Bull. Aust. Math. Soc. **109** (2024), 443–452
doi:10.1017/S0004972723000540

A NOTE ON THE GOORMAGHTIGH EQUATION CONCERNING DIFFERENCE SETS

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(Received 1 May 2023; accepted 12 May 2023; first published online 23 June 2023)

Abstract

Let p be a prime and let r, s be positive integers. In this paper, we prove that the Goormaghtigh equation $(x^m - 1)/(x - 1) = (y^n - 1)/(y - 1)$, $x, y, m, n \in \mathbb{N}$, $\min\{x, y\} > 1$, $\min\{m, n\} > 2$ with $(x, y) = (p^r, p^s + 1)$ has only one solution $(x, y, m, n) = (2, 5, 5, 3)$. This result is related to the existence of some partial difference sets in combinatorics.

2020 *Mathematics subject classification*: primary 11D61; secondary 05B10, 11J86.

Keywords and phrases: exponential Diophantine equation, Goormaghtigh equation, generalised Ramanujan–Nagell equation, Baker's method, partial difference set.

1. Introduction

Let $\mathbb{N}$ be the set of all positive integers. One hundred years ago, Ratat [27] and Rose and Goormaghtigh [28] conjectured that the equation

$$\frac{x^m - 1}{x - 1} = \frac{y^n - 1}{y - 1} \quad \text{for all } x, y, m, n \in \mathbb{N}, x \neq y, \min\{x, y\} > 1, \min\{m, n\} > 2, \tag{1.1}$$

has only two solutions $(x, y, m, n) = (2, 5, 5, 3)$ and $(2, 90, 13, 3)$ with $x < y$. Equation (1.1) is usually called the Goormaghtigh equation. The above conjecture is a very difficult problem in Diophantine equations. It was solved for some special cases (see [3, 5, 6, 9, 10, 12, 14–18, 22, 23, 26, 29–37]). But, in general, the problem is far from solved. The solution of (1.1) is closely related to some problems in number theory, combinatorics and algebra (see [1, 4, 13, 19, 21]). For example, while discussing the partial geometries admitting Singer groups in combinatorics, Leung *et al.* [19] found that the existence of partial difference sets in an elementary abelian 3-group is related to the solutions (x, y, m, n) of (1.1) with

$$(x, y) = (2^r, 3), \tag{1.2}$$

where r is a positive integer. In [19], they proved that (1.1) has no solutions (x, y, m, n) satisfying (1.2).

Let p be a prime and let r, s be positive integers. In this paper, we discuss the solutions (x, y, m, n) of (1.1) with

$$(x, y) = (p^r, p^s + 1). \tag{1.3}$$

Thus, we generalise the above-mentioned result in [19] to prove the following theorem.

THEOREM 1.1. *Equation (1.1) has only one solution* $(x, y, m, n) = (2, 5, 5, 3)$ *with (1.3).*

Combining Theorem 1.1 and [19, Corollary 37] with $q = \alpha + 1 = 2^s + 1$, we immediately obtain the following corollary which may be regarded as a generalisation of [19, Corollary 44].

COROLLARY 1.2. *Suppose that a proper partial geometry Π has at least two subgroup lines and that the parameters of the corresponding partial difference set have the form in* [19, (34)]. *Then, Π cannot be expressed as*

$$\Pi = \mathrm{pg}((2^s + 1)^u, (2^r - 1)(2^s + 1)^u + 2^s + 1, 2^s)$$

with $r, s, t \in \mathbb{N}$.

The organisation of the paper is as follows. In Section 2, we prove Theorem 1.1 in the case where $r \le s$ using an upper bound for the number of solutions of the generalised Ramanujan–Nagell equations due to Bugeaud and Shorey [8]. In Section 3, using a lower bound for linear forms in three logarithms due to Matveev [24], we show that if $r > s$ and $p^r > 3.436 \times 10^{15}$, then (1.1) has no solutions (x, y, m, n) with (1.3). Thus, the remaining case to be checked is $r > s$ and $p^r < 3.436 \times 10^{15}$. For this, we appeal to the reduction method due to Dujella and Pethő [11], based on [2, Lemma] by Baker and Davenport, to complete the proof of Theorem 1.1 in Section 4.

2. The case $r \le s$

LEMMA 2.1 [20]. *The equation*

$$\frac{X^k - 1}{X - 1} = Y^l \quad \text{for all } X, Y, k, l \in \mathbb{N}, X > 1, Y > 1, k > 2, l > 1, \tag{2.1}$$

has only two solutions, $(X, Y, k, l) = (3, 11, 5, 2)$ *and* $(7, 20, 4, 2)$ *with* $2 \mid l$.

Let D_1 and D_2 be coprime positive integers and let p be a prime with $p \nmid D_1 D_2$. Further, let $N(D_1, D_2, p)$ denote the number of solutions (X, Z) of the equation

$$D_1 X^2 + D_2 = p^Z \quad \text{for all } X, Z \in \mathbb{N}. \tag{2.2}$$

Combining the results in [7, 8], we immediately obtain the following two lemmas.

LEMMA 2.2. *We have* $N(D_1, D_2, 2) \le 1$*, except for the following cases:*

(i) $N(1, 7, 2) = 5$, $(X, Z) = (1, 3), (3, 4), (5, 5), (11, 7)$ *and* $(181, 15)$*;*
(ii) $N(3, 5, 2) = 3$, $(X, Z) = (1, 3), (3, 5)$ *and* $(13, 9)$*;*
(iii) $N(7, 1, 2) = 2$, $(X, Z) = (1, 3)$ *and* $(3, 6)$*;*

https://doi.org/10.1017/S0004972723000540 Published online by Cambridge University Press

(iv) $N(1, 2^{k+2} - 1, 2) = 2$, $(X, Z) = (1, k + 2)$ *and* $(2^{k+1} - 1, 2k + 2)$, *where* k *is a positive integer with* $k > 1$;
(v) $N(3, 29, 2) = 2$, $(X, Z) = (1, 5)$ *and* $(209, 17)$;
(vi) $N(5, 3, 2) = 2$, $(X, Z) = (1, 3)$ *and* $(5, 7)$;
(vii) $N(13, 3, 2) = 2$, $(X, Z) = (1, 4)$ *and* $(71, 16)$;
(viii) $N(21, 11, 2) = 2$, $(X, Z) = (1, 5)$ *and* $(79, 17)$; *and*
(ix) *if* $D_1a^2 = 2^k - \delta$ *and* $D_2 = 3 \cdot 2^k + \delta$, *where* a, k *are positive integers with* $k > 1$ *and* $\delta \in \{1, -1\}$, *then* $N(D_1, D_2, 2) = 2$, $(X, Z) = (a, k + 2)$ *and* $((2^{k+1} + \delta)a, 3k + 2)$.

LEMMA 2.3. *If* $p \neq 2$, *then* $N(D_1, D_2, p) \leq 1$, *except for the following cases:*

(i) $N(2, 1, 3) = 3$, $(X, Z) = (1, 1), (2, 2)$ *and* $(11, 5)$; *and*
(ii) *if* $4D_1a^2 = p^k - \delta$ *and* $4D_2 = 3p^k + \delta$, *where* a, k *are positive integers and* $\delta \in \{1, -1\}$, *then* $N(D_1, D_2, p) = 2$, $(X, Z) = (a, k)$ *and* $((2p^k + \delta)a, 3k)$.

PROPOSITION 2.4. *If* $r \leq s$, *then (1.1) has only one solution* $(x, y, m, n) = (2, 5, 5, 3)$ *with (1.3).*

PROOF. We now assume that (x, y, m, n) is a solution of (1.1) with (1.3). Then

$$\frac{p^{rm} - 1}{p^r - 1} = \frac{(p^s + 1)^n - 1}{p^s}. \tag{2.3}$$

When $r = s$, by (2.3),

$$\frac{p^{r(m+1)} - 1}{p^r - 1} = (p^r + 1)^n. \tag{2.4}$$

If $2 \mid n$, by (2.4), the equation (2.1) has a solution $(X, Y, k, l) = (p^r, p^r + 1, m + 1, n)$ with $2 \mid l$. However, since $m > 2$, by Lemma 2.1, this is impossible. So $2 \nmid n$ and $n \geq 3$.

Since $p^r + 1 > 2$ and $p^r \equiv -1 \pmod{(p^r + 1)}$, by (2.4),

$$0 \equiv (p^r - 1)(p^r + 1)^n \equiv p^{r(m+1)} - 1 \equiv (-1)^{m+1} - 1 \pmod{(p^r + 1)},$$

from which we get $2 \mid m + 1$. Hence, by (2.4),

$$\frac{(p^{2r})^{(m+1)/2} - 1}{p^{2r} - 1} = (p^r + 1)^{n-1}. \tag{2.5}$$

Recall that $2 \nmid n$ and $n \geq 3$. We see from (2.5) that if $(m + 1)/2 > 2$, then (2.1) has a solution $(X, Y, k, l) = (p^{2r}, p^r + 1, (m + 1)/2, n - 1)$ with $2 \mid l$. But, by Lemma 2.1 again, this is impossible. Therefore, since $2 \nmid m$ and $m \geq 3$, we get $m = 3$, and by (2.5),

$$\frac{(p^{2r})^{(m+1)/2} - 1}{p^{2r} - 1} = \frac{p^{4r} - 1}{p^{2r} - 1} = p^{2r} + 1 = (p^r + 1)^{n-1} \geq (p^r + 1)^2 > p^{2r} + 1,$$

which is a contradiction. Thus, (1.1) has no solutions (x, y, m, n) with (1.3) and $r = s$.

When $r < s$, by (2.3),

$$(p^r - 1)(p^s + 1)^n + (p^s - p^r + 1) = p^{rm+s}. \tag{2.6}$$

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Since $r < s$, $p^r - 1$, $p^s + 1$ and $p^s - p^r + 1$ are positive integers satisfying

$$\gcd((p^r - 1)(p^s + 1), p^s - p^r + 1) = 1, \quad p \nmid (p^r - 1)(p^s + 1)(p^s - p^r + 1). \tag{2.7}$$

If $2 \mid n$, by (2.6), the equation (2.2) has a solution

$$(X, Z) = ((p^s + 1)^{n/2}, rm + s)$$

for $(D_1, D_2) = (p^r - 1, p^s - p^r + 1)$. Notice that (2.2) has another solution $(X, Z) = (1, s)$ for $(D_1, D_2) = (p^r - 1, p^s - p^r + 1)$. So

$$N(p^r - 1, p^s - p^r + 1, p) \geq 2. \tag{2.8}$$

However, by (2.7), using Lemmas 2.2 and 2.3, (2.8) is false.

Similarly, if $2 \nmid n$, by (2.6), the equation (2.2) has a solution

$$(X, Z) = ((p^s + 1)^{(n-1)/2}, rm + s)$$

for $(D_1, D_2) = ((p^r - 1)(p^s + 1), p^s - p^r + 1)$. Moreover, (2.2) has another solution $(X, Z) = (1, r + s)$ for $(D_1, D_2) = ((p^r - 1)(p^s + 1), p^s - p^r + 1)$. So

$$N((p^r - 1)(p^s + 1), p^s - p^r + 1, p) \geq 2. \tag{2.9}$$

Applying Lemmas 2.2 and 2.3 to (2.9), we can only obtain

$$(p, r, x) = (2, 1, 2). \tag{2.10}$$

Therefore, by (1.3) and (2.10), we get $(D_1, D_2) = (5, 3)$ and $(x, y, m, n) = (2, 5, 5, 3)$. Thus, the proposition is proved. □

3. The case $r > s$

In this section, we assume that $r > s$ and (x, y, m, n) is a solution of (1.1) with (1.3).

LEMMA 3.1. *If $(p, s) \neq (2, 1)$, then $n > p^r$.*

PROOF. By (2.3),

$$\frac{p^{rm} - 1}{p^r - 1} = \sum_{i=0}^{m-1} p^{ri} = \sum_{j=1}^{n} \binom{n}{j} p^{s(j-1)} = \frac{(p^s + 1)^n - 1}{p^s},$$

from which we get

$$p^r \left(\frac{p^{r(m-1)} - 1}{p^r - 1} \right) = (n - 1) + \sum_{j=2}^{n} \binom{n}{j} p^{s(j-1)}. \tag{3.1}$$

Since $n > 2$ and $p \nmid (p^{r(m-1)} - 1)/(p^r - 1)$, we see from (3.1) that $p \mid n - 1$ and

$$p^r \parallel (n - 1) + \sum_{j=2}^{n} \binom{n}{j} p^{s(j-1)}. \tag{3.2}$$

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Let

$$p^t \,\|\, n-1 \tag{3.3}$$

and

$$p^{t_j} \,\|\, j \quad \text{for all } j = 2, \dots, n, \tag{3.4}$$

where t is a positive integer and t_j $(j = 2, \dots, n)$ are nonnegative integers. Then

$$t_j \le \frac{\log j}{\log p} \le j-1 \quad \text{for all } j = 2, \dots, n. \tag{3.5}$$

Notice that both symbols '≤' in (3.5) can be taken by equal signs '=' if and only if $(p, t_j, j) = (2, 1, 2)$. It follows from (3.5) that if $(p, t_j) \ne (2, 1)$, then

$$t_j < j-1 \quad \text{for all } j = 2, \dots, n. \tag{3.6}$$

Hence, since $\gcd(j, j-1) = 1$ and $(p, s) \ne (2, 1)$, by (3.3), (3.4) and (3.6),

$$\binom{n}{j} p^{s(j-1)} \equiv n(n-1)\binom{n-2}{j-2}\frac{p^{s(j-1)}}{(j-1)j} \equiv 0 \pmod{p^{t+1}} \quad \text{for all } j = 2, \dots, n. \tag{3.7}$$

Therefore, by (3.3) and (3.7),

$$p^t \,\|\, (n-1) + \sum_{j=2}^{n} \binom{n}{j} p^{s(j-1)}. \tag{3.8}$$

Comparing (3.2) and (3.8),

$$t = r. \tag{3.9}$$

Further, since $n > 1$, by (3.3) and (3.9), we obtain $n - 1 \ge p^r$ and $n > p^r$. The lemma is proved. □

Let $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ be the sets of all integers, rational numbers and real numbers, respectively. Let α be an algebraic number of degree d and let $\alpha^{(1)}, \dots, \alpha^{(d)}$ denote all the conjugates of α. Further, let

$$f(X) = a\prod_{i=1}^{d}(X - \alpha^{(i)}) \in \mathbb{Z}[X] \quad \text{for all } a \in \mathbb{N}$$

denote the minimal polynomial of α. Then

$$\mathrm{h}(\alpha) = \frac{1}{d}\Big(\log a + \sum_{i=1}^{d} \log \max\{1, |\alpha^{(i)}|\}\Big)$$

is called the absolute logarithmic height of α.

LEMMA 3.2 ([24, 25]). *Let α_1, α_2, α_3 be three distinct real algebraic numbers with* $\min\{\alpha_1, \alpha_2, \alpha_3\} > 1$ *and let b_1, b_2, b_3 be three positive integers with* $\gcd(b_1, b_2, b_3) = 1$.

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Further, let

$$\Lambda = b_1 \log \alpha_1 + b_2 \log \alpha_2 - b_3 \log \alpha_3.$$

If $\Lambda \neq 0$, *then*

$$\log |\Lambda| > -CD^2 A_1 A_2 A_3 \log(1.5eDB \log(eD)),$$

where

$$D = [\,\mathbb{Q}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{Q}\,], \quad D' = [\,\mathbb{R}(\alpha_1, \alpha_2, \alpha_3) : \mathbb{R}\,], \tag{3.10}$$

$$A_j \geq \max\{D\,\mathrm{h}(\alpha_j), |\log \alpha_j|\} \quad \text{for } j = 1, 2, 3, \tag{3.11}$$

$$B \geq \max \left\{ b_j \frac{A_j}{A_1} \,\middle|\, j = 1, 2, 3 \right\}, \tag{3.12}$$

$$C = \frac{5 \times 16^5}{6D'} e^3 (7 + 2D') \left(\frac{3e}{2}\right)^{D'} (26.25 + \log(D^2 \log(eD))). \tag{3.13}$$

Proposition 3.3. *If* $r > s$ *and* $p^r > 3.436 \times 10^{15}$, *then (1.1) has no solutions* (x, y, m, n) *with (1.3).*

Proof. By [19], the proposition holds for $(p, s) = (2, 1)$. We can therefore assume that $(p, s) \neq (2, 1)$. By (2.3),

$$(p^r - 1)(p^s + 1)^n = p^{rm+s} + (p^r - p^s - 1). \tag{3.14}$$

Since $p^r - p^s - 1 > 0$, taking the logarithms of both sides of (3.14),

$$\log(p^r - 1) + n \log(p^s + 1) = (rm + s) \log p + \log\left(1 + \frac{p^r - p^s - 1}{p^{rm+s}}\right). \tag{3.15}$$

Since $\log(1 + \varepsilon) < \varepsilon$ for any $\varepsilon > 0$, by (3.15),

$$\begin{aligned} 0 &< \log(p^r - 1) + n \log(p^s + 1) - (rm + s) \log p \\ &= \log\left(1 + \frac{p^r - p^s - 1}{p^{rm+s}}\right) < \frac{p^r - p^s - 1}{p^{rm+s}}. \end{aligned} \tag{3.16}$$

Take

$$\alpha_1 = p^r - 1, \quad \alpha_2 = p^s + 1, \quad \alpha_3 = p, \quad b_1 = 1, \quad b_2 = n, \quad b_3 = rm + s \tag{3.17}$$

and

$$\Lambda = \log(p^r - 1) + n \log(p^s + 1) - (rm + s) \log p. \tag{3.18}$$

By (3.16) and (3.18), we have $\Lambda > 0$ and

$$(rm + s) \log p + \log \Lambda < \log(p^r - p^s - 1) < \log(p^r - 1). \tag{3.19}$$

In order to apply Lemma 3.2, by (3.10), (3.11) and (3.17), we can choose the following parameters.

$$D = D' = 1, \tag{3.20}$$

$$A_1 = \log(p^r - 1), \quad A_2 = \log(p^s + 1), \quad A_3 = \log p. \tag{3.21}$$

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Further, by (3.12), (3.13), (3.16), (3.20) and (3.21),

$$B = \frac{(rm+s)\log p}{\log(p^r-1)}$$

and

$$C < 1.691 \times 10^{10}. \tag{3.22}$$

Applying Lemma 3.2 to (3.17) and (3.18), by (3.20)–(3.22),

$$\log \Lambda > -1.691 \times 10^{10}(\log(p^r-1))(\log(p^s+1))(\log p) \times \Big(1.406 + \log\Big(\frac{(rm+s)\log p}{\log(p^r-1)}\Big)\Big). \tag{3.23}$$

Substituting (3.23) into (3.19), we get

$$1 + 1.691 \times 10^{10}(\log(p^s+1))(\log p)\Big(1.406 + \log\Big(\frac{(rm+s)\log p}{\log(p^r-1)}\Big)\Big) > \frac{(rm+s)\log p}{\log(p^r-1)}. \tag{3.24}$$

Hence, since $(p,s) \neq (2,1)$ and $p^s + 1 \geq 4$, by (3.23), we can calculate that

$$\frac{(rm+s)\log p}{\log(p^r-1)} < 1.501 \times 10^{12}(\log(p^s+1))(\log p)(\log\log(p^s+1)). \tag{3.25}$$

On the other hand, by (3.16),

$$\frac{(rm+s)\log p}{\log(p^r-1)} > \Big(1 - \frac{p^r-p^s-1}{p^{rm+s}\log(p^r-1)}\Big) + \frac{n\log(p^s+1)}{\log(p^r-1)} > \frac{n\log(p^s+1)}{\log(p^r-1)}. \tag{3.26}$$

Since $\log p \leq (\log p^r)/2$ for $r \geq 2$, the combination of (3.25) and (3.26) yields

$$n < 1.501 \times 10^{12}(\log p)(\log(p^r-1))(\log\log(p^s+1)) < 7.505 \times 10^{11}(\log p^r)^2(\log\log p^r). \tag{3.27}$$

Further, since $(p,s) \neq (2,1)$, by Lemma 3.1, we have $n > p^r$. Hence, by (3.27),

$$p^r < 7.505 \times 10^{11}(\log p^r)^2(\log\log p^r). \tag{3.28}$$

Therefore, by (3.28), we obtain $p^r < 3.436 \times 10^{15}$. Thus, if $r > s$ and $p^r > 3.436 \times 10^{15}$, then (1.1) has no solutions (x,y,m,n) with (1.3). The proposition is proved. □

4. Proof of Theorem 1.1

We continue to assume that $r > s$ and that (x,y,m,n) is a solution of (1.1) with (1.3). Put $m' = rm + s$. By (3.25),

$$m' < 1.501 \times 10^{12}(\log(p^r-1))(\log(p^s+1))(\log\log(p^s+1)). \tag{4.1}$$

Since Proposition 3.3 implies that $p^s \le p^{r-1} < 1.718 \times 10^{15}$, we see from (4.1) that

$$m' < 6.702 \times 10^{15}. \tag{4.2}$$

On the other hand, we deduce from Lemma 3.1 and (3.26) that

$$m' > \frac{n\log(p^s+1)}{\log p} > \frac{p^r\log(p^r+1)}{\log p}. \tag{4.3}$$

Now, by (3.16),

$$0 < n - m'\kappa + \mu < AB^{-m'}, \tag{4.4}$$

where

$$m' = rm + s, \quad \kappa = \frac{\log p}{\log(p^s+1)}, \quad \mu = \frac{\log(p^r-1)}{\log(p^s+1)}, \quad A = \frac{p^r-p^s-1}{\log(p^s+1)}, \quad B = p.$$

LEMMA 4.1. *Let $\kappa, \mu, A > 0$ and $B \ge 1$ be real numbers and let M' be a positive integer. Let p/q be a convergent of the continued fraction expansion of κ such that $q > 6M'$, and put $\varepsilon = \|\mu q\| - M'\|\kappa q\|$, where $\|\cdot\|$ denotes the distance from the nearest integer. If $\varepsilon > 0$, then inequality (4.4) has no integer solution (n, m') satisfying*

$$\frac{\log(Aq/\varepsilon)}{\log B} \le m' \le M'.$$

PROOF. Since the assertion is identical with that of [11, Lemma 5a] if the middle term of inequalities (4.4) is multiplied by -1, the lemma is proved in the same way as [11, Lemma 5a]. □

By Proposition 3.3, (4.2) and (4.4), we may apply Lemma 4.1 with $M' = 6.702 \times 10^{15}$ in the ranges

$$2 \le p < \sqrt{R}, \quad 1 \le s < r < \log_p R$$

with $(p, s) \ne (2, 1)$, where $R = 3.436 \times 10^{15}$. For $7 \le p < \sqrt{R}$, the first step of reduction gives $m' \le 43$, which contradicts (4.3) with $p \ge 7$ and $r \ge 2$. For $p = 5$, the first and second steps of reduction give $m' \le 52$ and $m' \le 30$, respectively. The latter contradicts (4.3) with $p = 5$ and $r \ge 2$. For $p = 3$, the first and second steps of reduction give $m' \le 75$ and $m' \le 45$, respectively, which, together with (4.3), yields $r = 2$. For $p = 2$, the first and second steps of reduction give $m' \le 126$ and $m' \le 75$, respectively, from which by (4.3) we obtain $r \in \{3, 4\}$.

Thus, it remains to consider the cases where

$$(p, r, s) \in \{(2, 3, 2), (2, 4, 2), (2, 4, 3), (3, 2, 1)\}. \tag{4.5}$$

In view of the bounds for $m' = rm + s$ obtained above, it suffices to check that (3.14) with (4.5) has no solution (m, n) in the ranges $m \le 24$ and $n \le 34$, which can be easily done. Therefore, the theorem is proved.

https://doi.org/10.1017/S0004972723000540 Published online by Cambridge University Press

Acknowledgement

The authors thank the anonymous referee for suggesting the addition of a connection between Theorem 1.1 and the results in [19], which led to Corollary 1.2.

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