ON LATTICE PATHS WITH SEVERAL DIAGONAL STEPS

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1. In this note we consider the enumeration of unrestricted and restricted minimal lattice paths from $(0, 0)$ to (m, n), with the following $(\mu + 2)$ moves, μ being a positive integer. Let the line segment between two lattice points on which no other lattice point lies be called a step. A lattice path at any stage can have either (1) a vertical step denoted by S_0, or (2) a diagonal step parallel to the line $x = ty$ $(t = 1, \ldots, \mu)$, denoted by S_t, or (3) a horizontal step, denoted by $S_{\mu+1}$.

A special case of the enumeration problem for $\mu = 1$ and $m = n$ has been studied by Moser and Zayachkowaski in [4], whereas Rohatgi in [5] discussed the same for $\mu = 1$, and $m > n$.

2. For simplicity of presentation, we first derive the results for the case $\mu = 1$. Our considerations are based on a combinatorial approach which is capable of immediate extension for general μ.

When $\mu = 1$, there are three possible moves, i.e. S_0, a vertical step, or S_1, a diagonal step parallel to $x = y$ (briefly referred to in this section as a diagonal step), or S_2, a horizontal step. We define the following notations to be used subsequently in this section.

For non-negative integers $m, n, \alpha, \beta, \ell$ and r,

$S(m, n; r)$: any path from $(0, 0)$ to (m, n) having exactly r diagonal steps;

$N(m, n; r)$: the number of paths of the type $S(m, n; r)$;

$f(\alpha, n, \beta; r)$: the number of paths of the type $S(\alpha + \beta n, n; r)$, $\alpha > 0$, never touching the line $x = \beta y$;

$g(n, \beta; r)$: the number of paths of the type $S(\beta n, n; r)$, never touching the line $x = \beta y$ except at the end points;

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$f'(\alpha, n, \beta; r)$: the number of paths of the type $S(\alpha + \beta n, n; r)$, $\alpha \geq 0$, never crossing the line $x = \beta y$;

$A_\ell(m, n, \beta; r)$: the number of paths of the type $S(m, n; r)$ never crossing the line $x = \beta y - \ell$.

Note that the restricted enumeration of paths in [4] and [5] is discussed for $\beta = 1$.

Let the multinomial coefficient $\binom{x}{j_1, \ldots, j_k}$, represent

$$\frac{x(x-1)\ldots\left(x - \sum_{i=1}^{k} j_i + 1\right)}{\prod_{i=1}^{k} j_i!}.$$

Clearly

$$(1) \qquad N(m, n; r) = \begin{cases} \binom{m+n-r}{r, n-r}, & 0 \leq r \leq \min(m, n); \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 1.

(a)

$$(2) \quad f(\alpha, n, \beta; r) = \begin{cases} \dfrac{\alpha}{\alpha + (\beta+1)n - r} \dbinom{\alpha + (\beta+1)n - r}{r, n-r}, & 0 \leq r \leq n, \ n \geq 0, \ \alpha > 0; \\ 0, & \text{otherwise.} \end{cases}$$

(b)

$$(3) \quad g(n, \beta; r) = \begin{cases} \dfrac{1}{(\beta+1)n - r - 1} \dbinom{(\beta+1)n - r - 1}{r, n-r}, & 0 \leq r \leq n, \ n \geq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Proof. (a) Subtracting from total number of paths of the type $S(\alpha + \beta n, n; r)$, those paths that definitely cross or touch the line $x = \beta y$, we get the recurrence relation

$$(4)\quad f(\alpha, n, \beta; r) = N(\alpha+\beta n, n; r) - \sum_{i=1}^{n} \sum_{j=\max(0, r+i-n)}^{\min(r,i)} N(\beta i, i; j)\, f(\alpha, n-i, \beta; r-j).$$

The boundary conditions are as follows:

$$(5)\quad \begin{cases} f(0, n, \beta; r) = 0, & \text{unless } n = r = 0; \\ f(\alpha, 0, \beta; r) = 1, & \text{for } r = 0 \text{ and } \alpha \geqslant 0, \\ \qquad\qquad = 0, & \text{otherwise;} \\ \text{and} & \\ f(\alpha, n, \beta; r) = 0, & \text{if either } n \text{ or } r \text{ is a negative integer.} \end{cases}$$

We prove part (a) of the theorem by using induction. For $n = 0$, result (2) trivially follows from (5). For $n = 1$, actual enumeration shows that

$$f(\alpha, 1, \beta; r) = \begin{cases} \alpha, & 0 \leqslant r \leqslant 1,\ \alpha \geqslant 1; \\ 0, & \text{otherwise.} \end{cases}$$

Hence (2) is true for $n = 1$. By induction hypothesis and using the recurrence relation (4) we can write,

$$(6)\quad f(\alpha, n, \beta; r) = \binom{\alpha+(\beta+1)n-r}{r,\ n-r} \sum_{i=1}^{n} \sum_{j=\max(0, r+i-n)}^{\min(r,i)} \binom{(\beta+1)i-j}{j,\ i-j} \times \frac{\alpha}{\alpha+(\beta+1)(n-i)-(r-j)} \binom{\alpha+(\beta+1)(n-i)-(r-j)}{r-j,\ n-r-i+j}.$$

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To complete the proof we have to show that

$$(7)\quad \sum_{i=0}^{n} \sum_{j=\max(0,\ r+i-n)}^{\min(r,\ i)} \binom{(\beta+1)i - j}{j,\ i-j} \times \frac{\alpha}{\alpha+(\beta+1)(n-i)-(r-j)} \binom{\alpha+(\beta+1)(n-i)-(r-j)}{r-j,\ n-r-i+j} = \binom{\alpha+(\beta+1)n-r}{r,\ n-r}.$$

By an interchange of the order of summation, the left-hand side of (7), after some simplification, can be expressed as

$$\sum_{j=0}^{r} \sum_{s=0}^{n-r} \binom{(\beta+1)(s+j)-j}{j,\ s} \times \frac{\alpha}{\alpha+(\beta+1)(n-j-s)-(r-j)} \binom{\alpha+(\beta+1)(n-j-s)-(r-j)}{r-j,\ n-r-s}$$

which by summation formula (10) with $k = 2$, in [2], yields the right hand side of (7). This completes the proof for (a).

We remark here that the expression for $f(\alpha, n, \beta; r)$ satisfies the obvious recurrence relation

$$(8)\quad f(\alpha, n, \beta; r) = f(\alpha+\beta, n-1, \beta; r) + f(\alpha+\beta-1, n-1, \beta; r-1) + f(\alpha-1, n, \beta; r),$$

for $0 \leqslant r \leqslant n$, $\alpha > 0$ and $n \geqslant 0$, with the boundary conditions same as (5).

(b) We observe that $g(n, \beta; r)$ satisfies the relation

$$(9)\quad g(n, \beta; r) = f(\beta, n-1, \beta; r) + f(\beta-1, n-1, \beta; r-1),$$

for $0 \leqslant r \leqslant n$, $n \geqslant 1$, which by using (2) simplifies to the required expression (3). This completes the proof of the theorem.

Evidently one finds

$$(10)\quad f'(\alpha, n, \beta; r) = f(\alpha+1, n, \beta; r) = \frac{\alpha+1}{\alpha+1+(\beta+1)n-r}\binom{\alpha+1+(\beta+1)n-r}{r,\ n-r}$$

for $0 \leqslant r \leqslant n$, $\alpha \geqslant 0$.

Putting $\beta = 1$, and $\alpha = m - n$ in (2), (3) and (10), we obtain the expressions for $Q(m,n)$ and $Q'(m,n)$ defined in [5], as

$$(11)\quad Q(m,n) = \begin{cases} \sum_{r=0}^{n} \frac{m-n}{m+n-r}\binom{m+n-r}{r,\ n-r} & \text{for } m > n; \\ \sum_{r=0}^{n-1} \frac{1}{2n-r-1}\binom{2n-r-1}{r,\ n-r} & \text{for } m = n; \\ 0 & \text{for } m < n; \end{cases}$$

and

$$(12)\quad Q'(m,n) = \begin{cases} \sum_{r=0}^{n} \frac{m-n+1}{m+n-r+1}\binom{m+n-r+1}{r,\ n-r} & \text{for } m \geqslant n; \\ 0 & \text{for } m < n; \end{cases}$$

which provide the solutions to (1) and (2) in [5].

An expression for $A_\ell(m, n, \beta; r)$ can be obtained by an argument analogous to that for (4), which for general β cannot further be simplified. When $\beta = 1$,

$$(13)\quad A_\ell(m, n, 1; r) = N(m, n; r) - \sum_{i=\ell+1}^{n} \sum_{j=\max(0,r+i-n)}^{\min(i-\ell-1,r)} N(i-\ell-1, i; j)\, f'(m-n+\ell, n-i, \beta; r-j),$$

for $0 \leq r \leq \min(m, n)$, $m \geqslant n - 1$, which by an interchange of the order of summation and some elementary simplification reduces to

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$$(14) \qquad \binom{m+n-r}{r,\, n-r} - \sum_{j=0}^{r} \sum_{s=\ell+1}^{n-r} \binom{2s+j-\ell-1}{j,\, s-\ell-1}$$

$$\times \frac{m-n+\ell+1}{m-n+\ell+1+2(n-r-s)+r-j} \binom{m-n+\ell+1+2(n-r-s)+r-j}{r-j,\quad n-r-s}.$$

The second term in (14) sums up to $\binom{m+n-r}{r,\ n-r-\ell-1}$ by the use of (10) with $k = 2$ in [2].

Thus the expression for $A_\ell(m, n, 1; r)$ is

$$\binom{m+n-r}{r,\, n-r} - \binom{m+n-r}{r,\ n-r-\ell-1},$$

which can be written as

$$(15) \qquad \binom{m+n-r}{r} \left[\binom{m+n-2r}{n-r} - \binom{m+n-2r}{n-r-\ell-1} \right].$$

By the use of corollary (4) in [3], we have the following relation

$$(16) \qquad A_\ell(m, n, 1; r) = \binom{m+n-r}{r} A_\ell(m-r, n-r, 1; 0).$$

3. In this section we state results for general enumeration probler where $1 \leq \mu \leq \beta$. Let $f(\alpha, n, \beta; r_1, \ldots, r_\mu)$ represent the number of lattice paths from $(0, 0)$ to $(\alpha + \beta n, n)$, $\alpha > 0$, never touching the line $x = \beta y$ and having r_i steps of the type S_i $(i = 1, \ldots, \mu)$. Also denote by $g(n, \beta; r_1, \ldots, r_\mu)$ the number of lattice paths from $(0, 0)$ to $(\beta n, n)$, never touching the line $x = \beta y$ except at the end points and having r_i steps of the type S_i $(i = 1, \ldots, \mu)$.

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THEOREM 2.

(a) $f(\alpha, n, \beta; r_1, \ldots, r_\mu) =$

$$(17)\quad \begin{cases} \dfrac{\alpha}{\alpha + (\beta+1)n - \Sigma i r_i} \dbinom{\alpha + (\beta+1)n - \Sigma i r_i}{r_1, \ldots, r_\mu, \; n - \Sigma r_i}, \\ \qquad \underline{\text{for}} \;\; 0 \leqslant \Sigma r_i \leqslant n, \; 0 \leqslant r_i \leqslant n \; (i = 1, \ldots, \mu), \; n \geqslant 0, \; \alpha > 0; \\ 0, \qquad \underline{\text{otherwise}}; \end{cases}$$

(b) $g(n, \beta; r_1, \ldots, r_\mu) =$

$$(18)\quad \begin{cases} \dfrac{1}{(\beta+1)n - \Sigma i r_i - 1} \dbinom{(\beta+1)n - \Sigma i r_i - 1}{r_1, \ldots, r_\mu, \; n - \Sigma r_i} \\ \qquad \underline{\text{for}} \;\; 0 \leqslant \Sigma r_i \leqslant n, \; 0 \leqslant r_i \leqslant n \; (i = 1, \ldots, \mu), \; n \geqslant 1; \\ 0, \qquad \underline{\text{otherwise}}; \end{cases}$$

where Σ, _stands for_ $\sum_{1}^{\mu}$.

We conclude with a generalization of the result in [4], on paths ending at (km, kn), m, n being coprime (also see [1]).

In [1] it has been shown that the number of paths from (0, 0) to (km, kn) without diagonal steps, which do not cross the line nx = my are given by

$$(19)\qquad \phi_k = \Sigma^* \frac{F_1^{k_1}}{k_1!} \cdot \frac{F_2^{k_2}}{k_2!} \cdot \ldots$$

and of those which never touch the line nx = my except at the end points are given by

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(20) $$\psi_k = \Sigma^* (-1)^{1 + \Sigma_{i=1}^{\infty} k_i} \frac{F_1^{k_1}}{k_1!} \cdot \frac{F_2^{k_2}}{k_2!} \cdots,$$

where $$F_j = \frac{1}{(m+n)j} \binom{(m+n)j}{mj}$$

and Σ^* is the summation over all $k_i \geq 0$, $i = 1, 2, \ldots,$ subject to $\Sigma_{i=1}^{\infty} i k_i = k$.

We remark here that the results (19) and (20) also hold for the case with diagonal steps provided we modify the function F_j by

$$F_j' = \Sigma_{R'} \frac{1}{k(m+n) - \Sigma i r_i} \binom{k(m+n) - \Sigma i r_i}{r_1, \ldots, r_\mu, kn - \Sigma r_i}$$

where R' consists of restrictions:

$$\{0 \leq \Sigma r_i \leq kn,\ 0 \leq \Sigma i r_i \leq km,\ 0 \leq r_i \leq \min(km, kn)\ (i = 1, \ldots, \mu)\},$$

and r_i represents the number of steps of the type S_i. Σ stands for Σ_1^{μ} everywhere.

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