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A Common Extension of Arhangel′skiĭ′s Theorem and the Hajnal–Juhász Inequality

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Abstract. We present a result about G_δ covers of a Hausdorff space that implies various known cardinal inequalities, including the following two fundamental results in the theory of cardinal invariants in topology: $|X| \leqslant 2^{L(X)\chi(X)}$ (Arhangel'skiĭ) and $|X| \leqslant 2^{c(X)\chi(X)}$ (Hajnal–Juhász). This solves a question that goes back to Bell, Ginsburg and Woods's 1978 paper (M. Bell, J.N. Ginsburg and R.G. Woods, *Cardinal inequalities for topological spaces involving the weak Lindelöf number*, Pacific J. Math. **79**(1978), 37–45) and is mentioned in Hodel's survey on Arhangel'skiĭ's Theorem (R. Hodel, *Arhangel'skii's solution to Alexandroff's problem: A survey*, Topology Appl. **153**(2006), 2199–2217).

In contrast to previous attempts, we do not need any separation axiom beyond T_2.

1 Introduction

Two of the milestones in the theory of cardinal invariants in topology are the following inequalities.

Theorem 1 (Arhangel'skiĭ, 1969 [2,15]) *If X is a T_2 space, then $|X| \leq 2^{L(X)\chi(X)}$.*

Theorem 2 (Hajnal–Juhász, 1967 [13]) *If X is a T_2 space, then $|X| \leq 2^{c(X)\chi(X)}$.*

Here $\chi(X)$ denotes the *character* of X, $c(X)$ denotes the *cellularity* of X (which is the supremum of the cardinalities of the pairwise disjoint collections of non-empty open subsets of X), and $L(X)$ denotes the *Lindelöf degree* of X (which is the smallest infinite cardinal κ such that every open cover of X has a subcover of size at most κ).

The intrinsic difference between the cellularity and the Lindelöf degree makes it non-trivial to find a common extension of the two previous inequalities. The first attempt was made in 1978 by Bell, Ginsburg and Woods [5], who used the notion of weak Lindelöf degree. The weak Lindelöf degree of X, $\mathrm{wL}(X)$, is defined as the least infinite cardinal κ such that every open cover of X has a $(\leq\kappa)$-sized subcollection whose union is dense in X. Clearly, $\mathrm{wL}(X) \leq L(X)$, and we also have $\mathrm{wL}(X) \leq c(X)$, since every open cover without $< \kappa$-sized dense subcollections can be refined to a κ-sized pairwise disjoint family of non-empty open sets by an easy transfinite

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induction. Unfortunately, the Bell–Ginsburg–Woods result needs a separation axiom that is much stronger than Hausdorff.

Theorem 3 ([5]) *If X is a normal space, then $|X| \leq 2^{wL(X)\chi(X)}$.*

It is still unknown whether this inequality is true for regular spaces, but in [5] it was shown that it may fail for Hausdorff spaces. Indeed, the authors constructed Hausdorff non-regular first-countable weakly Lindelöf spaces of arbitrarily large cardinality. Some progress on the question of whether $|X| \leq 2^{wL(X)\cdot\chi(X)}$ for every regular space X can be found in [7], [9] and [12].

Arhangel'skiĭ [3] got closer to obtaining a common generalization of these two fundamental results by introducing a relative version of the weak Lindelöf degree, namely the cardinal invariant $wL_c(X)$, *i.e.*, the least infinite cardinal κ such that for any closed set $F \subseteq X$ and any family of open subsets of X $\mathcal{U}$ satisfying $F \subseteq \bigcup\mathcal{U}$ there is a subcollection $\mathcal{V} \in [\mathcal{U}]^{\leq\kappa}$ such that $F \subseteq \overline{\bigcup\mathcal{V}}$.

Theorem 4 ([3]) *If X is a regular space, then $|X| \leq 2^{wL_c(X)\chi(X)}$.*

O. Alas [1] showed that the previous inequality continues to hold for Urysohn spaces, but it is still open whether it is true for Hausdorff spaces.

In [4] Arhangel'skiĭ made another step forward by introducing the notion of strict quasi-Lindelöf degree, which allowed him to give a common refinement of *the countable case* of his 1969 theorem and of the Hajnal–Juhász inequality. He defined a space X to be *strictly quasi-Lindelöf* if for every closed subset F of X, for every open cover $\mathcal{U}$ of F and for every countable decomposition $\{\mathcal{U}_n : n < \omega\}$ of $\mathcal{U}$ there are countable subfamilies $\mathcal{V}_n \subset \mathcal{U}_n$ for every $n < \omega$ such that $F \subset \bigcup\{\overline{\bigcup\mathcal{V}_n} : n < \omega\}$. It is easy to see that every Lindelöf space is strictly quasi-Lindelöf and every ccc space is strictly-quasi Lindelöf. Arhangel'skiĭ proved that every strictly quasi-Lindelöf first-countable Hausdorff space has cardinality at most continuum.

However, Arhangel'skiĭ's approach cannot be extended to higher cardinals. Indeed, it is not even clear whether $|X| \leq 2^{\chi(X)}$ is true for every strictly quasi-Lindelöf space X. This inspired us to introduce the following cardinal invariants.

Definition 5

• The *piecewise weak Lindelöf degree of X*, $pwL(X)$, is defined as the minimum cardinal κ such that for every open cover $\mathcal{U}$ of X and every decomposition $\{\mathcal{U}_i : i \in I\}$ of $\mathcal{U}$, there are $(\leq\kappa)$-sized families $\mathcal{V}_i \subset \mathcal{U}_i$, for every $i \in I$ such that $X \subset \bigcup\{\overline{\bigcup\mathcal{V}_i} : i \in I\}$.

• The *piecewise weak Lindelöf degree for closed sets of X*, $pwL_c(X)$, is defined as the minimum cardinal κ such that for every closed set $F \subset X$, for every open family $\mathcal{U}$ covering F and for every decomposition $\{\mathcal{U}_i : i \in I\}$ of $\mathcal{U}$, there are $(\leq\kappa)$-sized subfamilies $\mathcal{V}_i \subset \mathcal{U}_i$ such that $F \subset \bigcup\{\overline{\bigcup\mathcal{V}_i} : i \in I\}$.

As a corollary to our main result, we will obtain the following bound, which is the desired common extension of Arhangel'skiĭ's Theorem and the Hajnal–Juhász inequality.

Theorem 6 *For every Hausdorff space X,* $|X| \le 2^{\mathrm{pwL}_c(X)\cdot\chi(X)}$.

For undefined notions we refer to [11]. Our notation regarding cardinal functions mostly follows [14]. To state our proofs in the most elegant and compact way we use the language of elementary submodels, which is well presented in [10].

2 A Cardinal Bound for the G_δ-Modification

The following proposition collects a few simple general facts about the piecewise weak Lindelöf number that will be helpful in the proof of the main theorem.

Proposition 7 *For any space X, we have the following:*

(i) $\mathrm{pwL}(X) \le \mathrm{pwL}_c(X)$.
(ii) $\mathrm{pwL}_c(X) \le L(X)$.
(iii) $\mathrm{pwL}_c(X) \le c(X)$.
(iv) *If X is* T_3*, then* $\mathrm{wL}_c(X) \le \mathrm{pwL}(X)$.

Proof The first two items are trivial. To prove the third, let F be a closed subset of X and $\mathcal{V} = \bigcup\{\mathcal{V}_i : i \in I\}$ an open collection satisfying $F \subseteq \bigcup\mathcal{V}$. Suppose $c(X) \le \kappa$. For every $i \in I$, let $\mathcal{C}_i$ be a maximal collection of pairwise disjoint non-empty open subsets of X such that for each $C \in \mathcal{C}_i$, there is some $V_C \in \mathcal{V}_i$ with $C \subseteq V_C$. By letting $\mathcal{W}_i = \{V_C : C \in \mathcal{C}_i\}$, the maximality of $\mathcal{C}_i$ implies that $\bigcup\mathcal{V}_i \subseteq \overline{\bigcup\mathcal{W}_i}$ and so $F \subseteq \bigcup\{\overline{\cup\mathcal{W}_i} : i \in I\}$. Since $|\mathcal{W}_i| \le |\mathcal{C}_i| \le \kappa$, we have $\mathrm{pwL}_c(X) \le \kappa$.

To prove the fourth item, assume X is a regular space and let κ be a cardinal such that $\mathrm{pwL}(X) \le \kappa$. Let F be a closed subset of X and $\mathcal{U}$ an open cover of F. If $\mathcal{U}$ covers X we are done. Otherwise use regularity to choose, for every $p \in X\setminus\bigcup\mathcal{U}$, an open set U_p such that $p \in U_p$ and $F \cap \overline{U}_p = \emptyset$. Note that $\mathcal{U} \cup \{U_p : p \in X\setminus F\}$ is an open cover of X, so by $\mathrm{pwL}(X) \le \kappa$, there is a κ-sized subfamily $\mathcal{V}$ of $\mathcal{U}$ such that $X \subset \overline{\bigcup\mathcal{V}} \cup \bigcup\{\overline{U_p} : p \in X\setminus F\}$. Hence $F \subset \overline{\bigcup\mathcal{V}}$, and we are done. ∎

Corollary 8 *If X is a regular space then* $|X| \le 2^{\mathrm{pwL}(X)\cdot\chi(X)}$.

Proof Combine Proposition 7(iv) with Arhangel'skiĭ's result that $|X| \le 2^{\mathrm{wL}_c(X)\cdot\chi(X)}$ for every regular space X. ∎

We say that $G \subset X$ is a G^c_κ-set if there is a family $\{U_\alpha : \alpha < \kappa\}$ of open subsets of X such that $G = \bigcap\{U_\alpha : \alpha < \kappa\} = \bigcap\{\overline{U_\alpha} : \alpha < \kappa\}$.

Theorem 9 *Let X be a Hausdorff space such that* $t(X)\cdot\mathrm{pwL}_c(X) \le \kappa$ *and X has a dense set of points of character* $\le \kappa$. *Then every cover of X by* G^c_κ*-sets has a* $\le 2^\kappa$*-sized subcollection whose union is dense in X.*

Proof Let $\mathcal{F}$ be a cover of X by G^c_κ-sets. Let θ be a large enough regular cardinal and M be a κ-closed elementary submodel of $H(\theta)$ such that $|M| = 2^\kappa$ and M contains everything we need (that is, $X, \mathcal{F} \in M$, $\kappa + 1 \subset M$, *etc.*).

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For every $F \in \mathcal{F}$ choose open sets $\{U_\alpha(F) : \alpha < \kappa\}$ such that $F = \bigcap\{U_\alpha(F) : \alpha < \kappa\} = \bigcap\{\overline{U_\alpha(F)} : \alpha < \kappa\}$. If $F \in \mathcal{F} \cap M$ we can assume that $\{U_\alpha(F) : \alpha < \kappa\} \in M$ and hence $\{U_\alpha(F) : \alpha < \kappa\} \subset M$.

Claim 1. $\mathcal{F} \cap M$ covers $\overline{X \cap M}$.

Proof of Claim 1 Let $x \in \overline{X \cap M}$. Since $\mathcal{F}$ is a cover of X we can find a set $F \in \mathcal{F}$ such that $x \in F$. Moreover, using $t(X) \leq \kappa$, we can find a κ-sized subset S of $X \cap M$ such that $x \in \overline{S}$. Note that $x \in \overline{U_\alpha \cap S}$ for every $\alpha < \kappa$. Moreover, by κ-closedness of M, the set $U_\alpha \cap S$ belongs to M. Set $B = \bigcap\{\overline{U_\alpha \cap S} : \alpha < \kappa\}$. Note that $x \in B \subset F$ and $B \in M$. Therefore $H(\theta) \models (\exists G \in \mathcal{F})(x \in B \subset G)$ and all the free variables in the previous formula belong to M. Therefore, by elementarity we also have that $M \models (\exists G \in \mathcal{F})(x \in B \subset G)$, and hence there exists a set $G \in \mathcal{F} \cap M$ such that $x \in G$, which is what we wanted to prove. ▲

Claim 2. $\mathcal{F} \cap M$ has dense union in X.

Proof of Claim 2 Suppose by contradiction that $X \nsubseteq \overline{\bigcup(\mathcal{F} \cap M)}$. Then we can fix a point $p \in X \setminus \overline{\bigcup(\mathcal{F} \cap M)}$ such that $\chi(p, X) \leq \kappa$. Let $\{V_\alpha : \alpha < \kappa\}$ be a local base at p.

Let $\mathcal{C} = \{U_\alpha(F) : F \in \mathcal{F} \cap M, \alpha < \kappa\}$. Note that $\mathcal{C}$ is an open cover of $\overline{X \cap M}$ and $\mathcal{C} \subset M$.

For every $x \in \overline{X \cap M}$, using Claim 1 we can choose a set $F_x \in \mathcal{F} \cap M$ such that $x \in F_x$. Since $p \notin F_x$, there is $\alpha < \kappa$ such that $p \notin \overline{U_\alpha(F_x)}$. Hence we can find an ordinal $\beta_x < \kappa$ such that $V_{\beta_x} \cap U_\alpha(F_x) = \emptyset$. This shows that $\mathcal{U} = \{U \in \mathcal{C} : (\exists \beta < \kappa)(U \cap V_\beta = \emptyset)\}$ is an open cover of $\overline{X \cap M}$. Let $\mathcal{U}_\alpha = \{U \in \mathcal{U} : U \cap V_\alpha = \emptyset\}$. Then $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ is a decomposition of $\mathcal{U}$, and hence we can find a κ-sized family $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ for every $\alpha < \kappa$ such that $\overline{X \cap M} \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$. Note that by κ-closedness of M the sequence $\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$ belongs to M and hence the previous formula implies that:

$$M \models X \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}.$$

So, by elementarity:

$$H(\theta) \models X \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}.$$

But that is a contradiction, because $p \notin \overline{\bigcup \mathcal{V}_\alpha}$, for every $\alpha < \kappa$. ▲

Since $|\mathcal{F} \cap M| \leq 2^\kappa$, Claim 2 proves that every cover of X by G^c_κ-sets has a 2^κ-sized subcollection whose union is dense in X, as we wanted. ■

As a first consequence, we derive the desired common extension of Arhangel'skiĭ's Theorem and the Hajnal–Juhász inequality.

Recall that the *closed pseudocharacter of the point x in X* ($\psi_c(x, X)$) is defined as the minimum cardinal κ such that there is a κ-sized family $\{U_\alpha : \alpha < \kappa\}$ of open neighbourhoods of x with $\bigcap\{\overline{U_\alpha} : \alpha < \kappa\} = \{x\}$. The closed pseudocharacter of X ($\psi_c(X)$) is then defined as $\psi_c(X) = \sup\{\psi_c(x, X) : x \in X\}$.

Corollary 10 *Let X be a Hausdorff space. Then $|X| \leq 2^{\mathrm{pwL}_c(X) \cdot \chi(X)}$.*

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Proof It suffices to note that in a Hausdorff space, $\psi_c(X) \cdot t(X) \leq \chi(X)$, and hence if κ is a cardinal such that $\chi(X) \leq \kappa$, then $\mathcal{G} = \{\{x\} : x \in X\}$ is a cover of X by G^c_κ-sets. Therefore by Theorem 9, $\mathcal{G}$ has a $\leq 2^\kappa$-sized subcollection $\mathcal{D}$ such that $D = \bigcup \mathcal{D}$ is dense in X. It turns out that D is a dense subset of X of cardinality $\leq 2^\kappa$. Since $|X| \leq (d(X))^{\chi(X)}$ for every Hausdorff space X, we have $|X| \leq 2^\kappa$, as desired. ∎

Remark Corollary 10 is a *strict* improvement of both Arhangel'skiĭ's Theorem and the Hajnal-Juhász inequality. Indeed, if S is the Sorgenfrey line and $A([0,1])$ the Aleksandroff duplicate of the unit interval, then the space $X = (S \times S) \oplus A([0,1])$ is first countable, $\mathrm{pwL}_c(X) = \aleph_0$, and $L(X) = c(X) = \mathfrak{c}$.

Recall that a space is initially κ-compact if every open cover of cardinality at most κ has a finite subcover (for $\kappa = \omega$ we obtain the usual notion of countable compactness). The following Lemma essentially says that if X is an initially κ-compact space such that $\mathrm{wL}_c(X) \leq \kappa$, then it satisfies the definition of $\mathrm{pwL}_c(X) \leq \kappa$ when restricted to decompositions of cardinality at most κ.

Lemma 11 *Let X be an initially κ-compact space such that $\mathrm{wL}_c(X) \leq \kappa$ and F be a closed subset of X. If $\mathcal{U}$ is an open cover of F and $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ is a κ-sized decomposition of $\mathcal{U}$, then there are κ-sized subfamilies $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ such that $F \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$.*

Proof Let $U_\alpha = \bigcup \mathcal{U}_\alpha$. Then $\{U_\alpha : \alpha < \kappa\}$ is an open cover of F of cardinality κ, so by initial κ-compactness there is a finite subset S of κ such that $F \subset \{U_\alpha : \alpha \in S\}$. Now let $\mathcal{W} = \bigcup\{\mathcal{U}_\alpha : \alpha \in S\}$. We then have $F \subset \bigcup \mathcal{W}$, and hence by $\mathrm{wL}_c(X) \leq \kappa$ we can find a κ-sized subfamily $\mathcal{W}'$ of $\mathcal{W}$ such that $F \subset \overline{\bigcup \mathcal{W}'}$. Now set $\mathcal{V}_\alpha = \{W \in \mathcal{W}' : W \in \mathcal{U}_\alpha\}$. Then $|\mathcal{V}_\alpha| \leq \kappa$ and $F \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$, as we wanted. ∎

Noticing that in the proof of Theorem 9 we only needed to apply the definition of $\mathrm{pwL}_c(X) \leq \kappa$ to decompositions of cardinality κ, Theorem 9 and Lemma 11 imply the following corollaries.

Corollary 12 ([8]) *Let X be an initially κ-compact space containing a dense set of points of character at most κ and such that $\mathrm{wL}_c(X) \cdot t(X) \leq \kappa$. Then every cover of X by G^c_κ-sets has a 2^κ-sized subcollection whose union is dense in X.*

Corollary 13 (Alas, [1]) *Let X be an initially κ-compact space with a dense set of points of character κ, such that $\mathrm{wL}_c(X) \cdot t(X) \cdot \psi_c(X) \leq \kappa$. Then $|X| \leq 2^\kappa$.*

3 Open Questions

Corollary 8 can be slightly improved by replacing regularity with the Urysohn separation property (that is, every pair of distinct points can be separated by disjoint closed neighbourhoods). Indeed, in a similar way as in the proof of Proposition 7(iv), it can be shown that if X is Urysohn then $\mathrm{wL}_\theta(X) \leq \mathrm{pwL}(X)$, where $\mathrm{wL}_\theta(X)$ is the weak Lindelöf number for *θ-closed sets* (see [6]). Moreover, $|X| \leq 2^{\mathrm{wL}_\theta(X) \cdot \chi(X)}$ for every Urysohn space X. However it is not clear whether regularity can be weakened to the Hausdorff separation property. That motivates the next question.

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Question 3.1 *Is the inequality* $|X| \le 2^{\mathrm{pwL}(X)\cdot\chi(X)}$ *true for every Hausdorff space X?*

Moreover, we were not able to find an example that distinguishes a countable piecewise weak Lindelöf number for closed sets from the strictly quasi-Lindelöf property.

Question 3.2 *Is there a strictly quasi-Lindelöf space X such that* $\mathrm{pwL}_c(X) > \aleph_0$*?*

Arhangel'skiĭ's notion of a strict quasi-Lindelöf space suggests a natural cardinal invariant. Define the strict quasi-Lindelöf number of X, $\mathrm{sqL}(X)$, to be the least cardinal number κ such that for every closed subset F of X, for every open cover $\mathcal{U}$ of F, and for every $\le\kappa$*-sized* decomposition $\{\mathcal{U}_\alpha : \alpha < \kappa\}$ of $\mathcal{U}$ there are κ-sized subfamilies $\mathcal{V}_\alpha \subset \mathcal{U}_\alpha$ such that $X \subset \bigcup\{\overline{\bigcup \mathcal{V}_\alpha} : \alpha < \kappa\}$. Obviously $\mathrm{sqL}(X) \le \mathrm{pwL}_c(X)$. It is not at all clear from our argument whether the piecewise weak-Lindelöf number for closed sets can be replaced with the strict quasi-Lindelöf number in Corollary 10.

Question 3.3 *Let X be a Hausdorff space. Is it true that* $|X| \le 2^{\mathrm{sqL}(X)\cdot\chi(X)}$*?*

Even the following special case of the above question seems to be open.

Question 3.4 *Let X be a strict quasi-Lindelöf space. Is it true that* $|X| \le 2^{\chi(X)}$*?*

Finally, it would be interesting to know whether the assumption about the existence of a dense set of points of *small* character can be removed from Theorem 9.

Question 3.5 *Let κ be an infinite cardinal and let X be a Hausdorff space such that* $t(X)\cdot\mathrm{pwL}_c(X) \le \kappa$*. Is it true that every cover of X by* G^c_κ*-sets has a* $\le 2^\kappa$*-sized subcollection whose union is dense in X?*

An affirmative answer to this question would imply that the answer to the following question is also positive.

Question 3.6 *Let X be a Hausdorff space. Is it true that*

$$|X| \le 2^{\mathrm{pwL}_c(X)\cdot t(X)\cdot\psi_c(X)}?$$

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