J. Austral. Math. Soc. Ser. B 34(1992), 174–198

PLANFORM EVOLUTION IN CONVECTION —AN EMBEDDED CENTRE MANIFOLD

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(Received 25 July 1990; revised 31 May 1991)

Abstract

The new motion of embedding a centre manifold in some higher-dimensional manifold leads to a practical approach to the rational low-dimensional approximation of a wide class of dynamical systems; it also provides a simple geometric picture for these approximations. In particular, I consider the problem of finding an approximate, but accurate, description of the evolution of a two-dimensional planform of convection. Inspired by a simple example, the straightforward adiabatic iteration is proposed to estimate an embedding manifold and arguments are presented for its effectiveness. Upon applying the procedure to a model convective planform problem I find that the resulting approximations perform remarkably well—much better than the traditional Swift-Hohenberg approximation for planform evolution.

1. Introduction

Convective fluid flow arising from unstably density gradients is one of the driving forces in the atmosphere, the oceans, and the earth's mantle. A difficult problem in fluid mechanics is to predict the large-scale motion of such convection—it is difficult even for Rayleigh numbers near the onset of the convection in the simple geometries used in laboratory studies. Small-scale problems, where the horizontal extent of the flow is of the same order as the vertical extent, may be simulated directly on a computer, see [1, 13] for example. But many geophysical fluid flows have a large horizontal extent compared to their height. These problems are just too enormous to be simulated directly; a simple, effective and accurate approximation has to be developed. A scheme which could do this for Rayleigh numbers relatively near the onset of convection is developed here; but for simplicity I only treat model problems.

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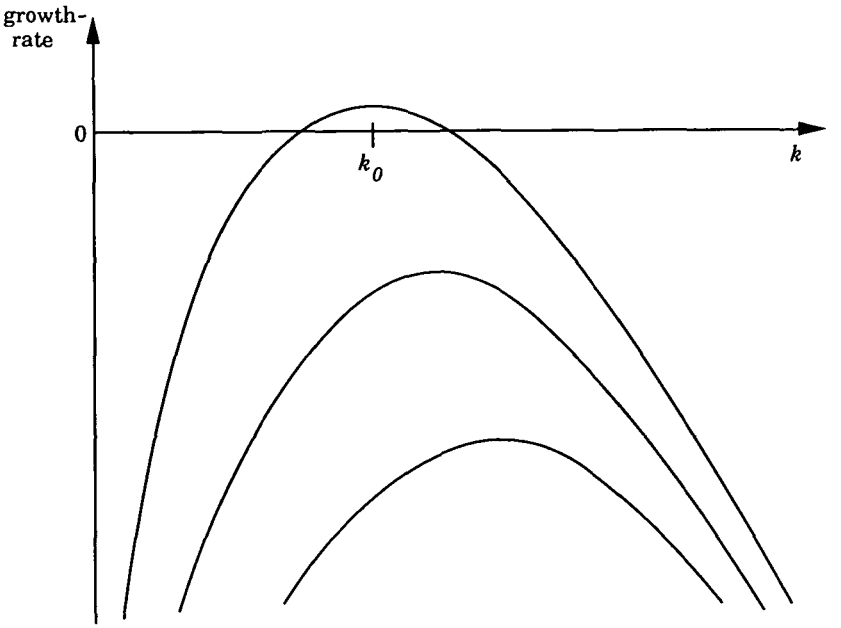

FIGURE 1. The spectrum of a typical convection problem near the onset of convection as a function of the magnitude of the horizontal wavenumber; it shows that most modes are strongly damped with only a limited range of modes near $k = k_0$ being active.

The treatment is based on centre manifold [4] and invariant manifold theory [23], but is significantly different due to intractable difficulties which arise if centre manifold theory is applied directly to such a large-scale flow.

For definiteness, imagine a laboratory experiment where a fluid, heated from below, is enclosed between two horizontal plates of large horizontal extent. On the length-scale of the vertical separation between the plates, the flow typically takes the form of convective rolls or possibly hexagons. On a large scale, the rolls or hexagons in different regions may be at different orientations, and will interact and evolve over a long time scale. This relatively slow evolution of how the rolls fit together in the experimental tank is the planform selection problem.

A linear picture of the evolution, for very small amplitudes, may be obtained by taking the horizontal Fourier transform of the problem and then solving the eigen-problem for the growth-rates of the various modes in the vertical. A typical picture of growth-rates for the vertical modes as a function of the magnitude, k, of the horizontal wavenumber $\mathbf{k}$ is shown in Figure 1,

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for a Rayleigh number just above the critical value. Note that most modes are very strongly damped and do not even appear in this figure. Linearly, there exists an annulus of wavenumbers in the **k**-plane whose amplitudes grow. Eventually their growth is arrested by the nonlinear processes in the fluid and the long-term dynamics is determined by the nonlinear interaction between modes whose wavenumbers lie in and near this annulus.

As the Rayleigh number is increased through the critical value, there is a bifurcation from the equilibrium state of no motion, to this complex evolution. It is thought that centre manifold theory should provide a rigorous and accurate low-dimensional model of these dynamics, as it does for other (though simpler) bifurcation problems [4, 5, 20]. Indeed, there is no great difficulty in applying centre manifold theory to the case of slowly-varying pure rolls, a one-dimensional planform problem, which was first done heuristically by Newell and Whitehead [17] and Segel [28]. These one-dimensional rolls may even be used to predict the evolution of a two-dimensional planform [8], provided that in any small locale the solution field is limited to look like a pure one-dimensional roll; the orientation and the amplitude of the rolls may then vary significantly over the whole field. This approach is similar to that employed by Howard and Kopell [14] for the reaction-diffusion equation, and it similarly suffers from having to use defects and/or shocks to fit together patches of curving roll solutions; it is also limited to solutions based on just one, or at most a few, pure rolls. However, attempts to apply centre manifold theory to a two-dimensional planform problem, where the underlying structure is that of many interacting rolls, encounters apparently intractable zero (or near-zero) divisors.[2] I should mention that there appears to be no difficulty in applying centre manifold theory to the case where the weakly unstable modes are themselves of very long wavelength [21], as occurs in fixed heat-flux convection [19]. In that case, the unstable modes are based upon a small disc centered on the origin in the wavenumber plane, rather than upon an annulus.

The zero-divisors arise in the following manner. Exactly at criticality the solution field, say $a(x, y, t)$, has marginal modes with wavenumber $k = |\mathbf{k}| = k_0$, and centre manifold theory asserts that near criticality these modes may be used to parameterise a low-dimensional manifold on which the long-term evolution takes place. Thus we would suggest that, to a leading approximation,

$$a = \int_0^{2\pi} A(\phi, t) e^{ik_0(x\cos\phi + y\sin\phi)}\, d\phi + \text{nonlinear corrections}, \qquad (1)$$

[2]The difficulties here with zero-divisors should not be confused with the classic "zero-divisor problem" in Hamiltonian mechanics, which was addressed by Kolmogoroff, Arnold and Moser [27, §6.1].

that is, a time-varying linear superposition of all the marginal modes plus some higher-order correction terms. The evolution of A determines the low-dimensional long-term evolution of the planform according to some rule

$$\frac{\partial A}{\partial t} = G(\phi, A). \tag{2}$$

This rule, and the corrections to the expression for a, are normally found by looking at the nonlinear interactions which may be cubic, say, and so we would have

$$a^3 \approx \int_0^{2\pi}\int_0^{2\pi}\int_0^{2\pi} A(\phi_1, t)A(\phi_2, t)A(\phi_3, t) \times e^{ik_0[x(\cos\phi_1+\cos\phi_2+\cos\phi_3)+y(\sin\phi_1+\sin\phi_2+\sin\phi_3)]}\, d\phi_1\, d\phi_2\, d\phi_3 . \tag{3}$$

on the right-hand side of a singular linear equation for the corrections to a. To solve this equation, we would eliminate all components in the right-hand side with wavenumber of magnitude k_0, as these are not in the range of the linear operator on the left-hand side, by an appropriate choice of G. Then the corrections could be found; see [20] for some simple examples of this process. Unfortunately, due to the nature of the wavenumber combinations in the triple integral in (3), there are components in the right-hand side with wavenumber having a magnitude arbitrarily close to k_0 and so the inverse of the linear operator, in effect, divides by numbers arbitrarily close to zero—which cannot be reasonable. This is a common obstacle in problems with a continuous spectrum of eigen values.

As another example, zero-divisors frequently occur in wave dynamics, where they signal resonant interactions. On the two-dimensional free surface of water where there is typically a continuous spectrum of propagating waves, these resonant interactions are almost sure to occur. However, a very useful, if extremely complicated, approximation for the four-wave resonant interaction in a continuous spectrum of deep water waves has been derived by Zakharov [33]. The Zakharov equation has been used in a number of studies of wave evolution, see [32, 7] for example, and I have commented later on its relation to the main thrust of this paper.

In some cases, a continuous spectrum near the marginal mode can be dealt with within centre manifold theory [21] to produce powerful approximations. For example, one can derive shear dispersion approximations [15] and a unified beam theory [25]. Moreover, these approximations may be derived complete with asymptotically appropriate initial conditions [22], boundary conditions [24], and forcing [6]. It is also straightforward to derive evolution equations for a one-dimensional field of convective rolls whose properties vary in the horizontal space dimension [unpublished work by the author]. The trick in all these applications is simply to allow the marginal mode, or

modes, to vary slowly in space and to use the effects of these slow variations as a perturbing term in the asymptotics. This works because, in essence, one sweeps all the modes which are near-marginal into neighbouring marginal modes. It is an effective approximation because the slowly-varying amplitude does not itself resolve the length-scale of the roll, it only needs to resolve the slow variations. Unfortunately, this approach does not seem to work for two-dimensional planform problems in convection (nor does it seem to work for a field of interacting waves with a continuous spectrum). To attempt this approach, the amplitude A in (1) would be made a slowly-varying function of the horizontal coordinates so that, instead of representing just the marginal modes with wavenumber of magnitude k_0, (1) could represent an annulus of wavenumbers $|k - k_0| < \varepsilon$ for some ill-defined ε. Consequently the components of the right-hand side (3) would either be inside the annulus, and therefore somehow be used to determine the long-term evolution as governed by G, or outside the annulus, and be used to determine the corrections to the shape of the centre manifold. But there are a couple of problems here: first, into which of a continuum of marginal modes $|\mathbf{k}| = k_0$ should a particular near-marginal mode be swept; and second, how is anyone to determine a definite and appropriate value for the width of the annulus ε? This last problem is especially uncomfortable—where in the continuous spectrum is the dividing line to be drawn between near-marginal modes which are deemed to be of long-term interest and those modes which are deemed not to be? All this is most unsatisfactory.

The current practice [18, 11, 8, 9, 3, 10] in the theoretical study of the evolution of convective planforms is to use the model equation

$$\frac{\partial A}{\partial t} = \varepsilon A - (k_0^2 + \nabla^2)^2 A - A^3 \tag{4}$$

where ε is a small parameter measuring the difference of the Rayleigh number from its critical value. This was originally derived by Swift and Hohenberg [30]. The rationale is that the spectrum of the marginal modes of this model equation can match those of the convection problem, and that the nonlinearity is typical for the symmetry and the need to stabilise finite-amplitude rolls. As a simplifying approximation, it does not have quite the same appeal as the Newell and Whitehead slowly-varying roll approximation, because it necessarily must resolve the length-scale of the rolls—this means that the numerical simulation of a very large sized planform is unrealistic. However, it does avoid the zero-divisor difficulties, and it does seem to model the evolution of convection to some extent. But there are many previously unresolved questions. Does it form an accurate model of convection? How can this sort of approximation be put on a rational basis, rather than relying on problem-specific heuristic arguments? How can systematic corrections be

calculated to such an equation? Is there a simple geometric picture to sum up the nature of this approximation? This paper answers these questions from the point of view of centre manifold theory.

The Swift-Hohenberg equation (4) resolves not only the marginal modes of wavenumber near k_0, but also a lot of other rapidly decaying modes. The crucial trick would seem to be that the model must get the dynamics of the marginal modes correct—the rapidly-decaying modes are deemed to be irrelevant to the evolution of the planform. Thus there are dynamical modes carried along in the model other than the desired ones of importance in the long-term evolution. Now, the model will be useful if its long-term evolution matches that of the convective system under study; that is, the evolution on the centre manifold of the model must be the same as that on the centre manifold of the original system. The geometric picture is then that the model must have embedded in it the centre manifold of the original system. This rational philosophy of approximation is explored in a simple example in Section 2. The simple procedure of "adiabatic iteration", a generalisation of adiabatic elimination as is used in synergetics [12], is discovered to form a sequence of low-dimensional models which more and more accurately embed the long-term dynamics of the original system.

The properties of this simple procedure to find an embedding manifold are then examined in Section 3. There it is shown how the iteration produces more accurate approximations to an embedding manifold. Furthermore, it is shown that under suitable conditions, all solutions (not too large) of a dynamical system approach exponentially quickly to a solution of the low-dimensional model system which is obtained on the embedding manifold. This last property justifies the use of the model system as an accurate approximation to the long-term dynamics of the dynamical system under study.

The adiabatic iteration proposed is simply adiabatic elimination iterated. Adiabatic elimination [12, Chapter 7] is promoted as *the* basic tool of synergetics in finding accessible models of physical phenomena from the primitive detailed equations. Thus the attractive geometric picture developed herein and the properties established in Section 3 are directly pertinent to all the work which has been based on synergetics. The problem of modelling the convective planform is an example of the derivation of "generalised Ginzburg-Landau equations for nonequilibrium phase transitions" [12, §7.7]—indeed, Figure 7.4(a) in [12] is essentially the same as Figure 1. This close connection ensures that the results developed in this paper will greatly enhance and extend the synergetics approach to modelling.

Then in Section 4 these ideas are applied to a simple planform selection problem. There I compare the evolution of the exact system with the leading

order approximation derived through the above ideas, and with the corresponding Swift-Hohenberg equation. I conclude that the Swift-Hohenberg equation is not an accurate model of the long-term evolution of the planform, at least for this problem. For qualitative and quantitative long-term prediction of the evolution, the approach of embedding the centre manifold performs remarkably well.

2. A simple example of embedding a centre manifold

Consider the simple three-dimensional dynamical system

$$\dot{x} = -y(x+z) \tag{5}$$

$$\dot{y} = -\mu y + x^2 - z^2 \tag{6}$$

$$\dot{z} = -z + xy \tag{7}$$

where μ is some constant. For my purposes, take $0 < \mu < 1$, so that the linearised decay rate of y is between that of x (undamped) and that of z (decay rate 1). To relate to the problem of planform selection in convection, imagine that x represents the critical convective modes; y represents the modes on the same branch of eigenvalues as x and so has decay rates which range from near zero to large and negative; and z is representative of the other, strongly damped, convective modes.

The system (5–7) possesses a centre manifold toward which all solutions lying sufficiently close to the origin are attached exponentially quickly [4]. Viewing this exponential approach as being an uninteresting transient, it is the long-term evolution on the centre manifold which is of interest, as it forms a low-dimensional, here one-dimensional, model [23, 16] of the dynamics of the full three-dimensional system. To find the centre manifold, I pose that it is described by

$$y = \eta(x), \quad z = \zeta(x), \tag{8}$$

which upon substitution into (5–7) leads to the two equations

$$\mu\eta = x^2 - \zeta^2 + \eta'\eta(x+\zeta), \qquad \zeta = x\eta + \zeta'\eta(x+\zeta) \tag{9}$$

for the centre manifold. Solving this pair of equations either by iteration or by substitution of a power series in x [4], I find that the centre manifold is given approximately by

$$\eta = \frac{1}{\mu}x^2 + \frac{2}{\mu^3}x^4 - \frac{12-2\mu+\mu^2}{\mu^5}x^6 + \frac{2(56+16\mu-3\mu^2-3\mu^3)}{\mu^7}x^8 + O(x^9) \tag{10}$$

$$\zeta = \frac{1}{\mu}x^3 + \frac{2+3\mu}{\mu^3}x^5 + \frac{12+18\mu+17\mu^2}{\mu^5}x^7 + O(x^9). \tag{11}$$

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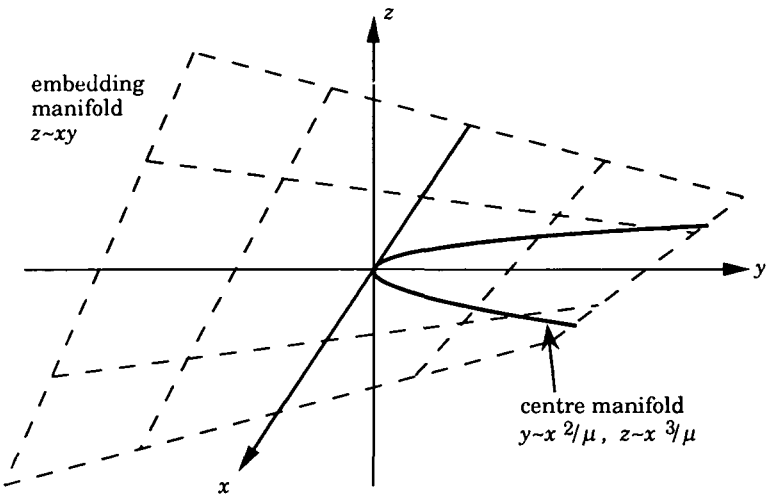

FIGURE 2. A manifold $z = xy$ which approximately embeds the centre manifold (10–11) of the full system (5–7).

The long-term evolution of the dynamical system is then modelled by

$$\dot{x} = -\eta(x+\zeta) = -\frac{1}{\mu}x^3 - \frac{2+\mu}{\mu^3}x^5 + O(x^7). \tag{12}$$

A problem here is that if μ is numerically small then these expressions give poor approximations to the centre manifold because of the divisions by the powers of μ. Geometrically, the centre manifold is highly curved, at least near the original where $y \sim x^2/\mu$ and $z \sim x^3/\mu$, and so it is hard to approximate.

However, we may embed the one-dimensional centre manifold in a relatively smooth two-dimensional surface, for example $z = xy$ as shown in Figure 2. This forms the basis for a different low-dimensional model of the full system. The idea is to find some relatively smooth surface $z = \mathcal{Z}(x, y)$, called the *embedding manifold*, which contains the centre manifold (10–11). Once this has been found, the *reduced system*

$$\dot{x} = -y(x + \mathcal{Z}(x, y)) \tag{13}$$

$$\dot{y} = -\mu y + x^2 - \mathcal{Z}(x, y)^2 \tag{14}$$

is taken to be a long-term model of the full system (5–7). The aim is to ensure that the reduced system (13–14) has the *same centre manifold* as the original full system.

Although the long-term behaviour of the original system may then be accurately modelled, the exponential transients in the reduced system (decaying

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roughly like $\exp(-\mu t)$) need not have any exact counterpart in the original. Now, if $z = \mathcal{Z}(x, y)$ was to be an invariant manifold [23] of the full system, then by substitution into (7) and using (5–6), it must satisfy

$$\mathcal{Z} = xy - \dot{z} = xy + \frac{\partial \mathcal{Z}}{\partial x} y(x + \mathcal{Z}) - \frac{\partial \mathcal{Z}}{\partial y}(-\mu y + x^2 - \mathcal{Z}^2). \tag{15}$$

However, for the centre manifolds to be the same we need only ensure that this equation is satisfied on the curve which is the centre manifold; the justification of this assertion and some subsequent steps is left to the next section. In practice an exact expression for the embedding manifold is unobtainable; all we seek is a sufficiently accurate approximation. One way to achieve a sequence of approximations is by iteration—starting with the guess $\mathcal{Z}^{(0)} = 0$ we substitute the current iterate $\mathcal{Z}^{(k)}$ into the right-hand side of (15) to then give the next iterate $\mathcal{Z}^{(k+1)}$. Here this results in the sequence

$$\mathcal{Z}^{(1)} = xy \tag{16}$$

$$\mathcal{Z}^{(2)} = (1+\mu)xy - x^3 + xy^2 + xy^3 + x^3y^2 \tag{17}$$

$$\begin{aligned}\mathcal{Z}^{(3)} = {} & (1+\mu+\mu^2)xy - (1+\mu)x^3 + (1+3\mu)xy^2 - 5x^3y \\ & + (2+5\mu+\mu^2)xy^3 - (6-\mu-4\mu^2-\mu^3)x^3y^2 - (1+4\mu+2\mu^2)x^5y \\ & + (1+\mu)x^7 + O(x(x^2+|y|)^4)\end{aligned} \tag{18}$$

A cross-section through these surfaces is plotted in Figure 3. Note that although the surfaces may be quite different from each other, near the centre manifold (10–11) they are all nearly the same; this agreement gets better closer to the origin. Intuitively it may be argued that this iteration forms a useful sequence of approximations because the right-hand side of (15), namely $xy - \dot{z}$, is small since xy is nonlinear, and on the centre manifold the evolution of z is necessarily slow. Indeed, the first approximation, that $\mathcal{Z} \approx \mathcal{Z}^{(1)} = xy$, occurs in various contexts [12, 31] as the so-called *adiabatic approximation* since it neglects all the time variations of the ignored modes.

However, the definitive test is to see whether the reduced system (13–14) does indeed have the same centre manifold as the original, to some order of accuracy. Using the initial approximation that $\mathcal{Z}^{(0)} = 0$ the reduced system becomes

$$\dot{x} = -xy, \qquad \dot{y} = -\mu y + x^2,$$

which has a centre manifold $y = Y^{(0)}(x) = \eta(x) + O(x^6)$. Using the first approximation, that $\mathcal{Z}^{(1)} = xy$, the reduced system becomes

$$\dot{x} = -xy(1+y), \qquad \dot{y} = -\mu y + x^2(1-y^2)$$

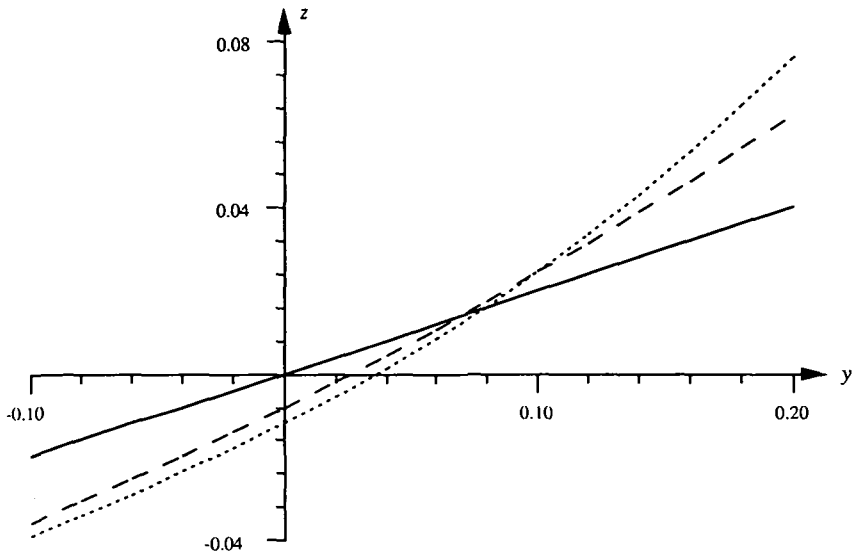

FIGURE 3. The $x = 0.2$ cross-sections of the approximate embedding manifolds for $\mu = 0.5$ showing the close agreement at the centre manifold (10–11) near $y \approx 0.08$: $\mathcal{Z}^{(1)}$ is the solid line; $\mathcal{Z}^{(2)}$ is dashed; $\mathcal{Z}^{(3)}$ is dotted.

which has a centre manifold $y = Y^{(1)}(x) = \eta(x) + O(x^8)$. Evidently the y location of the centre manifold of the kth version of the reduced equations has errors of $O(x^{2k+6})$. The errors in the z location are larger, with $\mathcal{Z}^{(k)}(x, Y^{(k)}) = \zeta(x) + O(x^{2k+3})$, but they still decrease by two orders in x at each iteration. Thus this *adiabatic iteration* does seem to provide increasingly accurate embeddings of the centre manifold of the original system (5–7).

One important property of this iteration occurs for small μ. Adjoining the equation $\dot{\mu} = 0$ to (5–7), a standard trick to study bifurcations [4] and slowly-varying solutions in space [21, 15, 25], we would find that there exists a centre manifold $z = Z(\mu, x, y)$ to the enlarged system. This centre manifold is guaranteed to exist and be a useful approximation sufficiently close to the origin in μxyz-space; that is, for small μ no matter what its sign. The relevance of this observation here is that one procedure to find this centre manifold $z = Z(\mu, x, y)$ is *precisely* the iteration scheme outlined above! Thus the iterates (16–18) based on (15) are guaranteed to be accurate for small μ, not only for the ultimate algebraic evolution but also for the evolution on a time-scale of order $1/\mu$. Furthermore, this accuracy is also achieved for small and positive μ in which case the embedding manifold

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approximates the centre-unstable manifold of the system. These properties are essential for the convective planform problem where there can be no clear distinction between the dynamically active, marginal nodes (x) and the weakly damped modes (y) which all lie on the top branch of eigenvalues shown in Figure 1.

Note that the iteration has other desirable features. There is no possibility of zero or near-zero divisors occurring in the scheme; the only division is by the coefficient of decay of the z mode, and this coefficient is always significantly different from zero for any sensible approximation. Also there is no need to distinguish between the variables x and y; this is crucial to its application in planform selection where marginal modes transform indistinguishably into decaying modes, and vice versa, as the wavenumber k is varied.

3. Adiabatic iteration to embed a centre manifold

Consider a general dynamical system in $\mathbb{R}^{l+m+n}$ in the following simple form

$$\dot{x} = Ax + f(x, y, z) \tag{19}$$

$$\dot{y} = By + g(x, y, z) \tag{20}$$

$$\dot{z} = Cz + h(x, y, z) \tag{21}$$

where $x \in \mathbb{R}^l$, $y \in \mathbb{R}^m$ and $z \in \mathbb{R}^n$. Furthermore, the eigenvalues of $l \times l$ A must have zero real-part and those of $m \times m$ B and $n \times n$ C are taken to have negative real parts. The vector functions f, g and h are strictly nonlinear functions which are differentiable some number of times near the origin (where they and all their first derivatives vanish).

Given the constraints on the eigenvalues of A, B and C it is apparent [4] that the system (19–21) possesses an l-dimensional centre manifold described by $y = \eta(x)$, $z = \zeta(x)$, at least near the origin, where

$$\begin{aligned} \frac{\partial \eta}{\partial x}(Ax + f(x, \eta, \zeta)) &= B\eta + g(x, \eta, \zeta), \\ \frac{\partial \zeta}{\partial x}(Ax + f(x, \eta, \zeta)) &= C\zeta + h(x, \eta, \zeta). \end{aligned} \tag{22}$$

Furthermore, for all initial conditions sufficiently close to the origin (at least provided the origin is stable) the solution of the original system approaches the centre manifold exponentially quickly, and once there it evolves according to

$$\dot{x} = Ax + f(x, \eta(x), \zeta(x)). \tag{23}$$

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This last equation is of immense value as a low-dimensional (with just l degrees of freedom) and slowly-evolving model of the full system (19–21). In practice, the requirement that the eigenvalues of A have precisely zero real part is not restrictive. If x represents all the desired modes in the model then a standard trick [4, 5, 20] is to parameterise their growth rates by a variable ε, such that some finite ε recovers the desired system, and that $\varepsilon = 0$ gives the modes a zero growth rate. Then appending the variable ε to those in x, and the equation $\dot{\varepsilon} = 0$ to (19) puts the desired equations in the requisite form.

The motivation for studying the properties of the dynamical system (19–21) is that it, in principle, has a similar eigen-structure to a convection problem. As in Section 2, the vector x represents the active marginal modes, the vector y represents the decaying modes which come from the same branch of eigen-solutions as the marginal modes, and the vector z represents all the other decaying modes in the system. Of course, for convection in a large container, all these variables are effectively infinite dimensional rather than the finite dimensional vectors that I have posed above; I leave the problem of how all this is justified in infinite dimensions to an analyst with a more appropriate background. Actually, it is the infinite dimensionality of convection which is the crux of the difficulty. If the system was of finite dimensions, then there would be little difficulty finding the centre manifold of the system as the approximation to its long-term evolution. The difficulty in infinite dimensional problems is that it may be impossible to distinguish between the variables which appear in x and those which appear in y. Although I make such a distinction in this section, it is only done to establish properties of the embedding of a centre manifold—ultimately I describe a procedure which does not treat differently the variables in x and y.

The aim is to find a manifold $z = \mathcal{Z}(x, y)$ which embeds the centre manifold, satisfying (22), of the original system (19–21). Once this has been found we would use the *reduced system*

$$\dot{x} = Ax + f(x, y, \mathcal{Z}) \tag{24}$$

$$\dot{y} = By + g(x, y, \mathcal{Z}) \tag{25}$$

as a low-dimensional (here $l + m$ dimensional) model of, or approximation to, the original system. It would be easier to solve than the original in at least two respects: firstly, it would have much fewer unknowns; and secondly, the most rapidly decaying modes typically occur in z which do not appear in the reduced system and this results in the reduced system being much less stiff to solve numerically.

There are a number of important properties of an embedding manifold, and its approximation, which can be established.

(1) *If* $\mathcal{Z}(x, y)$ *is differentiable near the origin and* $\mathcal{Z}(0, 0) = 0$ *then the reduced system* (24–25) *possesses a centre manifold* $y = Y(x)$ *which satisfies*

$$\frac{\partial Y}{\partial x}(Ax + f(x, Y, \mathcal{Z}(x, Y))) = BY + g(x, Y, \mathcal{Z}(x, Y)). \qquad (26)$$

This is a straightforward consequence of the restrictions which were placed upon A, B, f and g at the outset.

(2) *The manifold* $z = \mathcal{Z}(x, y)$ *embeds the centre manifold of the original system if and only if it satisfies*

$$C\mathcal{Z} = -h(x, y, \mathcal{Z}) + \frac{\partial \mathcal{Z}}{\partial x}(Ax + f(x, y, \mathcal{Z})) + \frac{\partial \mathcal{Z}}{\partial y}(By + g(x, y, \mathcal{Z})) \qquad (27)$$

on the centre manifold of the original system $y = \eta(x)$.

This is established via the chain rule:

$$\begin{aligned} & z = \mathcal{Z}(x, y) \text{ embeds the centre manifold} \\ \Leftrightarrow\ & \zeta = \mathcal{Z}(x, \eta(x)) \text{ and (22)} \\ \Leftrightarrow\ & \left(\frac{\partial \mathcal{Z}}{\partial x} + \frac{\partial \mathcal{Z}}{\partial y}\frac{\partial \eta}{\partial x}\right)(Ax + f) = C\mathcal{Z} + h \\ \Leftrightarrow\ & \frac{\partial \mathcal{Z}}{\partial x}(Ax + f) + \frac{\partial \mathcal{Z}}{\partial y}(B\eta + g) = C\mathcal{Z} + h. \end{aligned}$$

An interesting aspect of (27) is that it is the equation for an invariant manifold of the original system [23] based on the x and y modes. The difference here is that we only require it to be satisfied on the centre manifold, rather than for all x and y, and this makes finding approximate solutions to (27) considerably easier.

(3) *If the origin is stable then sufficiently near the origin the long-term behaviour of the original system* (5–7) *is identical, to within an exponentially decaying difference, to the long-term behaviour of a solution of the reduced system* (24–25).

The conditions that the origin be stable and that the statements are necessarily true only in some neighbourhood of the origin are immediate consequences of the conditions of Theorem 2 in Carr [4]. Applying this theorem I may assert that the solutions of the reduced system (24–25) approach its centre manifold $y = Y(x)$ exponentially quickly. This is the same as the centre manifold of the original system, since (27) is the equation for an invariant manifold which must contain the centre manifold, and to which solutions are exponentially attracted. Thus all solutions of the original system exponentially quickly approach a solution of the reduced system.

(4) If $A = 0$ then the *adiabatic iteration* procedure to find approximations to the embedding manifold $z = \mathcal{Z}(x, y)$ is as follows (the case when $A \neq 0$ will be commented on later). Start with the approximation $\mathcal{Z}^{(0)} = 0$, and

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then iterate by substituting the current approximation $\mathcal{Z}^{(k)}$ into the right-hand side of (27) and then solving $C\mathcal{Z}^{(k+1)} =$ right-hand side to give the next approximation; thus the first iterate is simply $\mathcal{Z}^{(1)} = -C^{-1}h(x, y, 0)$. At all stages this iteration is straightforward to do as the eigenvalues of C are all bounded away from zero. The main complication is that algebraic details increase very quickly, as may be seen in Section 2.

The exact centre manifold of the reduced system based upon the iterative approximation $\mathcal{Z}^{(k)}$ differs from the centre manifold of the original system by an amount which is $O(|\mathcal{Z}^{(k)} - \mathcal{Z}^{(k+1)}|)$ where this is evaluated on the centre manifold.

To deduce this let $y = \tilde{Y}(x)$ be the exact centre manifold of the reduced system (24–25) based on the iterate $z = \mathcal{Z}^{(k)}(x, y)$ for the embedding manifold; it must thus satisfy

$$\frac{\partial \tilde{Y}}{\partial x}\left(Ax + f\left(x, \tilde{Y}, \mathcal{Z}^{(k)}(x, \tilde{Y})\right)\right) = B\tilde{Y} + g\left(x, \tilde{Y}, \mathcal{Z}^{(k)}(x, \tilde{Y})\right). \quad (28)$$

On the approximate embedding manifold this curve has $z = \tilde{Z}(x) = \mathcal{Z}^{(k)}(x, \tilde{Y}(x))$. Consider

$$\begin{aligned} C\tilde{Z} + h(x, \tilde{Y}, \tilde{Z}) - \frac{\partial \tilde{Z}}{\partial x}(Ax + f(x, \tilde{Y}, \tilde{Z})) \\ = C\mathcal{Z}^{(k)} + h(x, \tilde{Y}, \mathcal{Z}^{(k)}) - \frac{\partial \mathcal{Z}^{(k)}}{\partial x}(Ax + f(x, \tilde{Y}, \mathcal{Z}^{(k)}) \\ - \frac{\partial \mathcal{Z}^{(k)}}{\partial y}\frac{\partial \tilde{Y}}{\partial x}(Ax + f(x, \tilde{Y}, \mathcal{Z}^{(k)}) \\ = C\mathcal{Z}^{(k)} + h(x, \tilde{Y}, \mathcal{Z}^{(k)}) - \frac{\partial \mathcal{Z}^{(k)}}{\partial x}(Ax + f(x, \tilde{Y}, \mathcal{Z}^{(k)}) \\ - \frac{\partial \mathcal{Z}^{(k)}}{\partial y}(B\tilde{Y} + g(x, \tilde{Y}, \mathcal{Z}^{(k)})) \\ = C[\mathcal{Z}^{(k)} - \mathcal{Z}^{(k+1)}] \end{aligned}$$

where these are evaluated on the centre manifold $y = \tilde{Y}(x)$. This result together with (28) shows that the centre manifold on the approximate embedding manifold, described by $y = \tilde{Y}(x)$ and $z = \tilde{Z}(x)$, satisfies (22) to an error $O(|\mathcal{Z}^{(k)} - \mathcal{Z}^{(k+1)}|)$. Thus by Theorem 3 in Carr [4] it approximates the centre manifold of the original system to the same order of error.

How quickly will the adiabatic iteration procedure give more accurate embeddings of the centre manifold? For example, in Section 2 it was noted that each iteration gave the centre manifold of the original system to two more orders of accuracy in x. Again only consider the case when $A = 0$ and

consider the iterates $\mathcal{Z}^{(k)} = \mathcal{Z}(x, y) + \Theta^{(k)}(x, y)$ where $\mathcal{Z}$ satisfies (27) exactly. Linearising for small Θ gives the iteration for $\Theta^{(k)}$ to be

$$C\Theta^{(k+1)} = \left\{ -\frac{\partial h}{\partial z} + \frac{\partial \mathcal{Z}}{\partial x}\frac{\partial f}{\partial z} + \frac{\partial \mathcal{Z}}{\partial y}\frac{\partial g}{\partial z} + \frac{\partial \bullet}{\partial x} f + \frac{\partial \bullet}{\partial y}(By + g) \right\} \Theta^{(k)} \quad (29)$$

where $\bullet$ is a place-holder for $\Theta^{(k)}$. We are interested in how this iteration behaves on the original centre manifold $y = \eta(x)$. Using r to measure distance from the origin, and recognising that dominantly f, g, h and $\mathcal{Z}$ are $O(r^2)$ this iteration looks like

$$C\Theta^{(k)} = \{O(r) + O(r^2) + O(r^2) + O(r) + (O(1) + O(r))\}\Theta^{(k)}.$$

The presence of the $O(1)$ from the term $\frac{\partial \bullet}{\partial y}By$ shows that this iteration does not produce iterates which only change at increasing orders of r; see the previous section where the coefficient of the xy term changes by an $O(1)$ amount at each iteration (unless μ is asymptotically small). However, we are only interested in the centre manifold $y = \eta(x)$ where $By + g = \frac{\partial \eta}{\partial x} f = O(r^3)$! Thus the iteration is actually

$$C\Theta^{(k)} = \{O(r) + O(r^2) + O(r^2) + O(r) + O(r^2)\}\Theta^{(k)}.$$

Furthermore, on the centre manifold y and z are actually of order x^2—they should be counted as quantities of order 2 in r rather than order 1. Doing this, I observe that if f is of order 3 and $\frac{\partial h}{\partial z}$ is of order 2, as occurs in (5–7), then the centre manifold of the reduced system will approximate the actual centre manifold by two more orders of accuracy at each iteration, as seen in Section 2.

This adiabatic iteration seems to be the simplest effective procedure to find an embedding manifold for the centre manifold. There is a lot of freedom in choosing the embedding manifold which is not apparent in this iteration; where this freedom is hidden and how it may be used to advantage is a problem for further study. Another problem for study is what should be done if A is not zero. If A were in Jordan form but still had purely-zero eigenvalues, then I would expect that simple modifications could be made to the iteration procedure in order to restore its usefulness. However, if A has pure imaginary eigenvalues, as occurs in the analysis of a Hopf bifurcation for example, I surmise that more significant changes would have to be made. These changes would involve the separation of the time dependence into a fast and a slow part as is needed in the sub-centre manifold [29] theory of slowly-varying waves [26], and as is used in the derivation of Zakharov's equation for interacting deep water waves [32, page 115].

In summary, I comment on what the iteration looks like when it is practically impossible to disentangle the x and y modes of the dynamical system—

the only reason for considering this approach in the first place. In this case the equations of the dynamical system may be in the *standard form*

$$\dot{x} = Ax + f(x, z) \tag{30}$$

$$\dot{z} = Cz + h(x, z) \tag{31}$$

where the previous x and y modes are now indistinguishably grouped into the vector x; the matrix A has some essentially zero eigenvalues (corresponding to the weakly damped, weakly growing and neutral modes) and some significantly negative eigenvalues; and the matrix C has all negative eigenvalues bounded away from zero. The procedure is then to seek an embedding manifold $z = \mathcal{Z}(x)$ by the adiabatic iteration

$$C\mathcal{Z}^{(k+1)} = -h(x, \mathcal{Z}^{(k)}) + \frac{\partial \mathcal{Z}^{(k)}}{\partial x}(Ax + f(x, \mathcal{Z}^{(k)})), \tag{32}$$

starting from $\mathcal{Z}^{(0)} = 0$. Then the approximate reduced system

$$\dot{x} = Ax + f(x, \mathcal{Z}^{(k)}(x)) \tag{33}$$

models the long-term behaviour of the dynamical system (30–31) as they both have the same centre manifold, to some order of accuracy in x. The order of accuracy should increase with increasing k; however, I have only shown this if the marginal eigenvalues of A have no imaginary part and have a complete eigenspace.

4. A specific planform selection problem

Consider the simple problem in two-spatial dimensions (x and y)

$$\frac{\partial \alpha}{\partial t} = r\alpha - (1 + \nabla^4)\alpha - 2\nabla^2\beta + \frac{1}{2}\beta(\alpha + \beta) \tag{34}$$

$$\frac{\partial \beta}{\partial t} = r\beta - 2\nabla^2\alpha - (1 + \nabla^4)\beta - \frac{1}{2}\alpha(\alpha + \beta), \tag{35}$$

where r is a forcing parameter analogous to the Rayleigh number in convection. For simplicity, the boundary conditions on α and β are that they are L-periodic in both x and y for some box length L. This periodicity unfortunately eliminates interesting planform shapes generated by side walls; however, it is easy to implement numerically and is sufficient for my illustrative purposes. If the boundary conditions were not periodic then the appropriate boundary conditions for the mathematical model would not be known. My experience with one-dimensional slowly-varying approximations [24, 25] shows that there are little known subtleties in deriving asymptotically correct boundary conditions, and these need further research.

These equations are not in the standard form (30–31) and so I change to the mean and difference variables $a = (\alpha + \beta)/2$ and $b = (\alpha - \beta)/2$. In these variables the governing partial differential equations become

$$\frac{\partial a}{\partial t} = ra - (1 + \nabla^2)^2 a - ab = \mathscr{L}_r a - ab \tag{36}$$

$$\frac{\partial b}{\partial t} = rb - (1 - \nabla^2)^2 b + a^2; \tag{37}$$

with both a and b being L-periodic in both x and y.

The linear picture is straightforward to establish. Linearly the a and b equations are decoupled and the horizontal structure of the eigenmodes is simply $\exp(i\mathbf{k}\cdot\mathbf{x})$ where $\mathbf{x} = (x, y)$. The corresponding eigenvalues, forming the spectrum of the linear problem, are

$$\lambda_a = r - (1 - k^2)^2, \qquad \lambda_b = r - (1 + k^2)^2 \tag{38}$$

where $k = |\mathbf{k}|$ as shown in Figure 4. As discussed earlier, there are two different types of eigenvalues. The λ_b branch is always strongly dissipative and the corresponding modes are expected to have simply a modifying role on the evolution of the solutions. On the other hand the λ_a branch is composed of marginal modes for wavenumbers k near 1, and dissipative modes for small and large wavenumbers. Thus at the onset of "convection", which occurs for $r > 0$, the horizontal width of a "convective roll" is roughly π units. The difficulty for a centre manifold derivation is that there is no clear dividing line between the two types of modes on the λ_a branch. Strictly speaking, because of the finite size of the convective problem, the linear spectrum is actually discrete and zero-divisors cannot occur. However, even in a relatively small container the gaps in the spectrum are quite small, which results in small (near zero) divisors; in the example described later, $L = 16\pi$ is used, for which there are only a few gaps in the spectrum as big as 0.03—it is effectively a continuous spectrum.

However, it is easy to derive an approximate reduced system for this system. I just pose that $b = \mathscr{B}(a)$, which upon substituting into (36–37) gives

$$\left[(1 - \nabla^2)^2 - r\right]\mathscr{B} = a^2 - \frac{\partial \mathscr{B}}{\partial a}\left[\mathscr{L}_r a - a\mathscr{B}\right] \tag{39}$$

where $\frac{\partial \mathscr{B}}{\partial a}$ is a Fréchet derivative. Now, this is the equation for an invariant manifold [23] based on the a modes and to solve it properly I should really have the term $\frac{\partial \mathscr{B}}{\partial a}[\mathscr{L}_r a]$ on the left-hand side; but this is very difficult to do. However, the analysis of the previous sections suggests that I may effectively approximate the embedded centre manifold of the system by an iteration based on the above equation. Define the Green's function $\mathscr{G}_r$ by

$$\left[(1 - \nabla^2)^2 - r\right]\mathscr{G}_r = \delta(\mathbf{x}). \tag{40}$$

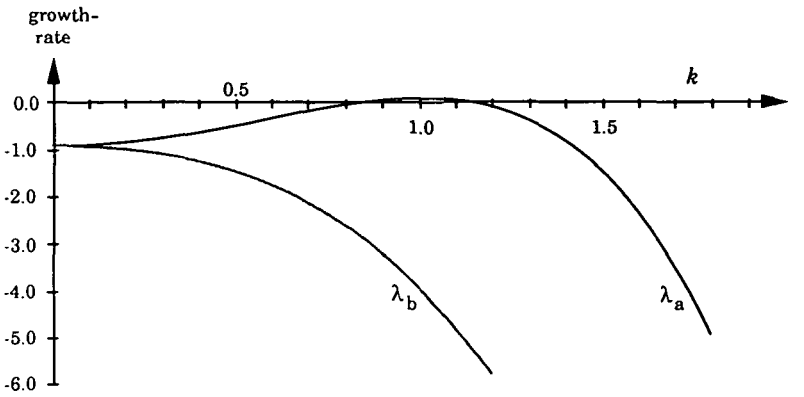

FIGURE 4. The spectrum of the model planform selection problem (34–35) for slightly supercritical forcing $r = 0.1$.

For example, $\mathscr{G}_0 = \frac{1}{4\pi^2} K_0(|\mathbf{x}|) \star K_0(|\mathbf{x}|)$ where $\star$ denotes convolution, and the Green's function for $r = 0.4$ is shown in Figure 5 (see next page). The iteration to find the embedding manifold starts with $\mathscr{B}^{(0)} = 0$ and then (39) gives

$$\mathscr{B}^{(1)} = \mathscr{G}_r \star a^2 \tag{41}$$

$$\mathscr{B}^{(2)} = \mathscr{G}_r \star \left\{ a^2 - 2\mathscr{G}_r \star \left[a\mathscr{L}_r a - a^2 \mathscr{G}_r \star a^2 \right] \right\} . \tag{42}$$

Thus, for example, the approximate reduced system based on the first iterate $\mathscr{B}^{(1)}$ is

$$\frac{\partial a}{\partial t} = \mathscr{L}_r a - a\mathscr{B}^{(1)} = ra - (1 + \nabla^2)^2 a - a\mathscr{G}_r \star a^2 . \tag{43}$$

This will be discussed as the principal model equation for the original system (34–35).

The unfamiliar feature of this evolution equation is the appearance of the non-local convolution operator $\mathscr{G}_r \star$. In particular, this implies that distant boundaries have a direct effect on the interior dynamics, albeit exponentially small. In convection this sort of non-local operator was first observed by Swift and Hohenberg [30] in their equation (A24). However, they very swiftly assumed, in the paragraph around their equation (A26), that it could be replaced with a localised operator. In essence they assumed that $\mathscr{G}_r$ could be replaced by a delta-function spike with the same volume underneath. Doing this here would suggest that the Swift-Hohenberg equation

$$\frac{\partial a}{\partial t} = \mathscr{L}_r a - a^3/(1 - r) = ra - (1 + \nabla^2)^2 a - a^3/(1 - r) \tag{44}$$

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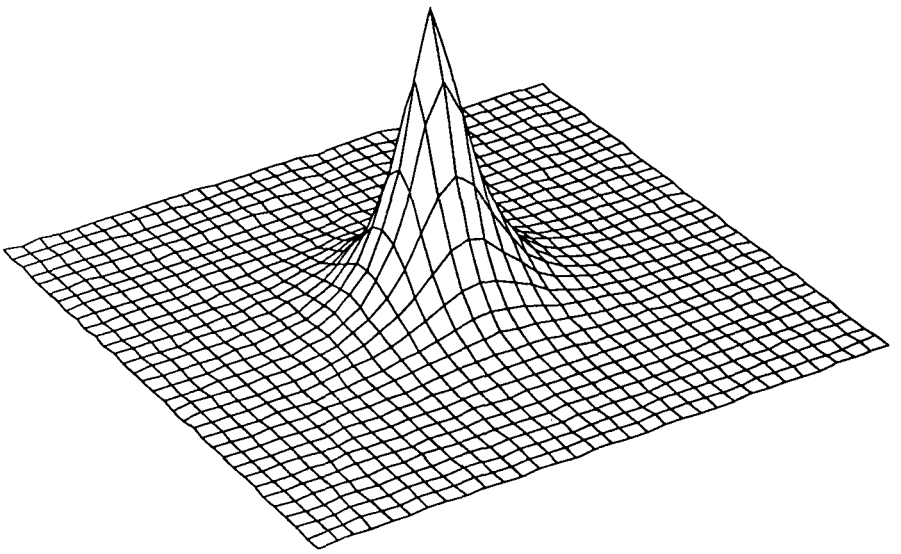

FIGURE 5. The Green's function $\mathcal{G}_r$ for $r = 0.4$ showing its radial symmetry and exponential decay. Note that each of the horizontal sides of the box are of length 8π, that is, eight marginal "convective rolls" long.

would be a useful approximation, in some sense, to the original problem. Indeed there has been a lot of research on planform selection [18, 11, 8, 9, 3, 10] which is based on the assumption that the Swift-Hohenberg equation does provide a reasonable model for convection. My numerical solutions for the various equations above indicate that the Swift-Hohenberg equation is deficient—the non-local nature of $\mathcal{G}_r\star$ is essential in the accurate modelling of the evolution of the convective planform.

As explained by Bestehorn *et al.*[2], the synergetics [12] approach to planform selection also leads to a non-local evolution equation. However, Bestehorn *et al* then proceed to also approximate the non-local operator by a localised differential operator by requiring that the interactions on the critical circle $|\mathbf{k}| = k_0$ be nearly correct.

The numerical solution of the evolution equations (36–37), (43) and (44) in a periodic box is straightforward. Using a box with sides of length $L = 16\pi$ (twice that shown in Figure 5 for $\mathcal{G}_r$), about 16 "convective rolls" may fit along each side. The unknowns a and b were discretised on a 64×64 grid so that wavenumbers $|k| \leq 4$ were resolved. All the derivatives were calculated spectrally, as was the convolution $\mathcal{G}_r\star$. The time integration may be effectively done by an implicit backward Euler method; I used a second order scheme with variable step size. To provide a genuine test, the initial

motion should lie on or near to the centre manifold so that the comparisons are not corrupted by the differing rapid transients in the various evolution equations. This was achieved by generating a field of white noise of amplitude proportional to $\sqrt{r}$ and then filtering it to leave only those wavenumber components whose linear growth-rate is non-negative; the resultant field was used as the initial condition for a. To start the original system, b was set to $\mathcal{G}_r \star a^2$, to ensure that its evolution commenced on the embedding manifold. Deriving asymptotically correct initial conditions for a model obtained by adiabatic iteration will require further research based on the ideas developed for centre manifold models [22].

I discuss only the set of simulations with $r = 0.4$; other simulations were similar. In some respects this is quite a large value of the parameter—it should be compared to the decay-rate of zero-wavenumber modes which is $r - 1 = -0.6$ and to the decay-rate of the b-mode near wavenumber 1 which is $r - 4 = -3.6$. In wavenumber space the annulus of linearly unstable modes is $0.80 < |k| < 1.28$. Seen in Figure 6 (see next page) are some snapshots of a during the evolution[3] of: the exact system (36–37); the first embedded centre manifold approximation (43); the Swift-Hohenberg approximation (44). Observe that the agreement between the first approximation and the exact system is extremely good: visually there is no difference at all; quantitatively the difference throughout most of the simulation was about 0.3%. Remarkably, in the very early stages of the evolution, $t < 10$, the difference is as high as a few percent, but the two systems then converge to be close to each other until the end of the simulations. On the other hand, the evolution of the Swift-Hohenberg approximation is markedly different to that of the exact system. Although they both seem to evolve ultimately towards a two-dimensional roll structure, they do it in significantly different ways. Firstly, the Swift-Hohenberg model convection is of a significantly lower amplitude than the exact system. Secondly, the Swift-Hohenberg model evolves to convective rolls much more quickly than the exact system which has a strong tendency to linger in a square pattern for a long time. Indeed, the last defect in the Swift-Hohenberg model has just disappeared at the last time shown of $t = 310$, while in the exact system and the embedded centre manifold approximation, the last defect does not disappear until soon after time $t = 1000$.

Another demonstration of the inaccuracy of the Swift-Hohenberg approximation may be obtained by investigating the behaviour of slowly-varying one-dimensional rolls. Consider rolls which are independent of y for small

[3] A complete sequence of raster fields, which may be animated in colour to show the evolution, may be obtained electronically from the author (email:aroberts@maths.adelaide.edu.au).

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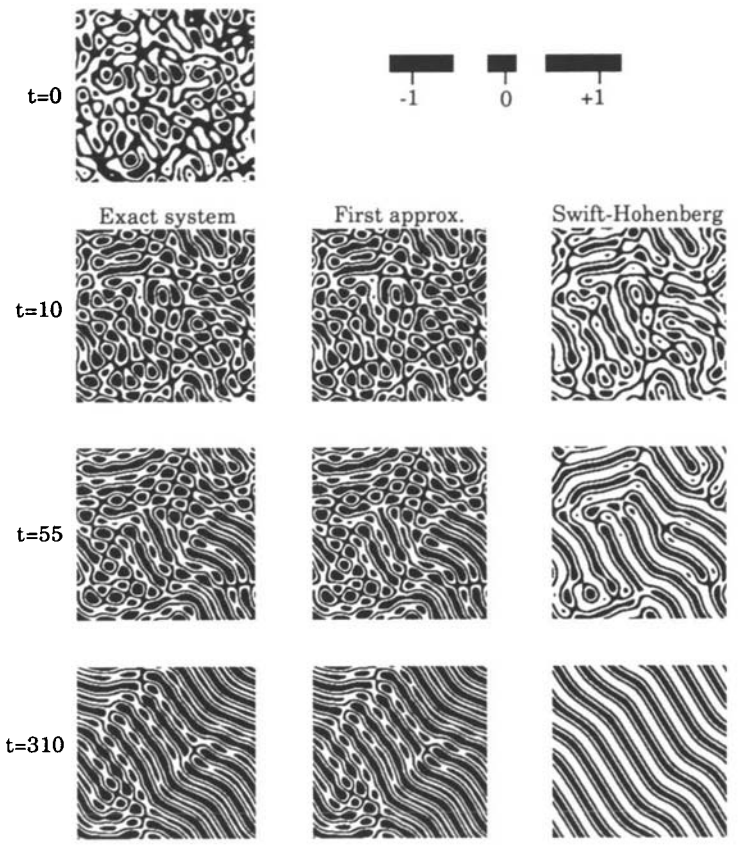

FIGURE 6. Comparison of the exact evolution of the planform with the two types of approximation: the non-local embedded centre manifold (first iteration), and the Swift-Hohenberg localising approximation. All simulations started at $t = 0$ with the initial planform shown at the top-left.

supercritical r, and whose complex amplitudes $A(x, t)$ vary slowly. An asymptotic analysis based on these ideas, first done in convection by Newell and Whitehead [17], predicts that a field of rolls governed by the Swift-Hohenberg equation (44) is approximately described by

$$a = Ae^{ix} + \overline{A}e^{-ix} - \frac{1}{64}A^3e^{i3x} - \frac{1}{64}\overline{A}^3e^{-i3x} \tag{45}$$

where $-$ denotes complex conjugation, and the evolution of the field is given by

$$\frac{\partial A}{\partial t} = rA + 4\frac{\partial^2 A}{\partial x^2} - 3|A|^2A\,. \tag{46}$$

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In contrast, the exact system (36–37) and the first approximation to the embedded centre manifold (43) describe a one-dimensional field of rolls, for the above regime, by *precisely* the same equations. In one dimension, the Green's function may easily be found to be

$$\mathscr{G}_r = \frac{1}{4\sqrt{r}}\left(\frac{1}{\lambda_1}e^{-\lambda_1|x|} - \frac{1}{\lambda_2}e^{-\lambda_2|x|}\right) \qquad (47)$$

$$\text{where } \lambda_1 = \sqrt{1-\sqrt{r}} \text{ and } \lambda_2 = \sqrt{1+\sqrt{r}}$$

$$\sim \frac{1}{4}(1+|x|)e^{-|x|} \text{ as } r \to 0. \qquad (48)$$

Thus the roll field for both exact and the first approximation is

$$a = Ae^{ix} + \bar{A}e^{-ix} - \frac{1}{1600}A^3e^{i3x} - \frac{1}{1600}\bar{A}^3e^{-i3x} \qquad (49)$$

with the component b correspondingly being

$$b = \frac{1}{25}A^2e^{i2x} + 2A\bar{A} + \frac{1}{25}\bar{A}^2e^{-i2x}, \qquad (50)$$

where the evolution of the field is

$$\frac{\partial A}{\partial t} = rA + 4\frac{\partial^2 A}{\partial x^2} - \frac{51}{25}|A|^2A. \qquad (51)$$

For example, the Swift-Hohenberg equation would predict that a steady field of rolls of wavenumber $k = 1$ would be $a \approx 2\sqrt{r/3}\cos x$, while the equation based on either of the embedded centre manifold or the exact system would predict rolls $a \approx 2\sqrt{25r/51}\cos x$ which are some 21% larger. Furthermore, the smoothing effect of $\mathscr{G}_r$ very nicely predicts the small size, $A^3/1600$, of the third harmonic in the wavefield a, while the Swift-Hohenberg equation predicts $A^3/64$ which is a factor of 25 too large. This last discrepancy can also be seen easily in Fourier transforms of the two-dimensional numerical simulations.

The quantitative errors seen for one-dimensional slowly-varying rolls may be dismissed as having little qualitative importance. However, my two-dimensional simulations suggest that these errors may be indicative of serious qualitative discrepancies in the two-dimensional evolution of the planform. The Swift-Hohenberg approximation performs poorly, while the embedded centre manifold approach, albeit with an unusual non-local operator, performs well—here it is virtually indistinguishable from the exact system.

5. Conclusion

This approach of embedding the centre manifold in a higher dimensional manifold seems to be a new and valuable idea in the rational approxima-

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tion of the evolution of dynamical systems. The simple "adiabatic iteration" scheme proposed to calculate the embedding manifold is relatively straightforward to apply and already appears in principle in the derivation of the Swift-Hohenberg equation. Furthermore, the iteration is based upon adiabatic elimination which is the fundamental tool of synergetics. Synergetics has been used to produce models in many branches of science [12, Chapters 7–11]; the properties outlined herein will also illuminate the nature of these models. Here I have placed this scheme within a simple geometric picture, that of embedding the centre manifold, so that its nature can be appreciated. Using this new geometric view it should be possible to adapt and generalise other analytic tools which are based on centre manifold theory to provide appropriate initial conditions [22], transformation of forcing [6] and, most interestingly for convection, boundary conditions [24]. The consequence is that this approach forms a practical and complete scheme for modelling.

The proposed adiabatic iteration seems to be rather inefficient to carry out beyond the first one or two iterations, due to the rapidly increasing algebraic detail, as seen in Section 2. This was not a difficulty in the convection problem studied in Section 4, as the embedding manifold produced by just the first iteration formed a very effective approximation. Moreover, the iteration does not seem to take advantage of the great freedom allowed within the principle that the centre manifold be embedded. There is perhaps some scope here to produce a modified scheme which is improved in some sense, although such improvements might only add complication. One hope is that a modified scheme may be found that avoids the non-local nature of the straightforward adiabatic iteration approximation. There is one property of the iteration, as posed, which is important and must not be lost: that the iteration reduces naturally to the scheme used to calculate a centre manifold whenever the growth-rate of a mode is nearly zero. This last property ensures that the scheme makes no sharp distriction between marginal and non-marginal modes, there is just a continuous transition.

This guiding principle of the scheme is directly akin to that employed in Zakharov's equation, which describes the interaction of deep-water waves. There, the approximate integral equation for four-wave interactions is only accurate when the waves are in resonance [32, page 116, 3rd paragraph]—the approximation process results in a specious four-wave interaction whenever the four waves are significantly out of resonance. It produces a reasonable model because the inaccurate interaction coefficients out of resonance are small when compared to the valid coefficients in resonance. In direct analogy with the convection problem, there is an ill-defined and gradual transition between the resonant and non-resonant interactions.

In view of the difficulties discussed in the Introduction, this centre man-

ifold embedding approach to calculating global approximations to the evolution of a planform seems to be the only viable one known. Furthermore, the results obtained for the model planform problem in Section 4 are impressive. However, I emphasise that the nonlocal nature of the resulting approximations, as seen in Section 4, seems to be essential to their accurate performance.

Acknowledgements

Much of this research was inspired by discussions with W. R. Young. The University of Adelaide and the Australian Research Council supported this research through research grants. I am also indebted to the National Center for Supercomputer Applications in the U.S.A. for their visualisation software.

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