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ON SUPERSOLVABILITY OF FINITE GROUPS

PIROSKA CSÖRGŐ
Eötvös University, Dept. of Algebra and Number Theory, Pàzmàny Pèter Sètàny 1/c, H-1117 Budapest, Hungary
e-mail: skacs.elte.hu

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Abstract. We prove a natural factorization of supersolvable groups and then we give another characterization of them in connection with the Fitting subgroup. Applying these theorems we describe the structure of some subclasses of supersolvable groups.

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Introduction. All groups considered in this note are finite. Recall that a group G is called *supersolvable* if its every chief factor is cyclic. We say following Kegel [**1**] that a subgroup of G is *S-quasinormal* in G if it permutes with every Sylow subgroup of G. $\pi(G)$ denotes the set of prime divisors of the order of the group G. Several authors examined the structure of a group under the assumption that some subgroups are well-situated in the group. They studied the influence of S-quasinormality of some subgroups of a finite group G which ensures the supersolvability of G. In this paper we prove a natural factorization of supersolvable groups. The corollary of this theorem is another characterization of supersolvable groups based on the structure of the Fitting subgroup. By using these results we describe the structure of these subclasses of supersolvable groups obtained under the assumption that some subgroups satisfy certain conditions.

Main results. Huppert proved [**2**, Satz 10.3, p. 724] the following theorem. If a finite group is the product of pairwise permutable cyclic subgroups, then it is supersolvable. Of course the converse of this statement is not even true in the class of nilpotent groups, since there are nonabelian groups of exponent p when $p > 2$. By studying it, we find that a supersolvable group can be decomposed as a product of cyclic subgroups of prime power order that are permutable if their orders are powers of different primes and those belonging to the same prime satisfy certain conditions.

THEOREM 1. *Let G be a group with $\pi(G) = \{p_1, \ldots, p_k\}$. Then G is supersolvable if and only if for all $p_i \in \pi(G)$ there is a Sylow p_i-subgroup P_i and cyclic subgroups P_{i_l} $(1 \leq l \leq t_i)$ of P_i such that*

(i) $P_i = P_{i_1} P_{i_2} \ldots P_{i_{t_i}}$,
(ii) $P_{i_1} \ldots P_{i_l} \lhd P_i$, *for all* $1 \leq l \leq t_i$,
(iii) $P_{i_l} \cdot P_{j_m} = P_{j_m} \cdot P_{i_l}$, *for all* $1 \leq i, j \leq k$, $i \neq j$, $1 \leq l \leq t_i$, $1 \leq m \leq t_j$.

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Moreover, for every chief series refining a Sylow tower there exists such a factorization of Sylow subgroups as given above.

For the proof we need the following result.

LEMMA 1. *Let A be an abelian normal Sylow p-subgroup of a group G. Let $a \in A$ be of order p such that $\langle a \rangle$ is normal in G and $A/\langle a \rangle$ is cyclic. Then either A is cyclic or $A = \langle a \rangle \times \langle b \rangle$, where $\langle b \rangle$ is normal in G.*

Proof. Assume that A is not cyclic. Obviously G/A acts on $A/\Phi(A)$. By Maschke's Theorem $A/\Phi(A) = \langle a\Phi(A) \rangle \times \langle b\Phi(A) \rangle$, where $\langle b\Phi(A) \rangle$ is G/A-invariant. As A is abelian, $A = \langle a \rangle \langle b \rangle$ holds. Since $|a| = p$ we conclude that here $|A : \langle b \rangle| = p$. Also $\langle b \rangle \supseteq \Phi(A)$, whence $\langle b \rangle$ is G/A-invariant. Consequently $\langle b \rangle \lhd G$. □

LEMMA 2. *Let G be a supersolvable group. Let P be a normal Sylow p-subgroup of G and H a p-complement. Then $P = A_1 \cdot A_2 \dots A_s$, where every A_i is a cyclic subgroup of P such that $H \leq N_G(A_i)$ and $A_1 A_2 \dots A_i \lhd P$, for all $1 \leq i \leq s$.*

Proof. We prove our statement by induction on the order of P. The supersolvability of G implies that there exists a subgroup A_1 of P of order p such that A_1 is normal in G. We can assume that $A_1 \neq P$. Obviously G/A_1 satisfies the conditions of our lemma. By induction on G/A_1 there exist cyclic subgroups $D_2/A_1, \dots, D_s/A_1$ of P/A_1 such that $HA_1/A_1 \leq N_{G/A_1}(D_i/A_1)$ and $(D_2/A_1) \dots (D_s/A_1) \lhd P/A_1 = D_2(A_1) \dots D_s(A_1)$, for all $2 \leq i \leq s$. Thus $H \leq N_G(D_i)$ and, since $A_1 \leq Z(P)$, the D_i are abelian. Applying Lemma 1 to D_i and A_1, we can see that for all i there is a cyclic subgroup A_i of D_i such that $D_i = A_1 \cdot A_i$ and A_i is normalized by H. The subgroups A_i have the required properties. □

Proof of Theorem 1. First we assume the supersolvability of G. Then G possesses an ordered Sylow tower by [**2**, Satz 9.1, p. 716]. Suppose $p_1 > p_2 > \dots > p_k$. Then for each p_i we have a Sylow p_i-subgroup P_i such that $P_i \leq N_G(P_j)$ for all $j < i$. Using the normality of P_i in $P_i P_{i+1} \dots P_k$ we apply Lemma 2 to P_i and $P_i P_{i+1} \dots P_k$. We get a factorization $P_i = P_{i_1} \dots P_{i_{t_i}}$, where P_{i_r} is a cyclic subgroup normalized by $P_{i+1} \dots P_k$ and $P_{i_1} \dots P_{i_r}$ is normal in P_i, for all $1 \leq r \leq t_i$. Let $1 \leq i, j \leq k$, $i \neq j$, $1 \leq l \leq t_i$, $1 \leq m \leq t_j$. Assume $i < j$. Then, as stated before, P_{i_l} is normalized by P_j, whence $P_{i_l} P_{j_m} = P_{j_m} P_{i_l}$ holds.

Conversely, assume that G is a group satisfying (i), (ii), and (iii) of the Theorem. Suppose $p_1 > p_2 > \dots > p_k$. Let $N = P_{11}$. By hypothesis we have $N \lhd P_1$ and $NP_{i_l} = P_{i_l} N$, for all $2 \leq i \leq k$ and $1 \leq l \leq t_i$. By Ito's Theorem [**2**, Satz 10.1, p. 722] we get that NP_{i_l} is supersolvable whence, since $p_1 > p_i$, $N \lhd NP_{i_l}$ follows. Therefore $P_i \leq N_G(N)$, for all $2 \leq i \leq k$. Thus we conclude $N \lhd G$. Since G/N obviously inherits the hypothesis, we have by induction that G/N is supersolvable. As N is cyclic, we find G is supersolvable. □

M. Asaad and M. Ramadan in [**3**, Theorem 3.3] proved the following result. Suppose that G is solvable and $\Phi(G) = 1$. Then G is supersolvable if and only if Fit G is the direct product of some normal subgroups of G of prime order.

Using our factorization on supersolvable groups we generalize the Theorem above. Not supposing $\Phi(G) = 1$ we give another characterization of supersolvable groups. For this aim we introduce the following concept.

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DEFINITION. A subgroup H of G is called *weak S-quasinormal* in G if, for every $p \in \pi(G)$, there is at least one Sylow p-subgroup of G that permutes with H.

REMARK. It follows from Theorem 1 that a supersolvable group is the product of some weak S-quasinormal cyclic subgroups of prime power orders.

THEOREM 2. *For a group G the following statements are equivalent.*

(a) *G is supersolvable.*

(b) $G' \leq \mathrm{Fit}\, G$ *and* $\mathrm{Fit}\, G$ *is the product of cyclic and weak S-quasinormal subgroups of G of prime power orders.*

(c) *There is a nilpotent normal subgroup N of G, such that* $G' \leq N$ *and N is the product of cyclic and weak S-quasinormal subgroups of G of prime power orders.*

Proof. (a) $\Longrightarrow$ (b) Let G be supersolvable. Then by [**2**, Satz 9.1, p. 716] G' is nilpotent, whence $\mathrm{Fit}\, G \geq G'$. The supersolvability implies the existence of an ordered Sylow tower. Let $\pi(G) = \{p_1, \ldots, p_k\}$ with $p_1 > p_2 \ldots > p_k$ and let $P_1 P_2 \ldots P_i$ $(i = 1, \ldots, k)$ be a Sylow tower of G. Clearly there exists a chief series refining our Sylow tower such that it contains $P_1 P_2 \ldots P_{i-1}(P_i \cap \mathrm{Fit}\, G)$ for all $1 \leq i \leq k$. Applying Theorem 1, it is easy to see that $P_i \cap \mathrm{Fit}\, G$ is the product of weak S-quasinormal cyclic subgroups.

(b) $\Longrightarrow$ (c) This is trivial, because we may choose $N = \mathrm{Fit}\, G$.

(c) $\Longrightarrow$ (a) Hypothesis (c) is obviously inherited by all quotient groups. Let G be a group of minimal order that is not supersolvable but satisfies (c). By the minimality we conclude that G has a unique minimal normal subgroup M. As G is solvable, $\Phi(G) = 1$ and $M = \mathrm{Fit}\, G$. Obviously $N = M = G'$ is an elementary abelian p-group, for some prime p, and G/N is a p'-group, so that N is the Sylow p-subgroup of G. By the conditions $N = N_1 \cdot N_2 \cdot \ldots \cdot N_t$ with cyclic and weak S-quasinormal subgroup N_i of G. The weak S-quasinormality implies that for every $q \neq p$ there is a Sylow q-subgroup Q such that $QN_i = N_iQ$ for all $1 \leq i \leq t$. Since N_i is subnormal in G, obviously $N_i \lhd N_iQ$. As $N_i \lhd N$, we find $N_G(N_i) = G$, whence $N_i = N$. We have N_i is cyclic and consequently G is supersolvable, a contradiction. □

We try to weaken these conditions to give another characterization of supersolvable groups.

THEOREM 3. *Let G be a group with* $G' \leq \mathrm{Fit}\, G$. *Then G is supersolvable if and only if there exists a normal subgroup H of G such that G/H is supersolvable and* $\mathrm{Fit}\, H$ *is the product of cyclic and weak S-quasinormal subgroups of G.*

Proof. (1) Assume that G is supersolvable. We may choose $H = \mathrm{Fit}\, G$. Using our Theorem 2, we conclude that H satisfies the conditions.

(2) Let G be a group of minimal order that is not supersolvable, but has got a normal subgroup H with the required properties. We now aim to show that $\Phi(G) = 1$. Assume $\Phi(G) \neq 1$. Since $\mathrm{Fit}\, G \geq G'$, G is solvable, whence $\mathrm{Fit}\, G \neq \Phi(G)$. Clearly $\mathrm{Fit}\, G \cap H = \mathrm{Fit}\, H$. If $\Phi(G) \cap H = \mathrm{Fit}\, G \cap H$, using again $\mathrm{Fit}\, G \geq G'$, we conclude that $H\Phi(G)/\Phi(G)$ is abelian and $H\Phi(G)/\Phi(G) \cap \mathrm{Fit}\, G/\Phi(G) = 1$. We have $H\Phi(G)/\Phi(G) \lhd G/\Phi(G)$ and further $\mathrm{Fit}(G/\Phi(G)) = \mathrm{Fit}\, G/\Phi(G)$. As G is solvable, $C_{G/\Phi(G)}(\mathrm{Fit}(G/\Phi(G))) \leq \mathrm{Fit}(G/\Phi(G))$. It follows that $H \leq \Phi(G)$, whence $G/\Phi(G)$ is supersolvable. Using Huppert's Theorem [**2**, Satz VI.8.6] we get that G is super-

solvable, contradicting the minimality of G. Thus $\Phi(G) \cap H \neq \operatorname{Fit} G \cap H = \operatorname{Fit} H$. Obviously $G/\Phi(G)$ satisfies the conditions of our theorem. The minimality of G yields the supersolvability of $G/\Phi(G)$. Using again Huppert's Theorem we find that G is supersolvable, a contradiction. Thus $\Phi(G) = 1$.

The supersolvability of G/H implies the existence of the following chain: $\operatorname{Fit} G \cap H = \operatorname{Fit} H = F_0 \lhd F_1 \lhd \ldots \lhd F_k = \operatorname{Fit} G$ such that $F_i \lhd G$ and F_i/F_{i-1} is of prime order for all $1 \leq i \leq k$. Assume F_i/F_{i-1} is of order p. Let H be a Hall subgroup of G with $\pi(H) = \pi(G) \setminus \{p\}$. Then H acts on F_i and F_{i-1}. Using Maschke's Theorem we get $F_i = F_{i-1} \times \langle b_i \rangle$ and $H \leq N_G(\langle b_i \rangle)$ so that $\langle b_i \rangle$ is weak S-quasinormal in G. As we have $\operatorname{Fit} G \cap H$ is the product of cyclic weak S-quasinormal subgroups of G, we conclude that $\operatorname{Fit} G$ is the product of cyclic weak S-quasinormal subgroups of G. Applying our Theorem 2 we find that G is supersolvable. This is the final contradiction. □

For the study of the structure of some subclasses of supersolvable groups we prove the following result.

Theorem 4. *Let G be a supersolvable group and U a normal p-subgroup of G with $p \neq 2$. Then every minimal subgroup of U is normal in U and S-quasinormal in G if and only if there is a chain $1 = U_0 \lhd U_1 \lhd \ldots \lhd U_k = U$ with $U_i \lhd G$, $|U_i/U_{i-1}| = p$ for every $1 \leq i \leq k$ and $\Omega_1(U) = U_l \leq Z(U)$, for some $1 \leq l \leq k$. Moreover, for every $g \in G$ with $(|g|, p) = 1$, there exists a natural number t_g with $1 \leq t_g \leq p-1$ such that $a^g = a^{t_g}$, where a is an arbitrary element of $D = \bigtimes_{i=1}^{k}(U_i/U_{i-1})$.*

Proof. Assume that every minimal subgroup of U is normal in U and S-quasinormal in G. Then these minimal subgroups are clearly in $Z(U)$; that is $\Omega_1(U) \leq Z(U)$. By the supersolvability of G there is a chain $1 = U_0 \lhd U_1 \lhd \ldots \lhd U_k = U$ such that $U_i \lhd G$ and U_i/U_{i-1} is of order p, for all $1 \leq i \leq k$. Moreover $U_l = \Omega_1(U)$, for some $1 \leq l \leq k$. Let $g \in G$ with $g \neq 1$, $(|g|, p) = 1$ and $u \in \Omega_1(U)$, $u \neq 1$. As $\langle u \rangle$ is subnormal in G, and g is the product of elements of prime power orders, the S-quasinormality implies $g \in N_G(\langle u \rangle)$. Thus we have $u^g = u^{t_g}$, for some natural number t_g with $1 \leq t_g \leq p-1$. Let $v \in \Omega_1(U)$, $1 \neq v \neq u$. Similarly $v^g = v^l$ where $1 \leq l \leq p-1$. Since $(uv)^g = (uv)^k$, for some natural number $1 \leq k \leq p-1$, using $\Omega_1(U) \leq Z(U)$ we find $k = t_g = l$. Thus $w^g = w^{t_g}$, for all $w \in \Omega_1(U)$. Apply our factorization Theorem 1 to $U\langle g \rangle$ and to the chief series $1 = U_0 \lhd U_1 \lhd \ldots \lhd U_k = U \lhd U\langle g \rangle$. Then for every $i \leq i \leq k$ there exists $a_i \in U_i$ such that $U_{i-1}\langle a_i \rangle = U_i$ and $g \in N_G(\langle a_i \rangle)$. Suppose $|a_i| = p^{m_i}$ and $a_i{}^g = a_i{}^{t_i}$, for some natural number $1 \leq t_i \leq p^{m_i} - 1$. Since $(a_i{}^{p^{m_i-1}})^g = (a^{p^{m_i-1}})^{t_g}$ we conclude $t_i \equiv t_g\ (p)$. Moreover $\bar{a}_i^g = \bar{a}_i^{t_g}$ is true for all $1 \leq i \leq k$ and $\bar{a}_i \in U_i/U_{i-1}$. Consequently $a^g = a^{t_g}$ follows for an arbitrary element a of $D = \bigtimes_{i=1}^{k}(U_i/U_{i-1})$.

Assume conversely that G has got a chain with the required properties. As $\Omega_1(U) \leq Z(U)$, every minimal subgroup of U is normal in U. Let $g \in G$ with $g \neq 1$, $(|g|, p) = 1$. Applying Lemma 2 to $\Omega_1(U)\langle g \rangle$, we get $\Omega_1(U) = \langle b_1 \rangle \times \ldots \times \langle b_l \rangle$ with $g \in N_G(\langle b_i \rangle)$ for all i. Using the hypothesis $b_i{}^g = b_i{}^{t_g}$ follows, whence $b^g = b^{t_g}$ holds for every $b \in \Omega_1(U)$. It easily follows from this fact that every minimal subgroup of U is S-quasinormal in G. □

In [**4**, Theorem 1] M. Asaad and the author have shown the following result. Let $\mathcal{F}$ be a saturated formation containing the class of supersolvable groups. Suppose

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that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every subgroup of $\operatorname{Fit} H$ of prime order or order 4 is S-quasinormal in G, then $G \in \mathcal{F}$. A corollary of this theorem [**4**, Corollary 4] is the following result. If G is solvable and every subgroup of $\operatorname{Fit} G$ of prime order or order 4 is S-quasinormal in G, then G is supersolvable. Examining the structure of this subclass of supersolvable groups we obtain the following result.

Theorem 5. *Let G be a solvable group. Then every subgroup of* $\operatorname{Fit} G$ *of prime order or order* 4 *is S-quasinormal in G if and only if $G = M(N \times K)$, where M is a nilpotent normal subgroup of odd order, N is a nilpotent subgroup, K is a nilpotent Hall subgroup such that $M \cap (N \times K) = 1$,* $\operatorname{Fit} G \cap N = 1$ *and every minimal subgroup of M is normal in M and S-quasinormal in G.*

Thus we can apply our Theorem 4 to the Sylow subgroups of M, and we get a complete characterization.

For the proof we need the following results.

Lemma 3. *Let U be a 2-group in a group G, $a \in N_G(U)$ with $(|a|, 2) = 1$ and a normalizes every minimal subgroup of U and every cyclic subgroup of order* 4. *Then $a \in C_G(U)$.*

Proof. Let h be an element of U of order 4. By the conditions either $h^a = h$ or $h^a = h^3$. If $h^a = h^3$, then $a^2 \in C_G(h)$. Since a is of odd order, we conclude $a \in C_G(h)$. Consequently $a \in C_G(\Omega_2(U))$, which yields $a \in C_G(U)$. □

Lemma 4. *Let P be a normal p-subgroup of a solvable group G with an odd prime p. Suppose every minimal subgroup of P is S-quasinormal in G. One of the following holds:*

(1) *every minimal subgroup of P is normal in P,*

(2) *$Q \leq C_G(P)$ for every Sylow q-subgroup Q of G with $q \neq p$.*

Proof. Assume there exists an element x_0 of P such that $|x_0| = p$ and $\langle x_0 \rangle$ is not normal in P. The solvability of G implies the existence of a Hall subgroup H with $\pi(H) = \pi(G) \setminus \{p\}$. As P is normal in G, $\langle x_0 \rangle$ is subnormal in G. From the S-quasinormality of $\langle x_0 \rangle$ we easily conclude that $H \leq N_G(\langle x_0 \rangle)$. Let H_1 be the normal closure of H in G. Obviously $H_1 \leq N_G(\langle x_0 \rangle)$. As $\langle x_0 \rangle$ is not normal in G, we find that $H_1 \cap P = P_0 \neq P$. Using $P \lhd H_1 P$ and $H_1 \lhd H_1 P$ we have that the elements of H fix the elements of P/P_0 by conjugation. Applying Glauberman's Theorem [**5**] we get that there exists $v \in P \setminus P_0$ such that $H \leq C_G(v)$. Clearly the elements of H normalize every minimal subgroup of P. Applying [**4**, Lemma 4] $H \leq C_G(P)$ holds. Let Q be a Sylow q-subgroup of G with $q \neq p$. As $Q \leq H^z$ for some $z \in G$ and P is normal in G, our statement follows. □.

Proof of Theorem 5. Suppose every subgroup of $\operatorname{Fit} G$ of order prime or 4 is S-quasinormal in G. It follows from [**4**, Corollary 4] that G is supersolvable, whence G' is nilpotent and $G' \leq \operatorname{Fit} G$. Using [**2**, Satz 3.10 p. 271] $G = H\operatorname{Fit} G$ for some nilpotent subgroup H. If P is an arbitrary Sylow subgroup of $\operatorname{Fit} G$, denote by P^* the unique Sylow subgroup of the subgroup HP containing P. Denote by $\mathcal{S}$ the set of those Sylow subgroups P of $\operatorname{Fit} G$ for which HP is nilpotent. Define $\mathcal{S}^* = \{P^*/P \in \mathcal{S}\}$.

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Suppose $P_1{}^*, P_2{}^* \in \mathcal{S}^*$. From the above we can easily conclude that $P_1{}^* \leq C_G(P_2{}^*)$. Let K be the direct product of the elements of $\mathcal{S}^*$. We have $\mathrm{Fit}\, G = M \times (K \cap \mathrm{Fit}\, G)$ and $H = N \times (H \cap K)$, for some nilpotent subgroup N and a nilpotent Hall subgroup M. Let B be the Sylow 2-subgroup of $\mathrm{Fit}\, G$ and h an arbitrary element of odd order of H. By the conditions, using Lemma 3, $h \in C_G(B)$ holds, whence BH is nilpotent; consequently M is of odd order. We have $NK = N \times K$, $M \lhd G$, $NM \lhd G$, and so we find $G = M(N \times K)$. Assume Q is a Sylow subgroup for an odd prime of $\mathrm{Fit}\, G$ with $H \cap Q \neq 1$. Obviously $H = (H \cap Q^*) \times T$ for some Hall subgroup T of H. Hence $T \leq C_G(H \cap Q)$ and $T \leq N_G(Q)$. By using our hypothesis and applying [**4**, Lemma 4] we get $T \leq C_G(Q)$. Thus in this case $Q^* \in \mathcal{S}^*$, that is $Q^* \leq K$. This fact implies $M \cap N = 1$ and $M \cap (N \times K) = 1$. Let R be an arbitrary Sylow subgroup of M. By our Lemma 4 every minimal subgroup of R is normal in R and consequently in M too, and is S-quasinormal in G. Obviously $\mathrm{Fit}\, G \cap N = 1$.

Assume conversely $G = M(N \times K)$ has the required properties. Clearly $M \leq \mathrm{Fit}\, G \leq MK$. Let D be a subgroup of $\mathrm{Fit}\, G$ of prime order or order 4. Supposing $D \leq M$, D is S-quasinormal in G. If $D \leq K$, using the structure of G, it is easy to see the S-quasinormality of D in G. □

Asaad, Ramadan and Shaalan proved the following result [**6**, Corollary 4.3]. If G is a solvable group and every maximal subgroup of any Sylow subgroup of $\mathrm{Fit}\, G$ is S-quasinormal in G, then G is supersolvable.

The study of the structure of this subclass of supersolvable groups has led to the following result.

THEOREM 6. *Let G be a solvable group. Then every maximal subgroup of a Sylow subgroup of* $\mathrm{Fit}\, G$ *is S-quasinormal in G if and only if, for every Sylow p-subgroup P of* $\mathrm{Fit}\, G$ *and for every element g of G with* $(|g|, p) = 1$*, there exists a natural number t_g such that* $1 \leq t_g \leq p-1$ *and* $\bar{a}^g = \bar{a}^{t_g}$ *is true, for all* $\bar{a} \in P/P \cap \Phi(G)$.

Proof. Assume every maximal subgroup of any Sylow subgroup of $\mathrm{Fit}\, G$ is S-quasinormal in G. Then, by [**6**, Corollary 4.3], G is supersolvable. Let P be an arbitrary Sylow p-subgroup of $\mathrm{Fit}\, G$ and $g \in G$ with $(|g|, p) = 1$. Applying our Lemma 2 to P and $P\langle g \rangle$, we find $P = A_1 \ldots A_k$, where A_i is a cyclic subgroup of P, normalized by $\langle g \rangle$ and $A_1 A_2 \ldots A_i \lhd P$, for all $1 \leq i \leq k$. We have $\Phi(P) \leq \Phi(G)$ and $P/P \cap \Phi(G)$ is an elementary abelian p-group. Consequently $P/P \cap \Phi(G) = \tilde{P} = \tilde{A}_1 \times \ldots \times \tilde{A}_k$, where $\tilde{A}_i = A_i/A_i \cap \Phi(G)$ and $\tilde{A}_i = \langle \bar{a}_i \rangle$. Obviously for every $1 \leq i \leq k$ there exists $1 \leq t_i \leq p-1$ such that $\bar{a}_i^g = \bar{a}_i{}^{t_i}$. It will suffice to show that $t_i = t_1$, for all $1 < i \leq k$. Suppose $t_l \neq t_m$, for some $1 \leq l, m \leq k$, $l \neq m$. Let $\tilde{A}$ be the product of every such $\tilde{A}_i$, where i is different from l and m. Let $\tilde{B} = \langle \bar{a}_l \bar{a}_m \rangle$. Clearly $\tilde{A}\tilde{B}$ is a maximal subgroup in $\tilde{P}$. It follows from the conditions that g acts on $\tilde{P}/\tilde{A}\tilde{B}$. As $\tilde{P} = \tilde{A}\tilde{B} \cdot \tilde{A}_l = \tilde{A}\tilde{B} \cdot \tilde{A}_m$ and $\bar{a}_l^g = \bar{a}_l^{t_l}$ and $\bar{a}_m^g = \bar{a}_m^{t_m}$, we find $t_l = t_m$, a contradiction. Putting $t_g = t_1$, this part of our Theorem follows.

The converse of this theorem is trivial. □

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