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A NOTE ON THE EXISTENCE OF A SOLUTION OF THE FALKNER–SKAN EQUATION

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1. **Introduction.** We are concerned with the existence proof of solution of the Falkner–Skan equation

$$f'''+ff''+\lambda(1-f'^2) = 0 \quad \lambda > 0 \tag{1}$$

subject to boundary conditions

$$\begin{aligned} f = f' &= 0 \quad \text{at } t = 0 \\ f' &= 1 \quad \text{at } t = \infty. \end{aligned} \tag{2}$$

The first existence and uniqueness proof based on a fixed point theorem was given by Weyl [4] in 1942, with the added assumption that $f' > 0$. In 1960, Coppel [1] proved the existence (and uniqueness with the assumption $0 < f' < 1$) by considering trajectories in the three-dimensional phase space.

In this note, we prove the existence of at least one solution without assuming the condition $1 > f' > 0$ by a "shooting method" which is commonly used in the numerical solution of two point boundary value problems. The method consists of seeking an appropriate initial condition for f'' so that the solution of the resulting initial value problem has the correct limiting behaviour when t is large.

The advantage is that we need only consider the behaviour of the trajectory in the f'–t plane. This method of proving existence is apparently first used by Ho and Wilson [2] and later by McLeod and Serrin [3]. The approach used here is essentially that of Serrin.

2. **The sets S^+ and S^-.** We consider equation (1) with the initial values

$$f(0) = f'(0) = 0; \qquad f''(0) = \beta, \tag{3}$$

where $-\infty < \beta < \infty$. We define the sets S^+ and S^- of values of β as follows:

$$\beta \in S^+ \text{ if } \exists t^+ > 0 \ni f'(t^+) > 1$$

and

$$f' > 0 \quad \text{for} \quad 0 < t < t^+.$$

$$\beta \in S^- \text{ if } \exists t^- > 0 \ni f'(t^-) < -k; \quad 0 < k < 1,$$

and

$$f' < 1 \quad \text{for} \quad 0 < t < t^-.$$

LEMMA 1. *The sets S^+ and S^- are disjoint and open.*

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Proof. That S^+ and S^- are disjoint is obvious. Since solutions of (1) depend continuously on their initial values, it is clear that S^+ and S^- are open.

LEMMA 2. *If* $\beta \geq e(1+\lambda)$ *then* $\beta \in S^+$.

Proof. First observe that if $0 < f' < 1$ in the open interval $(0, \tau)$ where $\tau \leq 1$, then for $0 < t < \tau$, we have

$$0 < \int_0^t f\,dn < 1.$$

Let $J = (0, \tau)$ be the maximal open interval with $\tau \leq 1$ in which $0 < f' < 1$. We can show $f' > t$ in $0 < t \leq \tau$. Rewrite equation (1) as

$$(f'' e^{\int_0^t f\,dn})' = \lambda(f'^2 - 1)\, e^{\int_0^t f\,dn} \tag{4}$$

Hence

$$(f'' e^{\int_0^t f\,dn})' > -\lambda e$$

and

$$f'' e^{\int_0^t f\,dn} > \beta - \lambda e, \quad 0 < t \leq \tau.$$

Using $\beta \geq e(1+\lambda)$, we have

$$f'' e > e, \quad 0 < t \leq \tau$$

and hence

$$f' > t, \quad 0 < t \leq \tau.$$

It follows from the definition of J that $\tau < 1$, and from $f'' > 1$ for $0 < t \leq \tau$ that $\beta \in S^+$.

LEMMA 3. *If* $\beta \leq 0$, *then* $\beta \in S^-$.

Proof. Clearly since $f'''(0) = -\lambda$, $\beta \notin S^+$. If further, $\beta \notin S^-$, we must have $-1 < -k < f'$ for all t. We can then show that f'' does not change sign. If f'' becomes positive, then f' must have a local minimum at which $-k < f'$, which violates equation (1). Hence f'' remains negative, implying f' is monotonic decreasing. That f' is bounded from below implies that $\lim_{t\to\infty} f' = -K$, $-k < -K < 0$, while f'' and f''' tend to zero. It then follows from equation (1) that

$$\lim_{t\to\infty} ff'' = \lambda(K^2 - 1) \neq 0.$$

Since $|f| \leq$ const. t, we have $|f''| \geq$ const.$/t$ for all sufficiently large t. Thus f'' is not integrable, contradicting the fact that f' has a finite limit. Hence, $\beta \in S^-$.

3. **The existence of a solution.** Since S^+ and S^- are disjoint, nonempty open sets, their complement D is also nonempty. Further, $\beta \in D$ implies that the solution of (1) and (3) can be continued for all $t > 0$ with $-1 < -k < f' < 1$.

LEMMA 4. *If* $\beta \in D$, $f'(\infty) = 1$.

Proof. We first show that f' is monotone. If f' is not monotone, f'' must change sign. Initially, we have $\beta > 0$. Supposing f'' changes sign at t_1, then the fact that f' cannot have a local minimum for $-1 < -k < f' < 1$ implies that $f'' < 0$ for $t > t_1$. A similar argument as in the proof of Lemma 3 shows that a contradiction is obtained. Hence f' is monotone increasing. Since f' is bounded from above, the limit of f' exists. If $\lim_{t\to\infty} f' \neq 1$, a contradiction is again obtained using the same argument as in the proof of Lemma 3. Hence $f'(\infty) = 1$.

We have thus proved the following result.

THEOREM. *The differential equation* (1) *subject to boundary conditions* (2) *has at least one solution.*

REMARKS. We have obtained an estimate for $f''(0)$ of the solution as

$$0 < f''(0) < e(1+\lambda).$$

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