_n(x) \leqq x^N \quad \text{for } x \in (c_n, \infty).$$

Set $d(x) = x^N - q_n(x)$. Since $d^{(n+1)}(x) > 0$ for $x > 0$, Rolle's theorem implies that $d(x)$ has at most $n + 1$ positive zeros. On the other hand, by a well known theorem on uniform approximation (see e.g. [**2**, p. 75; **6**, p. 20]) the function $d(x)/x^{2N}$ attains its maximum deviation at least $n + 2$ times on $[1, c_n]$ with alternating signs. Hence $d(x)$ has exactly $n + 1$ positive zeros, say $x_\nu (\nu = 0, 1, \ldots, n)$, all lying in $[1, c_n]$. Since $d(x)$ becomes positive for $x \to \infty$ the second inequality in (36) is certainly true.

Next, denote by $q_n[x_0, x_1, \ldots, x_\nu]$ the νth divided difference of $q_n(x)$ with respect to the points $x_0, x_1, \ldots, x_\nu$. Since $q_n(x_\nu) = x_\nu{}^N (\nu = 0, 1, \ldots, n)$ and since the first n derivatives of x^N are positive on $(0, \infty)$ it follows that (cf. [**5**, p. 249, (9)])

$$q_n[x_0, x_1, \ldots, x_\nu] > 0 \quad (\nu = 0, 1, \ldots, n).$$

Thus, representing $q_n(x)$ by Newton's interpolation formula (see e.g. [**5**, p. 248, (7)]) we obtain the first inequality in (36).

Now, taking into account (34), (36), and the definition of $g_n(x)$ and $q_n(x)$

we get

$$\mu_n^2 = \int_1^\infty \frac{1}{x^{4N}} (x^N - g_n(x))^2 dx$$

$$\leqq \int_1^\infty \frac{1}{x^{4N}} (x^N - q_n(x))^2 dx \leqq \int_1^{c_n} 4\lambda_n^2 dx + \int_{c_n}^\infty \frac{1}{x^{2N}} dx$$

$$< 4c_n\lambda_n^2 + \frac{1}{2N-1} \cdot \frac{1}{c_n^{2N-1}} = \frac{2N}{2N-1} 4c_n\lambda_n^2 < 4\lambda_n^{2-1/N}.$$

Therefore,

$$\lim_{n\to\infty}{}' (\mu_n)^{1/n} \leqq \lim_{n\to\infty}{}' (\lambda_n)^{1/n}.$$

This completes the proof of our theorem since, clearly,

$$\lambda_{0,n}((cz+d)^{-N}) = (1/d^N)\lambda_{0,n}\left(\left(\frac{cz}{d}+1\right)^{-N}\right) = (1/d^N)\lambda_{0,n}((z+1)^{-N}).$$

4. Proof of inequality (6). Finally, as promised, we shall briefly prove the inequality (6). With the help of an appropriate Möbius transformation we see that

$$\gamma_n = \inf_{\substack{p(z)\in\mathscr{P}_n \\ q(z)\in\mathscr{P}_n}} \left\{ \max_{-1\leqq x\leqq 1} \left| \left(\frac{1+x}{2d}\right)^{n+1} - \frac{p(x)}{q(x)} \right| \right\}.$$

It is clear that the infimum is attained and so there exist polynomials $p(x)$ and $q(x)$ in $\mathscr{P}_n$ with

$$\max_{-1\leqq x\leqq 1} |q(x)| = 1$$

and

(38) $|(1+x)^{n+1}q(x) - p(x)| \leqq \gamma_n(2d)^{n+1}|q(x)|$

for all $x \in [-1, 1]$. Putting

$$g(x) = (1+x)^{n+1}q(x) - p(x)$$

we obtain from (37) and (38) that

$$\max_{-1\leqq x\leqq 1} |g(x)| \leqq \gamma_n(2d)^{n+1}.$$

Therefore, by an inequality of W. Markoff (see e.g. [**1**, p. 300])

(39) $$\max_{-1\leqq x\leqq 1} |g^{(n+1)}(x)| \leqq \frac{\gamma_n}{2} (4d)^{n+1} \frac{(3n+1)!}{(2n)!}.$$

Next, putting

(40) $h(x) = (1+x)^{n+1}q(x)$

we have

$$h^{(n+1)}(x) \equiv g^{(n+1)}(x)$$

and

$$h(-1) = h'(-1) = \ldots = h^{(n)}(-1) = 0.$$

Hence

$$h(x) = \int_{-1}^{x}\int_{-1}^{t_n}\ldots\int_{-1}^{t_2}\int_{-1}^{t_1} g^{(n+1)}(t)dtdt_1\ldots dt_n$$

from which it follows that

$$|h(x)| \leqq \frac{(1+x)^{n+1}}{(n+1)!}\max_{-1\leqq t\leqq 1}|g^{(n+1)}(t)|$$

on the unit interval. The inequalities (39) and (40) now lead us to

$$|q(x)| \leqq \frac{\gamma_n}{2}(4d)^{n+1}\binom{3n+1}{2n}$$

which would contradict (37) if (6) were false.

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