AXIOMS FOR ABSOLUTE GEOMETRY. III

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Introduction. This paper is a continuation of [**1**; **2**]. In [**2**], I stated that I had been unable to construct examples of planes satisfying various conditions. Some of the examples that I have since constructed are given below. A discussion of one-dimensional absolute geometries, with examples, will be given in a separate paper. The relevant parts of [**1**] and [**2**] are [**1**, § 1, § 2 up to 2.4; **2**, § 2]. We shall use the notation and terminology of [**1**; **2**]; the axioms **C1*–C4*** and **C4**** (referred to below) can all be found in [**1**].

We shall show here that spaces of dimension greater than 1 exist, both Archimedean and non-Archimedean, satisfying **C1*–C4***, in which not all points are isometric, and that **C4**** does not follow from **C1*–C4*** in non-Archimedean geometries of dimension greater than 1. In some of the examples, as in the example in [**2**, § 2(c)], not every segment with isometric end-points has a mid-point.

1. Planes in which not all points are isometric.

We shall denote the sets of integers, rational numbers, and algebraic numbers by Z, Q, and A, and the set of all rational numbers of the form $p/3^r$ (p an integer, r a non-negative integer) by S. If a set is countable and totally ordered, with neither a least nor a greatest element, and if between any two distinct elements of the set there lies another element of the set, we shall say that the set is *rationally ordered.*

1.1. (i) *Q, A, and S are rationally ordered.*

(ii) [**3**, pp. 209, 202] *Between any two rationally ordered sets there exists a one-to-one order-preserving correspondence.*

We shall call the element $p/3^r$ of S *odd* or *even* according as p is odd or even; $p/3^r$ need not be in its lowest terms [**1**, p. 162]. Then clearly we have the following result.

1.2. *Between any two distinct elements of S there exist an odd element and an even element.*

We now construct various planes.

Example (a). Let the points of a plane ρ consist of all ordered pairs (a, b), where $a, b \in A$, and let the lines of ρ consist of all points satisfying linear equations with coefficients in A. We shall regard ρ as an *ordered* plane in the usual (Euclidean) way.

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It follows from 1.1 and 1.2 that we can assign *odd* or *even* parity to each element of A in such a way that between any two distinct elements of A there exist an odd element and an even element. (There are many ways of doing this; we choose one particular way.) We then assign *odd* or *even* parity to the point (a, b) of ρ according as its Euclidean distance from the origin O, namely $\sqrt{(a^2 + b^2)}$, is odd or even.

1.3. *Any line l of ρ has the property that between any two points of l there exist an odd point and an even point of l; also l is rationally ordered.*

Proof. Let the Euclidean distances from O of the distinct points P, Q of l be p, q. If $p \neq q$, there exists an odd (even) element r of A between p and q; the circle with centre O and radius r meets l at an odd (even) point between P and Q. If $p = q$, let M bisect PQ; then OM is perpendicular to PQ, therefore the distance of M from O is less than p. Hence there exist an odd and an even point between M and P, and hence between P and Q.

Clearly l is rationally ordered.

We can now easily adapt the proof of 1.1(ii), as given in [**3**], to prove the following result.

1.4. *If l is a line of ρ, then there exists a one-to-one order-preserving and parity-preserving mapping of l onto S.*

Thus we can label each point of l by means of an element of S. If $A, B \in l$ and if the labels of A and B are a and b, then we define the *length* of AB to be $|AB| = |b - a|$. If we perform this labelling of points independently for each line of ρ, we can define the length of any segment of ρ. Note that the label of a point depends on the particular line or segment under consideration; if A and C are points of the line k, then $|AC|$ is defined in terms of the labels of A and C *with respect to k*.

Definition. $AB \equiv CD$ if $|AB| = |CD|$ and if A and C have the same parity.

1.5. (i) *If $[ABC]$, then $|AB| + |BC| = |AC|$;*

(ii) *if A and C have the same parity, and if $|AB| = |CD|$, then B and D have the same parity;*

(iii) *axioms* **C1*–C4*** *are satisfied;*

(iv) *A and C are isometric if and only if they have the same parity;*

(v) *ρ is Archimedean;*

(vi) *the segment AC has a mid-point if and only if A and C are isometric.*

These results are easily verified; (ii) follows since S has the properties *even + even = odd + odd = even, odd + even = odd.*

Note that in this example and in the following ones it is important to start by fixing the parity of all points; the parity cannot be assigned on each line independently.

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Example (b). Let the points of a plane π consist of all ordered pairs (a, b) of rational numbers, and let the lines of π consist of all points satisfying linear equations with rational coefficients. We shall regard π as an *ordered* plane in the usual way. Let ϕ be a positive irrational number; we assign a parity to each point of π as follows. A point of π is an *odd* (*even*) point if it lies outside the square with vertices $(\pm n\phi, \pm n\phi)$ and inside the square with vertices $(\pm(n + 1)\phi, \pm(n + 1)\phi)$ for some odd (even) non-negative integer n; $(0, 0)$ is an *even* point. Then π is divided into a series of concentric open sets of even and odd points alternately, as in Figure 1A, in which the sets of odd points are shaded. The sides of the squares in the figure, being irrational points, do not belong to π. Let l be a line of π; l cannot pass through a vertex of any square, thus l is divided into a series of open sets of even and odd points alternately. Let us label these sets $\ldots, s_{-2}, s_{-1}, s_0, s_1, s_2, \ldots$ consecutively, as in Figure 1B, the label s_0 being given to any one of the even sets.

If $A, B \in s_m$, we write $A < B$ if $[XABY]$, where $X \in s_{m-1}$ and $Y \in s_{m+1}$ (Figure 1C). Then s_m is rationally ordered; hence there exists a one-to-one order-preserving mapping from s_m onto Q (1.1). If the point A of s_m corresponds in this mapping to $a \in Q$, we attach the label (m, a) to A. The parity of A is the same as the parity of m.

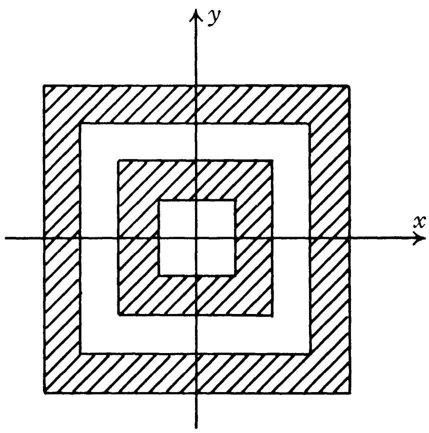

FIGURE 1A

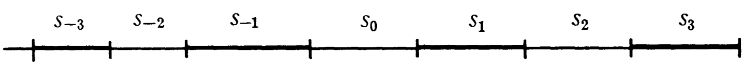

FIGURE 1B

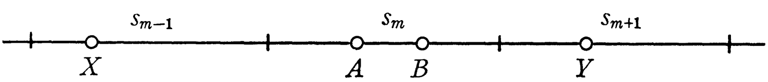

FIGURE 1C

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The set of labels of points of l is the set $\mathscr{L} = \{(m, a): m \in Z, a \in Q\}$. If we define $(m, a) + (n, b) = (m + n, a + b)$, then $\mathscr{L}$ is an additive Abelian group. We define (m, a) to be positive if $m > 0$ or if $m = 0$ and $a > 0$, and we define, as usual, $(m, a) > (n, b)$ if $(m, a) - (n, b)$ is positive, i.e., if $m > n$ or if $m = n$ and $a > b$. Also if $\alpha \in \mathscr{L}$ we define

$$
\begin{aligned}
|\alpha| &= \alpha \quad &&\text{if } \alpha \geqq 0,\\
&= -\alpha \quad &&\text{if } \alpha < 0.
\end{aligned}
$$

This order defined on $\mathscr{L}$ is the same as the geometrical order of the corresponding points of l.

If the points A and B of l have labels α and β, we define the *length* of AB to be $\lambda AB = |\beta - \alpha|$. If we perform this labelling of points independently for each line of π, always using $\mathscr{L}$ for the set of labels, we can define the length of any segment of π. As in the previous example, the label of a point depends on the particular line or segment under consideration; if A and C are points of the line k, then λAC is defined in terms of the labels of A and C *with respect to* k.

Definition. $AB \equiv CD$ *with respect to* λ if $\lambda AB = \lambda CD$ and if A and C have the same parity.

1.6. *In the plane π, with the above definition of congruence with respect to λ,*
(i) *if $[ABC]$, then $\lambda AB + \lambda BC = \lambda AC$;*
(ii) *if A and C have the same parity, and if $\lambda AB = \lambda CD$, then B and D have the same parity;*
(iii) *axioms* **C1*–C4*** *are satisfied;*
(iv) *A and C are isometric if and only if they have the same parity;*
(v) *π is non-Archimedean, but axiom* **C4**** *is satisfied;*
(vi) *the segment AC has a mid-point if and only if A and C are isometric.*

These results are easily verified.

Example (c). Let l_0 be a particular line of π. We give a new definition of length in π as follows, denoting the new length of AB by μAB.

If line $AB \neq l_0$, then $\mu AB = \lambda AB$;
if line $AB = l_0$, then

$$
\begin{aligned}
\mu AB &= \lambda AB + (0, 1) \quad &&\text{if } A \text{ is even, } B \text{ odd,}\\
&= \lambda AB - (0, 1) \quad &&\text{if } A \text{ is odd, } B \text{ even,}\\
&= \lambda AB &&\text{if } A \text{ and } B \text{ are both odd or both even.}
\end{aligned}
$$

Definition. $AB \equiv CD$ *with respect to* μ if $\mu AB = \mu CD$ and if A and C have the same parity.

1.7. *In the plane π, with the above definition of congruence with respect to μ,*
(i) *if $[ABC]$, then $\mu AB + \mu BC = \mu AC$;*

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(ii) *if A and C have the same parity, and if $\mu AB = \mu CD$, then B and D have the same parity;*

(iii) *axioms* **C1*–C4*** *are satisfied;*

(iv) *A and C are isometric if and only if they have the same parity;*

(v) *π is non-Archimedean, and axiom* **C4**** *is not satisfied;*

(vi) *the segment AC has a mid-point if and only if A and C are isometric.*

To prove (i) we simply have to enumerate various cases on l_0. To prove (v), let A and B be the points of l_0 with labels $(0, 0)$, $(1, 0)$ with respect to l_0, and let C and D be the points of a line $l \neq l_0$ with labels $(0, 0)$ and $(1, 1)$ with respect to l. Then $\mu AB = \mu CD = (1, 1)$, hence $AB \equiv CD$ with respect to μ; but $\mu BA = (1, -1)$, $\mu DC = (1, 1)$, thus $BA \not\equiv DC$. Hence **C4**** is not satisfied. The remaining results are easily verified.

As a consequence of 1.7 (v) we have a further interesting property, which is best illustrated by an example. Let A and P be the points of l_0 with labels $(0, 0)$ and $(2, 0)$ with respect to l_0, and let C and R be the points of $l \neq l_0$ with labels $(0, 0)$ and $(2, 0)$ with respect to l. The mid-points of AP and CR are M and N with labels $(1, -1)$, $(1, 0)$. We have (with respect to μ) $AP \equiv CR$, but $AM \not\equiv CN$, $MA \not\equiv NC$.

Example (d). We can set up a one-to-one order-preserving mapping from s_m onto S (1.1) instead of onto Q (ignoring the parity assigned to each element of S in (a)). Then the set of labels of points of π is the set

$$\mathscr{M} = \{(m, a) \colon m \in Z, a \in S\}.$$

If we define length and congruence as in (b), but using $\mathscr{M}$ instead of $\mathscr{L}$, then 1.6 is still true (but now $\lambda AB \in \mathscr{M}$) except for (vi); if A and C have labels $(0, 0)$ and $(0, \frac{1}{3})$ with respect to the line AC, then A and C are isometric but AC has no mid-point.

Example (e). If we use S instead of Q, as in (d), and then define length as in (c), but using $\mathscr{M}$ instead of $\mathscr{L}$, then 1.7 is still true (but now $\mu AB \in \mathscr{M}$) except for (vi).

There are clearly many ways of defining the odd and even points of π initially in the examples (b)–(e). Consider one more way. If $(a, b) \in \pi$, where a and b are rational, and if $(2m - 1)\phi < a < (2m + 1)\phi$, $(2n - 1)\phi < b < (2n + 1)\phi$, where ϕ is a positive irrational number as in Example (b), then we define (a, b) to be *even* if m and n are both even or both odd, and *odd* otherwise. Thus, π is divided into an "infinite chessboard" of even and odd points. This will *not* serve our purpose, since all points of the line $y = x$ are even, but if we translate the chessboard through a rational distance in the direction of the x-axis (or the y-axis) the difficulty is overcome, since no vertex then lies on a rational line.

Similar methods can be used to construct examples in spaces of higher dimension.

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